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Cubic of Positive Implicative Ideals in KU-Semigroup

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Abstract

In this paper, we define a cubic positive implicative-ideal, a cubic implicative-ideal and a cubic commutative-ideal of a semigroup in KU-algebra as a generalization of a fuzzy (positive implicative-ideal, an implicative-ideal and a commutative-ideal) of a semigroup in KU-algebra. Some relations between these types of cubic ideals are discussed. Also, some important properties of these ideals are studied. Finally, some important theories are discussed. It is proved that every cubic commutative-ideal, cubic positive implicative-ideal, and cubic implicative-ideal are a cubic ideal, but not conversely. Also, we show that if Θ is a cubic positive implicative-ideal and a cubic commutative-ideal then Θ is a cubic implicative-ideal. Some examples of the opposite direction of the previous theories are obtained.

Keywords A KU-semigroup, cubic ideal, cubic k-ideal, cubic positive implicative ideal, cubic implicative ideal, cubic commutative ideal.

1. **Introduction**

The structure of KU-algebra was studied by Prabpayak and Leerawat [1,2]. They gave a homomorphism of KU-algebras and proved some important theories. The idea of a fuzzy set was initialed in 1965 by the author Zadeh [3]. Since then this concept has been applied in many different branches of mathematic. The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy groups by Rosenfeld [4]. Fuzzy p-ideals and fuzzy H-ideals in BCI-algebras are introduced in [5]. Jun et al, in [6] studied of the Fuzzy Implicative ideals of BCK-algebras. Also, Mostafa et al [7] introduced the notion of fuzzy KU-ideals of KU-algebras and they investigated several basic properties which are associated to fuzzy KU-ideals. Many Mathematicians have studied "a fuzzy" for some algebraic structures, see [8-12]. There are several kinds of fuzzy set extensions in the fuzzy set theory, for example, interval-valued fuzzy sets and bipolar-valued fuzzy sets, which were introduced in [13-17].
The concept of cubic subalgebras /ideals in BCK/BCI-algebras was introduced by Jun et al. [18, 19]. They discussed the relationship between a cubic subalgebra and a cubic ideal. And then, Yaqoob et al [20] introduced the notion of cubic KU-algebra which is a generalization of the concept of fuzzy KU-ideals of KU-algebras. After that, Kareem and Hasan[21] introduced the

notion of a KU-algebra with semigroup which is called a KU-semigroup and defined some types of ideals in this concept. Also, they studied the fuzzy ideals of a KU-semigroup. After that, Kareem and Hasan [22] introduced the Cubic ideals of a semigroup in KU-algebra and defined some types of ideals in this concept. Senapati et al[23,24] introduced the two concepts which are cubic ideal and implicative ideal. Some authors introduced a cubic set of different structures. See [25-30].

In this work, the notion of cubic (positive implicative, implicative and commutative)-ideal are discussed and the relationship among these types are studied.

2. Basic Concepts

Definition (1) [1]**.** An algebra$(N,*,0)$ is named a KU-algebra if, for all $\varsigma, \omega, \kappa \in N$,

(ku_1) $(\varsigma * \omega) * [(\omega * \kappa) * (\varsigma * \kappa)] = 0$,

(ku_2) $\varsigma * 0 = 0$,

(ku_3) $0 * \varsigma = \varsigma$,

(ku_4) $\varsigma * \omega = 0$ and $\omega * \varsigma = 0$ implies $\varsigma = \omega$,

(ku_5) $\varsigma * \varsigma = 0$.

The binary relation $\leq$ on N is define by $\varsigma \leq \omega \Leftrightarrow \omega * \varsigma = 0$.

Theorem(2)[2]. Let$(N,*,0)$ be a KU-algebra. Then for all $\varsigma, \omega, \kappa \in N$:

(1) If $\varsigma \leq \omega$ imply$\omega * \kappa \leq \varsigma * \kappa$.

(2) $\varsigma * (\omega * \kappa) = \omega * (\varsigma * \kappa)$.

(3) $(\omega * \varsigma) * \varsigma \leq \omega$.

(4)$\left((\omega * \varsigma) * \varsigma\right) * \varsigma)) = \omega * \varsigma$.

Definition(3) [21]. A nonempty set N with $*, \circ$ and 0 is called a KU-semigroup if

(I)The set N with $*$ and 0 is a KU-algebra.

(II) The set Nwith $\circ$ and 0 is a semigroup.

(III) $\varsigma \circ (\omega * \kappa) = (\varsigma \circ \omega) * (\varsigma \circ \kappa)$ and$(\varsigma * \omega) \circ \kappa = (\varsigma \circ \kappa) * (\omega \circ \kappa)$, for all $\varsigma, \omega, \kappa \in N$.

Example (4) [21]. Let $N = \{0,1,2,3\}$ with two operations $*$ and $\circ$ defined by **Table 1**:

Table 1. a KU-semigroup

$*$	0	1	2	3
0	0	1	3	2
1	0	0	0	2
2	2	0	0	1
3	0	0	0	0

$\circ$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

It follows that $(N,*,\circ,0)$ is a KU-semigroup.

We recall, an interval valued fuzzy set $\tilde{\mu}$ in N is defined as $\tilde{\mu} = \{\langle ς, [\mu^L(ς), \mu^U(ς)], ς \in N\rangle\}$, where the ordinary fuzzy sets $\mu^L: N \to [0,1]$ and $\mu^U: N \to [0,1]$ are called a lower fuzzy set and an upper fuzzy set of $\tilde{\mu}$ respectively.

Also, we recall some definitions of a cubic subset of a KU-semigroup from [22].

Definition(5) [18]. The cubic set Θ of a non-empty set N is $\Theta = \{\langle ς, \tilde{\mu}_\Theta(ς), \lambda_\Theta(ς)\rangle : ς \in N\}$, which is briefly indicated by $\Theta = \langle \tilde{\mu}_\Theta, \lambda_\Theta\rangle$ where $\tilde{\mu}_\Theta(ς) = [\tilde{\mu}_\Theta{}^L(ς), \tilde{\mu}_\Theta{}^U(ς)]$ is an interval valued fuzzy set in N and $\lambda_\Theta(ς)$ is a fuzzy set in N.The set $\{ς \in N : \tilde{\mu}_\Theta(ς) \geq \tilde{t}, \lambda_\Theta(ς) \leq \alpha\}$ is called a cubic level set of $\Theta = \langle \tilde{\mu}_\Theta, \lambda_\Theta\rangle$, where $[0,0] \leq \tilde{t} \leq [1,1]$ and $\alpha \in [0,1]$.

Definition(6) [22]. The cubic set Θ of N is named a cubic sub KU-semigroup if for all $ς, ω \in N$,

1. $\tilde{\mu}_\Theta(ς * ω) \geq rmin\{\tilde{\mu}_\Theta(ς), \tilde{\mu}_\Theta(ω)\}, \lambda_\Theta(ς * ω) \leq max\{\lambda_\Theta(ς), \lambda_\Theta(ω)\}$.
2. $\tilde{\mu}_\Theta(ς \circ ω) \geq rmin\{\tilde{\mu}_\Theta(ς), \tilde{\mu}_\Theta(ω)\}, \quad \lambda_\Theta(ς \circ ω) \leq max\{\lambda_\Theta(ς), \lambda_\Theta(ω)\}$.

Definition(7) [22]. The cubic set Θ of N is named a ***cubic ideal***, if for all $ς, ω \in N$.

(CI₁) $\tilde{\mu}_\Theta(0) \geq \tilde{\mu}_\Theta(ς)$ and $\lambda_\Theta(0) \leq \lambda_\Theta(ς)$.

(CI₂) $\tilde{\mu}_\Theta(ω) \geq rmin\{\tilde{\mu}_\Theta(ς * ω), \tilde{\mu}_\Theta(ς)\}, \quad \lambda_\Theta(ω) \leq max\{\lambda_\Theta(ς * ω), \lambda_\Theta(ς)\}$.

(CI₃) $\tilde{\mu}_\Theta(ς \circ ω) \geq rmin\{\tilde{\mu}_\Theta(ς), \tilde{\mu}_\Theta(ω)\}, \quad \lambda_\Theta(ς \circ ω) \leq max\{\lambda_\Theta(ς), \lambda_\Theta(ω)\}$.

Definition(8) [22]. The cubic set Θ of N is named a ***cubic k-ideal*** if for all $ς, ω, κ \in N$

$(\boldsymbol{Ck_1})$ $\tilde{\mu}_\Theta(0)) \geq \tilde{\mu}_\Theta(ς)$, and $\lambda_\Theta(0) \leq \lambda_\Theta(ς)$.

$(\boldsymbol{Ck_2})$ $\tilde{\mu}_\Theta(ς * κ) \geq rmin\{\tilde{\mu}_\Theta(ς * (ω * κ)), \tilde{\mu}_\Theta(ω)\}$.

$\lambda_\Theta(ς * κ) \leq max\{\lambda_\Theta(ς * (ω * κ)), \lambda_\Theta(ω)\}$.

$(\boldsymbol{Ck_3})$ $\tilde{\mu}_\Theta(ς \circ ω) \geq rmin\{\tilde{\mu}_\Theta(ς), \tilde{\mu}_\Theta(ω)\}, \lambda_\Theta(ς \circ ω) \leq max\{\lambda_\Theta(ς), \lambda_\Theta(ω)\}$.

Theorem(9) [22]. Let $(N, *, \circ, 0)$ be a KU-semigroup. A non-empty subset Θ is a cubic *k*-ideal if and only if it is a cubic ideal of N .

3. Cubic Positive Implicative-Ideals of N

Definition(10). The cubic set Θ of N is named ***a cubic positive implicative-ideal*** if for all $ς, ω, κ \in N$

$(\boldsymbol{Cp_1})$ $\tilde{\mu}_\Theta(0)) \geq \tilde{\mu}_\Theta(ς)$, and $\lambda_\Theta(0) \leq \lambda_\Theta(ς)$.

$(\boldsymbol{Cp_2})$ $\tilde{\mu}_\Theta(κ * ω) \geq rmin\{\tilde{\mu}_\Theta(κ * (ς * ω)), \tilde{\mu}_\Theta(κ * ς)\}$,

$\lambda_\Theta(κ * ω) \leq max\{\lambda_\Theta(κ * (ς * ω)), \lambda_\Theta(κ * ς)\}$.

$(\boldsymbol{Cp_3})$ $\tilde{\mu}_\Theta(ς \circ ω) \geq rmin\{\tilde{\mu}_\Theta(ς), \tilde{\mu}_\Theta(ω)\}, \lambda_\Theta(ς \circ ω) \leq max\{\lambda_\Theta(ς), \lambda_\Theta(ω)\}$.

Definition(11). The cubic set Θ of N is named a ***cubic implicative-ideal*** if for all $ς, ω, κ \in N$

$(\boldsymbol{CV_1})$ $\tilde{\mu}_\Theta(0)) \geq \tilde{\mu}_\Theta(ς)$, and $\lambda_\Theta(0) \leq \lambda_\Theta(ς)$.

$(\boldsymbol{CV_2})$ $\tilde{\mu}_\Theta((ς * ω) * ς) \geq rmin\{\tilde{\mu}_\Theta(κ * ((ς * ω) * ς)), \tilde{\mu}_\Theta(κ)\}$,

$\lambda_\Theta((\varsigma * \omega) * \varsigma) \leq max\,\{\lambda_\Theta(\kappa * ((\varsigma * \omega) * \varsigma)\,, \lambda_\Theta(\kappa)\}$.

$(\boldsymbol{CV_3}) \tilde{\mu}_\Theta(\varsigma \circ \omega) \geq rmin\,\{\tilde{\mu}_\Theta(\varsigma)\,, \tilde{\mu}_\Theta(\omega)\}$, $\lambda_\Theta(\varsigma \circ \omega) \leq max\{\lambda_\Theta(\varsigma)\,,\ \lambda_\Theta(\omega)\}$.

Definition(12). The cubic set Θ of N is named a ***cubic commutative-ideal*** if for all $\varsigma, \omega, \kappa \in$ N

$(\boldsymbol{CC_1})$ $\tilde{\mu}_\Theta(0)) \geq \tilde{\mu}_\Theta(\varsigma)$, and $\lambda_\Theta(0) \leq \lambda_\Theta(\varsigma)$.

$(\boldsymbol{CC_2})$ $\tilde{\mu}_\Theta\big((\varsigma * \omega) * \omega) * \varsigma\big) \geq rmin\{\tilde{\mu}_\Theta\big(\omega * (\kappa * \varsigma)\big),\ \ \tilde{\mu}_\Theta(\kappa)\}$,

$\lambda_\Theta\big((\varsigma * \omega) * \omega) * \varsigma\big) \leq max\,\{\lambda_\Theta\big(\omega * (\kappa * \varsigma)\big)\,, \lambda_\Theta(\kappa)\}$.

$(\boldsymbol{CC_3})$ $\tilde{\mu}_\Theta(\varsigma \circ \omega) \geq rmin\,\{\tilde{\mu}_\Theta(\varsigma)\,, \tilde{\mu}_\Theta(\omega)\}$, $\lambda_\Theta(\varsigma \circ \omega) \leq max\,\{\lambda_\Theta(\varsigma)\,,\ \lambda_\Theta(\omega)\}$.

Lemma(13). In a cubic positive implicative-ideal Θ of N, if $\varsigma \leq \omega$, then

$\tilde{\mu}_\Theta(\varsigma) \geq \tilde{\mu}_\Theta(\omega)$ and $\lambda_\Theta(\varsigma) \leq \lambda_\Theta(\omega)$, for all $\varsigma, \omega \in$ N.

Proof. Since $\varsigma \leq \omega \Rightarrow\ \omega * \varsigma = 0$, since Θ is a cubic positive implicative-ideal, then

$\tilde{\mu}_\Theta(\kappa * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\kappa * (\omega * \varsigma)), \tilde{\mu}_\Theta(\kappa * \omega)\}$, put κ=0

$\tilde{\mu}_\Theta(0 * \varsigma) \geq r\,min\{\tilde{\mu}_\Theta(0 * (\omega * \varsigma)), \tilde{\mu}_\Theta(0 * \omega)\}$,

$\tilde{\mu}_\Theta(\varsigma) \geq r\,min\{\tilde{\mu}_\Theta(0 * 0), \tilde{\mu}_\Theta(\omega)\} = \tilde{\mu}_\Theta(\omega)$, and

$\lambda_\Theta(\kappa * \varsigma) \leq max\,\{\lambda_\Theta(\kappa * (\omega * \varsigma)), \lambda_\Theta(\kappa * \omega)\}$, put κ=0

$\lambda_\Theta(0 * \varsigma) \leq max\{\,\lambda_\Theta(0 * (\omega * \varsigma)\,, \lambda_\Theta(0 * \omega)\}$,

$\lambda_\Theta(\varsigma) \leq max\{\,\lambda_\Theta(0 * 0)\,,\ \lambda_\Theta(\omega)\} = \lambda_\Theta(\omega)$. ■

Theorem(14). In a cubic positive implicative-ideal Θ of N, if $\varsigma * \omega \leq \kappa$, then

$\tilde{\mu}_\Theta(\omega) \geq rmin\{\tilde{\mu}_\Theta(\kappa)\,, \tilde{\mu}_\Theta(\varsigma)\}$, and $\lambda_\Theta(\omega) \leq max\,\{\lambda_\Theta(\kappa)\,, \lambda_\Theta(\varsigma)\}$, for all $\varsigma, \omega, \kappa \in$ N.

Proof. Suppose $\varsigma * \omega \leq \kappa$holds, then by Lemma**(13)** we get

$\tilde{\mu}_\Theta(\varsigma * \omega) \geq \tilde{\mu}_\Theta(\kappa)$, and $\lambda_\Theta(\varsigma * \omega) \leq \lambda_\Theta(\kappa)$.

Since Θ is a cubic positive implicative-ideal, that is

$\tilde{\mu}_\Theta(\kappa * \omega) \geq rmin\{\tilde{\mu}_\Theta(\kappa * (\varsigma * \omega)), \tilde{\mu}_\Theta(\kappa * \varsigma)\}$, put $\kappa = 0$

$\tilde{\mu}_\Theta(0 * \omega) \geq rmin\{\tilde{\mu}_\Theta(0 * (\varsigma * \omega)\big), \tilde{\mu}_\Theta(0 * \varsigma)\}$,

$\tilde{\mu}_\Theta(\omega) \geq rmin\{\tilde{\mu}_\Theta(\varsigma * \omega), \tilde{\mu}_\Theta(\varsigma)\}$, but $\tilde{\mu}_\Theta(\varsigma * \omega) \geq \tilde{\mu}_\Theta(\kappa)$, then

$\tilde{\mu}_\Theta(\omega) \geq rmin\{\tilde{\mu}_\Theta(\kappa),\ \tilde{\mu}_\Theta(\varsigma)\}$, and

$\lambda_\Theta(\kappa * \omega) \leq max\{\lambda_\Theta(\kappa * (\varsigma * \omega)), \lambda_\Theta(\kappa * \varsigma)\}$, put $\kappa = 0$, we obtain

$\lambda_\Theta(\omega) \leq max\{\lambda_\Theta((\varsigma * \omega)), \lambda_\Theta(\varsigma)\}$, but $\lambda_\Theta(\varsigma * \omega) \leq \lambda_\Theta(\kappa)$, then

$\lambda_\Theta(\omega) \leq max\{\lambda_\Theta(\kappa),\ \lambda_\Theta(\varsigma)\}$, which is the required.■

Theorem(15). Every cubic positive implicative-ideal Θ of N is a cubic ideal.

Proof. Let Θ be a cubic positive implicative-ideal, then

$\tilde{\mu}_\Theta(\kappa * \omega) \geq rmin\{\tilde{\mu}_\Theta(\kappa * (\varsigma * \omega))\,, \tilde{\mu}_\Theta(\kappa * \varsigma)\}$,

$\lambda_\Theta(\kappa * \omega) \leq max\,\{\lambda_\Theta(\kappa * (\varsigma * \omega)), \lambda_\Theta(\kappa * \varsigma)\}$, put $\kappa = 0$ we get

$\tilde{\mu}_\Theta(0 * \omega) \geq rmin\{\tilde{\mu}_\Theta(0 * (\varsigma * \omega)), \tilde{\mu}_\Theta(0 * \varsigma)\}$,

$\tilde{\mu}_\Theta(\omega) \geq rmin\{\tilde{\mu}_\Theta(\varsigma * \omega), \tilde{\mu}_\Theta(\varsigma)\} \dots \dots . . (1)$, and

$\lambda_\Theta(0 * \omega) \leq max\ \{\lambda_\Theta(0 * (\varsigma * \omega)), \lambda_\Theta(0 * \varsigma)\},$

$\lambda_\Theta(\omega) \leq max\ \{\lambda_\Theta(\varsigma * \omega), \lambda_\Theta(\varsigma)\} \dots \dots \dots (2)$

From (1) and (2) , Θ is a cubic ideal.

The following example shows the converse of this theorem is not true, in general.

Example(16). Let $\mathrm{N} = \{0, a, b\}$ with two operations $*$ and $\circ$ defined by **Table 2**:

Table 2. a cubic ideal

*	0	a	b
0	0	0	0
a	0	a	0
b	0	0	b

∘	0	A	b
0	0	A	b
a	0	0	a
b	0	0	0

It follows that (N,∗,∘ ,0) is a KU-semigroup and define Θ by:

$$\tilde{\mu}_\Theta(\varsigma) = \begin{cases} [0.5,0.8]\ , & if\ \ \varsigma = 0 \\ [0.1,0.2]\ , & if\ \ \varsigma = a \\ [0.1,0.3]\ , & if\ \ \varsigma = b \end{cases} \quad \text{and} \quad \lambda_\Theta(\varsigma) = \begin{cases} 0.1, & if\ \ \varsigma = 0 \\ 0.5, & if\ \ \varsigma = a \\ 0.3, & if\ \ \varsigma = b \end{cases}$$

And then we can prove that {0,a,b} is a cubic ideal but not a cubic positive implicative-ideal, since

$\tilde{\mu}_\Theta(0 * a) = [0.1\ , 0.2\} \leq rmin\{\tilde{\mu}_\Theta(0 * (b * a)), \tilde{\mu}_\Theta(0 * b)\} = [0.1,0.3].$

Corollary(17). Every cubic positive implicative-ideal in a KU-semigroup is a cubic *k*-ideal.

Proof.

By referring to Theorem (15), we get Θ is a cubic ideal and by referring to Theorem (9) and upon it Θ is achieved the required.

Lemma(18). If Θ is a cubic ideal of N**,** then Θ is a cubic positive implicative-ideal if and only if the following conditions hold

(a) $\tilde{\mu}_\Theta\big((\kappa * \varsigma) * (\kappa * \omega)\big) \geq \tilde{\mu}_\Theta\big(\kappa * (\varsigma * \omega)\big)\ and\ \lambda_\Theta\big((\kappa * \varsigma) * (\kappa * \omega)\big) \leq \lambda_\Theta\big(\kappa * (\varsigma * \omega)\big).$

Proof. Since Θ is a cubic ideal of N, then

$\tilde{\mu}_\Theta(\kappa * \omega) \geq rmin\{\tilde{\mu}_\Theta\big((\kappa * \varsigma) * (\kappa * \omega)\big)\ , \tilde{\mu}_\Theta(\kappa * \varsigma)\}$ and by above condition, we have

$\tilde{\mu}_\Theta(\kappa * \omega) \geq rmin\{\tilde{\mu}_\Theta(\kappa * (\varsigma * \omega))\ , \tilde{\mu}_\Theta(\kappa * \varsigma)\}$, also

$\lambda_\Theta(\kappa * \omega) \leq max\{\lambda_\Theta\big((\kappa * \varsigma) * (\kappa * \omega)\big)\ , \lambda_\Theta(\kappa * \varsigma)\}$ and by above condition, we have

$\lambda_\Theta(\kappa * \omega) \leq max\{\lambda_\Theta(\kappa * (\varsigma * \omega))\ , \lambda_\Theta(\kappa * \varsigma)\}$, which is $\boldsymbol{Cp_2}$, the conditions $\boldsymbol{Cp_1}$ and $\boldsymbol{Cp_3}$ are verify from Definition(10).

On the contrary, by Theorem (15)**,**Θ is a cubic ideal.

Let $\alpha = \varsigma * \omega$ $\; and \;$ $\beta = (\kappa * \varsigma) * \omega$, then

$\tilde{\mu}_{\Theta}(\kappa * (\alpha * \beta)) = \tilde{\mu}_{\Theta}(\kappa * ((\varsigma * \omega) * ((\kappa * \varsigma) * \omega)) \geq \tilde{\mu}_{\Theta}(\kappa * ((\kappa * \varsigma) * \varsigma))$ by **ku1**

$=\tilde{\mu}_{\Theta}(0)$ by Theorem(2) . So $\tilde{\mu}_{\Theta}(\kappa * (\alpha * \beta)) = \tilde{\mu}_{\Theta}(0)$ and

$\lambda_{\Theta}(\kappa * (\alpha * \beta)) = \lambda_{\Theta}(\kappa * ((\varsigma * \omega) * (\kappa * \varsigma) * \omega)) \geq \lambda_{\Theta}(\kappa * ((\kappa * \varsigma) * \varsigma))$ by **ku1**

$= \lambda_{\Theta}(0)$ by Theorem (2). So $\lambda_{\Theta}(\kappa * (\alpha * \beta)) = \lambda_{\Theta}(0)$.

Now by using Theorem (2) and the condition $\boldsymbol{Cp_2}$ from definition of a cubic positive implicative-ideal we obtain:

$\tilde{\mu}_{\Theta}\big((\kappa * \varsigma) * (\kappa * \omega)\big) = \tilde{\mu}_{\Theta}(\kappa * (\kappa * \varsigma) * \omega) = \tilde{\mu}_{\Theta}(\kappa * \beta)$

$\geq rmin\{\tilde{\mu}_{\Theta}\big(\kappa * (\alpha * \beta)\big), \tilde{\mu}_{\Theta}(\kappa * \alpha)\} = rmin\{\tilde{\mu}_{\Theta}(0), \tilde{\mu}_{\Theta}(\kappa * \alpha)\} = \tilde{\mu}_{\Theta}(\kappa * \alpha)$

$=\tilde{\mu}_{\Theta}(\kappa * (\varsigma * \omega))$, and

$\lambda_{\Theta}\big((\kappa * \varsigma) * (\kappa * \omega)\big) = \lambda_{\Theta}(\kappa * (\kappa * \varsigma) * \omega) = \lambda_{\Theta}(\kappa * \beta)$

$\leq max\{\lambda_{\Theta}\big(\kappa * (\alpha * \beta)\big), \lambda_{\Theta}(\kappa * \alpha)\} = max\{\lambda_{\Theta}(0), \lambda_{\Theta}(\kappa * \alpha)\} = \lambda_{\Theta}(\kappa * \alpha)$

$=\lambda_{\Theta}(\kappa * (\varsigma * \omega))$, which is the condition (a).■

Theorem(19). Every cubic commutative-ideal Θ of N is a cubic ideal.

Proof. By definition of a cubic commutative-ideal, we have ***CI1*** and ***CI3*** are fulfilled

(C$\boldsymbol{C_2}$) $\tilde{\mu}_{\Theta}\big((\varsigma * \omega) * \omega) * \varsigma\big) \geq rmin\{\tilde{\mu}_{\Theta}\big(\omega * (\kappa * \varsigma)\big), \;\; \tilde{\mu}_{\Theta}(\kappa)\}$, put $\omega = 0$, we get

$\tilde{\mu}_{\Theta}\big((\varsigma * 0) * 0) * \varsigma\big) \geq rmin\{\tilde{\mu}_{\Theta}\big(0 * (\kappa * \varsigma)\big), \;\; \tilde{\mu}_{\Theta}(\kappa)\}$,

$\tilde{\mu}_{\Theta}(\varsigma) \geq rmin\{\tilde{\mu}_{\Theta}(\kappa * \varsigma), \;\; \tilde{\mu}_{\Theta}(\kappa)\}$, and

$\lambda_{\Theta}\big((\varsigma * \omega) * \omega) * \varsigma\big) \leq max\,\{\lambda_{\Theta}\big(\omega * (\kappa * \varsigma)\big), \lambda_{\Theta}(\kappa)\}$, put $\omega = 0$

$\lambda_{\Theta}\big((\varsigma * 0) * 0) * \varsigma\big) \leq max\,\{\lambda_{\Theta}\big(0 * (\kappa * \varsigma)\big), \lambda_{\Theta}(\kappa)\}$,

$\lambda_{\Theta}(\varsigma) \leq max\,\{\lambda_{\Theta}(\kappa * \varsigma), \lambda_{\Theta}(\kappa)\}$, which is a ***CI2*** condition, therefore Θ is a cubic ideal.■

Example (20). Let $N = \{0, a, b\}$ with two operations $*$ and $\circ$ defined by the tables in Example (16). Define $\tilde{\mu}_{\Theta}(\nu)$ and $\lambda_{\Theta}(\nu)$ as follows:

$$\tilde{\mu}_{\Theta}(\varsigma) = \begin{cases} [0.5\,,0.9] & if \quad \varsigma = 0, a \\ [0.2, 0.4] & if \quad \varsigma = b \end{cases} \quad and \quad \lambda_{\Theta}(\varsigma) = \begin{cases} 0.2 & if \quad \varsigma = 0, a \\ 0.5 & if \quad \varsigma = b \end{cases}$$

Then we can prove that Θ is a cubic ideal but not a cubic commutative-ideal, since

$\tilde{\mu}_{\Theta}\big(((b * 0) * 0) * b)\big) \leq rmin\{\tilde{\mu}_{\Theta}\big(0 * (a * b)\big), \tilde{\mu}_{\Theta}(a)\}$,

$\tilde{\mu}_{\Theta}(b) = [0.2\,, 0.4] \leq \tilde{\mu}_{\Theta}(a) = [0.5\,, 0.9]$ and

$\lambda_{\Theta}\big(((b * 0) * 0) * b)\big) \geq max\{\lambda_{\Theta}\big(0 * (a * b)\big), \lambda_{\Theta}(a)\}$,

So $\lambda_{\Theta}(b) = 0.5 \geq \lambda_{\Theta}(a) = 0.2$, this is also a wrong phrase.

So we conclude that the converse of the previous theorem is not true in general, as we will clarify in the following theorem

Theorem(21). A cubic ideal Θ of N is a cubic commutative-ideal if and only if it is satisfies the following inequalities:

(a) $\tilde{\mu}_\Theta(\omega * \varsigma) \leq \tilde{\mu}_\Theta((\varsigma * \omega) * \omega) * \varsigma)$.

(b) $\lambda_\Theta(\omega * \varsigma) \geq \lambda_\Theta((\varsigma * \omega) * \omega) * \varsigma)$, for all $\varsigma, \omega \in N$.

Proof. Let Θ is a cubic ideal which satisfies **(a)** and **(b)**, then

$\tilde{\mu}_\Theta(\omega * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\kappa * (\omega * \varsigma)), \tilde{\mu}_\Theta(\kappa)\}$, by substituting (**a**) and using Theorem(2), we obtain

$\tilde{\mu}_\Theta((\varsigma * \omega) * \omega) * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\omega * (\kappa * \varsigma)), \quad \tilde{\mu}_\Theta(\kappa)\}$ and

$\lambda_\Theta(\omega * \varsigma) \leq max\ \{\lambda_\Theta(\kappa * (\omega * \varsigma)), \lambda_\Theta(\kappa)\}$, substituting **(b)** and using Theorem **(2)** we get

$\lambda_\Theta((\varsigma * \omega) * \omega) * \varsigma) \leq max\ \{\lambda_\Theta(\omega * (\kappa * \varsigma)), \lambda_\Theta(\kappa)\}$, then Θ is a cubic commutative-ideal.

On the contrary, suppose $\Theta = \langle \tilde{\mu}_\Theta, \lambda_\Theta \rangle$ is a cubic commutative ideal, so

$\tilde{\mu}_\Theta((\varsigma * \omega) * \omega) * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\omega * (\kappa * \varsigma)), \tilde{\mu}_\Theta(\kappa)\}$, put $\kappa = 0$

$\tilde{\mu}_\Theta((\varsigma * \omega) * \omega) * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\omega * (0 * \varsigma)), \tilde{\mu}_\Theta(0)\}$,

$\tilde{\mu}_\Theta((\varsigma * \omega) * \omega) * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\omega * \varsigma), \tilde{\mu}_\Theta(0)\}$,

$\tilde{\mu}_\Theta((\varsigma * \omega) * \omega) * \varsigma) \geq \tilde{\mu}_\Theta(\omega * \varsigma)$, which is **(a)** likewise put $\kappa = 0$ in

$\lambda_\Theta((\varsigma * \omega) * \omega) * \varsigma) \leq ma\,x\{\lambda_\Theta(\omega * (\kappa * \varsigma)), \lambda_\Theta(\kappa)\}$, so we get

$\lambda_\Theta((\varsigma * \omega) * \omega) * \varsigma) \geq \lambda_\Theta(\omega * \varsigma)$.■

Theorem(22). Every cubic implicative-ideal of N is a cubic ideal.

Proof. By Definition(11), we have ***CI₁*** and ***CI₃*** are hold, then by

(CV$_2$)$\tilde{\mu}_\Theta((\varsigma * \omega) * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\kappa * ((\varsigma * \omega) * \varsigma)),\ \tilde{\mu}_\Theta(\kappa)\}$, take ω=0

$\tilde{\mu}_\Theta((\varsigma * 0) * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\kappa * ((\varsigma * 0) * \varsigma)),\ \tilde{\mu}_\Theta(\kappa)\}$,

$\tilde{\mu}_\Theta(\varsigma) \geq rmin\{\tilde{\mu}_\Theta(\kappa * \varsigma),\ \tilde{\mu}_\Theta(\kappa)\}$, also take ω=0 in

$\lambda_\Theta((\varsigma * \omega) * \varsigma) \leq max\ \{\lambda_\Theta(\kappa * ((\varsigma * \omega) * \varsigma), \lambda_\Theta(\kappa)\}$, we get

$\lambda_\Theta(\varsigma) \leq max\ \{\lambda_\Theta(\kappa * \varsigma), \lambda_\Theta(\kappa)\}$.■

Theorem(23). A cubic ideal of N is a cubic implicative if it satisfies the following:

$\tilde{\mu}_\Theta(\varsigma) \geq \tilde{\mu}_\Theta((\varsigma * \omega) * \varsigma)$ and $\lambda_\Theta(\varsigma) \leq \lambda_\Theta((\varsigma * \omega) * \varsigma)$, *for all* $\varsigma, \omega \in N$

Proof. Suppose Θ is a cubic ideal satisfies the above two inequalities, hence

$\tilde{\mu}_\Theta(\varsigma) \geq \tilde{\mu}_\Theta((\varsigma * \omega) * \varsigma) \geq rmin\{\tilde{\mu}_\Theta(\kappa * ((\varsigma * \omega) * \varsigma)),\ \tilde{\mu}_\Theta(\kappa)\}$ and

$\lambda_\Theta(\varsigma) \leq \lambda_\Theta((\varsigma * \omega) * \varsigma) \leq max\{\lambda_\Theta(\kappa * ((\varsigma * \omega) * \varsigma)), \lambda_\Theta(\kappa)\}$.■

Theorem(24). If Θ of N is a cubic positive implicative-ideal and a cubic commutative-ideal, then Θ is a cubic implicative-ideal.

Proof. By Theorem (21) and Lemma (18), we have:

$$\tilde{\mu}_\Theta\big((\varsigma * \omega) * \varsigma) * \varsigma\big) \geq \tilde{\mu}_\Theta((\varsigma * (\varsigma * \omega)) * (\varsigma * \omega)) \ \geq \tilde{\mu}_\Theta(\varsigma * ((\varsigma * \omega) * \omega))$$

$= \tilde{\mu}_\Theta\big((\varsigma * \omega) * (\varsigma * \omega)\big)$ and by Theorem(2)

$= \tilde{\mu}_\Theta(0)$. Also,

$$\lambda_\Theta\big((\varsigma * \omega) * \varsigma) * \varsigma\big) \leq \lambda_\Theta((\varsigma * (\varsigma * \omega)) * (\varsigma * \omega)) \leq \lambda_\Theta(\varsigma * (\omega * (\varsigma * \omega)))$$

$= \lambda_\Theta\big((\varsigma * \omega) * (\varsigma * \omega)\big)$ and by Theorem(2)

$= \lambda_\Theta(0)$

By two Theorems (15) and (19) and Θ is a cubic-ideal, then

$\tilde{\mu}_\Theta(\varsigma) \geq rmin\{\tilde{\mu}_\Theta\big(((\varsigma * \omega) * \varsigma\big) * \varsigma\big), \tilde{\mu}_\Theta((\varsigma * \omega) * \varsigma)\}$

$\geq rmin\{\tilde{\mu}_\Theta(0)\ ,\ \tilde{\mu}_\Theta\big((\varsigma * \omega) * \varsigma)\big\} = \tilde{\mu}_\Theta((\varsigma * \omega) * \varsigma)$, and

$\lambda_\Theta(\varsigma) \leq max\{\lambda_\Theta((\varsigma * \omega) * \varsigma) * \varsigma), \lambda_\Theta((\varsigma * \omega) * \varsigma)\} \leq max\{\lambda_\Theta(0)\ ,\ \lambda_\Theta\big((\varsigma * \omega) * \varsigma)\big\} = \lambda_\Theta((\varsigma * \omega) * \varsigma)$ so from Theorem(23), we get the required.■

4. Conclusion

Through this work, we present the definitions of the cubic (commutative, implicative, positive implicative)-ideals and study some relationships among these types. Also, some important theories are discussed. The main purpose of our future work is to study some concepts such as hyper cubic-ideal, filter cubic-ideal and intuitionistic cubic-ideals. Also, we can study the notion of the graph theory of cubic-ideal in a KU-semigroup.

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Conflict of Interest

The authors declare that they have no conflicts of interest

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