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Using a New General Complex Integral Transform for Solving Population Growth and Decay Problems

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Abstract

The Population growth and decay issues are one of the most pressing issues in many sectors of study. These issues can be found in physics, chemistry, social science, biology, and zoology, among other subjects.

We introduced the solution for these problems in this paper by using the SEJI (Sadiq- Emad- Jinan) integral transform, which has some mathematical properties that we use in our solutions. We also presented the SEJI transform for some functions, followed by the inverse of the SEJI integral transform for these functions.

After that, we demonstrate how to use the SEJI transform to tackle population growth and decay problems by presenting two applications that demonstrate how to use this transform to obtain solutions.

Finally, we conclude that the SEJI transform can readily solve the problems of population increase and decay, and that the action of this integral transform in overcoming these challenges can be explained through applications.

Keywords: decay problems, inverse of SEJI transform, population growth, SEJI transform.

1. Introduction:

There are many integral transforms have been solved the population growth and decay problems, such as Sawi, Mohand, Kamal, Shehu, Elzaki and complex SEE transform [1-7].

The equation of population growth is a first order linear ordinary differential equation [8-12].

$$\frac{dN}{dt} = \mu N. \qquad (1)$$

With initial condition

$$N(t_0) = N_{0.} \quad (2)$$

Where μ is a positive real number, N is the population value at time t and N_0 is the initial population at time t_0 .

This equation is known as the Malthusian law of population growth.

The decay problems of the core are defined as first order linear ordinary differential equations [13-15].

$$\frac{dM}{dt} = -\eta M. \quad (3)$$

With initial condition

$$M(t_0) = M_0. \quad (4)$$

Where M is the value of the matter at time $t, \eta \in \mathbb{R}^+$ and M_0 is the initial matter at time t_0.

The negative sign in the equation (3) points the mass of the core decreases with time, thus the time derivative of M denoted by $\frac{dM}{dt}$ have to be negative.

We define the SEJI integral transform as follows [16]:

$$T_g^c\{f(t); s\} = F_g^c(s) = p(s) \int_{t=0}^{\infty} e^{-iq(s)t} f(t)dt,$$

where $t \geq 0, p(s) \neq 0$ and $q(s)$ are positive functions of parameter s, $i = \sqrt{-1}$.

The SEJI integral transform of the function $f(t), t \geq 0$ has a condition that $f(t)$ is a piecewise continuous and $f(t)$ of exponential order, which these sufficient conditions for the existence of SEJI integral transform for $f(t)$.

2. The Linearity Property of SEJE Transform

We can prove easily the linearity property of the SEJE transform.

Let $T_g^c\{f(t)\} = F_g^c(s)$ and $T_g^c\{h(t)\} = H_g^c(s)$, then for every α and β are constants

$$T_g^c\{\alpha f(t) \pm \beta h(t)\} = \alpha F_g^c(s) \pm \beta H_g^c(s)$$

Proof:

$$\begin{aligned} T_g^c\{\alpha f(t) \pm \beta h(t)\} &= p(s) \int_{t=0}^{\infty} e^{-iq(s)t} \big(\alpha f(t) \pm \beta h(t)\big)dt, \\ &= p(s) \int_{t=0}^{\infty} e^{-iq(s)t} \big(\alpha f(t)\big)dt \pm p(s) \int_{t=0}^{\infty} e^{-iq(s)t} \big(\beta h(t)\big)dt, \\ &= \alpha p(s) \int_{t=0}^{\infty} e^{-iq(s)t} f(t)dt \pm \beta p(s) \int_{t=0}^{\infty} e^{-iq(s)t} h(t)dt, \\ &= \alpha F_g^c(s) \pm \beta H_g^c(s). \quad \blacksquare \end{aligned}$$

Also, we can prove this property for the inverse of SEJI integral transform.

$$f(t) = T_g^{c^{-1}}\{F_g^c(s)\} = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} \frac{i}{p(s)} e^{iq(s)t} F_g^c(s)ds,$$

where δ is positive fixed number, $t \geq 0$. i is complex number ($i^2 = -1$).

Let $T_g^{c^{-1}}\{F_g^c(s)\} = f(t)$ and $T_g^{c^{-1}}\{H_g^c(s)\} = h(t)$ then for every α and β are constant

$$T_g^{c^{-1}}\{\alpha F_g^c(s) \pm \beta H_g^c(s)\} = \alpha f(t) \pm \beta h(t) \ .$$

Proof:

$$T_g^{c^{-1}}\{\alpha F_g^c(s) \pm \beta H_g^c(s)\} = \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{i}{p(s)} e^{iq(s)t}\left(\alpha F_g^c(s) \pm \beta H_g^c(s)\right) ds,$$

$$= \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{i}{p(s)} e^{iq(s)t}\left(\alpha F_g^c(s)\right) ds \pm \frac{1}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{i}{p(s)} e^{iq(s)t}\left(\beta H_g^c(s)\right) ds,$$

$$= \frac{\alpha}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{i}{p(s)} e^{iq(s)t} F_g^c(s) ds \pm \frac{\beta}{2\pi i}\int_{\delta-i\infty}^{\delta+i\infty} \frac{i}{p(s)} e^{iq(s)t} H_g^c(s) ds,$$

$$= \alpha f(t) \pm \beta h(t). \quad \blacksquare$$

3. The SEJI Integral Transform for Some Fundamental Functions: [16]

In the following a list the SEJE transform for some important functions:

Table 1. The SEJE transform for some basic functions.

Functions $f(t)$	$T_g^c\{f(t)\} = F_g^c(s)$	**Functions $f(t)$**	$T_g^c\{f(t)\} = F_g^c(s)$
1. 1	$\frac{-ip(s)}{q(s)}$	6. $e^{\alpha t}$, α is a constant	$-p(s)\left[\frac{\alpha}{\alpha^2 + (q(s))^2} + i\frac{q(s)}{\alpha^2 + (q(s))^2}\right]$, $q(s) > a.$
2. t	$\frac{-\ p(s)}{[q(s)]^2}$	7. $\sin(\alpha t)$	$\frac{-\alpha\, p(s)}{(q(s))^2 - \alpha^2}$, $q(s) > \lvert\alpha\rvert$
3. t^2	$\frac{2ip(s)}{[q(s)]^3}$	8. $\cos(\alpha t)$	$\frac{-i\, p(s)\, q(s)}{(q(s))^2 - \alpha^2}$, $q(s) > \lvert\alpha\rvert.$
4. t^n, $n \in N$	$(-i)^{n+1}\frac{n!\ p(s)}{[q(s)]^{n+1}}$	9. $\sinh(\alpha t)$	$\frac{-\alpha\, p(s)}{(q(s))^2 + \alpha^2}$, $q(s) > 0.$
5. $t^n, n > -1$	$(-i)^{n+1}\frac{\Gamma(n+1)\ p(s)}{[q(s)]^{n+1}}$	10. $\cosh(\alpha t)$	$\frac{-i\, p(s)\, q(s)}{(q(s))^2 + \alpha^2}$, $q(s) > 0.$

4. The Inverse of SEJI Integral Transform for Some Fundamental Functions: [16]

In the following a list of the inverse of the SEJI integral transform for some important functions:

Table 2. The inverse of SEJE transform for some basic functions.

$F_g^c(s)$	$f(t) = T_g^{c^{-1}}\{F_g^c(s)\}$	$F_g^c(s)$	$f(t) = T_g^{c^{-1}}\{F_g^c(s)\}$
1. $\frac{-ip(s)}{q(s)}$	1	6. $-p(s)\left[\frac{\beta}{\beta^2+(q(s))^2} + i\frac{q(s)}{\beta^2+(q(s))^2}\right]$, $q(s) > \beta$.	$e^{\beta t}$, β is a constant
2. $\frac{-p(s)}{[q(s)]^2}$	t	7. $\frac{-\beta\, p(s)}{(q(s))^2-\beta^2}$, $q(s) > \|\beta\|$	$\sin(\beta t)$
3. $\frac{2ip(s)}{[q(s)]^3}$	$\frac{t^2}{2!}$	8. $\frac{-i\, p(s)\, q(s)}{(q(s))^2-\beta^2}$, $q(s) > \|\beta\|$.	$\cos(\beta t)$
4. $(-i)^{n+1}\frac{p(s)}{[q(s)]^{n+1}}$, $n \in N$	$\frac{t^n}{n!}$	9. $\frac{-\beta\, p(s)}{(q(s))^2+\beta^2}$, $q(s) > 0$.	$\sinh(\beta t)$
5. $(-i)^{n+1}\frac{p(s)}{[q(s)]^{n+1}}$, $n > -1$	$\frac{t^n}{\Gamma(n+1)}$	10. $\frac{-i\, p(s)\, q(s)}{(q(s))^2+\beta^2}$, $q(s) > 0$.	$\cosh(\beta t)$

5.The SEJI Integral Transform of Derivatives of a Function $f(t)$: [16]

In [10] gave the application of SEJI integral transform to derivative of $f(t)$.

Let $T_g^c\{f(t)\} = F_g^c(s)$, we get:

i. $T_g^c\{f'(t)\} = iq(s)F_g^c(s) - f(0)p(s)$.

ii. $T_g^c\{f''(t)\} = (iq(s))^2 F_g^c(s) - p(s)f'(0) - iq(s)p(s)f(0)$.

iii. $T_g^c\{f^{(n)}(t)\} = (iq(s))^n F_g^c(s) - p(s)\left[\sum_{k=1}^{n}(iq(s))^{k-1} f^{(n-k)}(0)\right]$.

6.The Method of SEJI Integral Transform for Solving The Population Growth Problem:

In this section, we illustrate the method of solution the population growth problem in equation (1) and (2) by using the SEJI integral transform.

Take the SEJI integral transform for equation (1):

$$T_g^c\left\{\frac{dN}{dt}\right\} = T_g^c\{\mu N\}. \quad (5)$$

As we know, SEJE transform of derivative of function, so we get:

$$iq(s)N_g^c(s) - N(0)p(s) = \mu N_g^c(s). \quad (6)$$

By applying the initial condition (2) and on simplification, we get:

$$N_g^c(s)[iq(s) - \mu] = N_0 p(s).$$

Then,

$$N_g^c(s) = \frac{N_0 p(s)}{[iq(s) - \mu]}. \quad (7)$$

For obtaining the value of $N(t)$ we take the inverse SEJE integral transform on both sides of (7), so we have:

$$N(t) = T_g^{c^{-1}}\{N_g^c(s)\} = N_0\, T_g^{c^{-1}}\left\{\frac{p(s)}{[iq(s) - \mu]}\right\},$$

$$N(t) = N_0 e^{\mu t}. \quad \forall t \geq 0$$

where $N(t)$ be the value of population at time t.

7.The Method of SEJE Transform for Solving The Decay Problem:

In this section, we introduce the method of solution the decay problem in equations (3) and (4) by using the SEJE transform.

Take the SEJE transform for equation (3):

$$T_g^c\left\{\frac{dM}{dt}\right\} = T_g^c\{-\eta M\}. \tag{8}$$

As we know, SEJE transform of derivative of function, so we have:

$$iq(s)M_g^c(s) - M(0)p(s) = -\eta M_g^c(s). \tag{9}$$

By applying the initial condition (4) and on simplification, we get:

$$M_g^c(s)[iq(s) + \eta] = M_0 p(s),$$

we obtain:

$$M_g^c(s) = \frac{M_0 p(s)}{[iq(s) + \eta]}. \tag{10}$$

For obtaining the value of $M(t)$ we take the inverse SEJE transform on both sides of (10), so we have:

$$M(t) = T_g^{c^{-1}}\{M_g^c(s)\} = M_0\, T_g^{c^{-1}}\left\{\frac{p(s)}{[iq(s) + \eta]}\right\},$$

$M(t) = M_0 e^{-\eta t}$. $\forall t \geq 0$

Where $M(t)$ is the value of substance at time t.

8.Applications:

We have now some applications to explain the effect of SRJE integral transform in solving population growth and decay problems.

Application 8.1:

A country's population expands at a rate proportional to the number of people currently residing there. If the population doubles in three years and reaches 10,000 in five years, calculate the number of individuals who lived in the country at the start [10].

We can write this information by the following equation:

$$\frac{dN}{dt} = \mu N. \tag{11}$$

Where N is the number of people living in the country at time t and μ is a proportionality rate. Assume that N_0 is the country's initial population at time $t = 0$.

Now, we will apply the same steps in equation (11),

$$T_g^c\left\{\frac{dN}{dt}\right\} = \mu T_g^c\{N\}. \tag{12}$$

Since $N = N_0$ when $t = 0$, so we get:

$$N_g^c(s) = \frac{N_0 p(s)}{[iq(s) - \mu]}. \tag{13}$$

By applying inverse of SEJE transform for (13),

$$N(t) = T_g^{c^{-1}}\{N_g^c(s)\} = N_0\, T_g^{c^{-1}}\left\{\frac{p(s)}{[iq(s) - \mu]}\right\},$$

$$N(t) = N_0 e^{\mu t}. \tag{14}$$

At time $t = 3, N = 2N_0$, by using in (14), we find:

$$2N_0 = N_0 e^{3\mu} \Rightarrow e^{3\mu} = 2$$

$$\mu = 0.231.$$

Now at time $t = 5, \mathrm{N} = 10^4$, using in (14), we get:

$$10^4 = N_0 e^{3(0.231)},$$

$$N_0 = 3151.$$

Where N_0 is the desired initial value of people living in the country.

Application 8.2:

If there is originally 100 mg. of radioactive material present and after three hours it is noticed that the radioactive material has lost 20% of its original mass, calculate the half-life of the radioactive substance [10] .

We write these information by the following equation:

$$\frac{dM}{dt} = -\eta M, \tag{15}$$

where η is the rate of proportionality and M represents the value of radioactive material at time. Assume that at time $t = 0$ and $M_0$is the initial value of p radioactive material.

Now, we will apply the same steps in equation (15),

$$T_g^c\left\{\frac{dM}{dt}\right\} = -\eta T_g^c\{M\}. \tag{16}$$

Since $M = M_0$ when $t = 0$, so we get:

$$M_g^c(s) = \frac{M_0 p(s)}{[iq(s) + \eta]}. \tag{17}$$

By applying inverse of SEJE integral transform for (17),

$$M(t) = T_g^{c^{-1}}\{M_g^c(s)\} = M_0\, T_g^{c^{-1}}\left\{\frac{p(s)}{[iq(s) + \eta]}\right\},$$

$$M(t) = M_0 e^{-\eta t}. \tag{18}$$

At time $t = 0, M = M_0 = 100$, by using in (8.8), we find:

$$M(\mathrm{t}) = 100e^{\eta t}. \tag{19}$$

Now at time $\mathrm{t} = 5,$ the radioactive material has lost 20 present of its original mass 100 mg. therefore,

$M = 100 - 20 = 80$ is using in (18), we get:

$$80 = 100e^{-3\eta},$$

$$\eta = 0.07438. \tag{20}$$

we wanted t when $M = \frac{M_0}{2} = 50$, then from (19), we get:

$$50 = 100e^{-\eta t}. \tag{21}$$

By substituting the value of η in (21),

$50 = 100e^{-0.07438t} \Rightarrow t = 9.32$ hours,

which t is the desired half-life of the radioactive material.

9.Conclusion

In this study, we show how to improve the SEJI integral transform approach for solving population growth and decay problems. The effect of the SEJI integral transform for tackling these difficulties is explained through the applications. So, we can use the suggested transform to solve the population growth and decay problems for the different organisms.

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