



The Completion of Generalized 2-Inner Product Spaces

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Abstract

A complete metric space is a well-known concept. Kreyszig shows that every non-complete metric space W can be developed into a complete metric space \widehat{W} , referred to as completion of W .

We use the b-Cauchy sequence to form \widehat{W} which “is the set of all b-Cauchy sequences equivalence classes”. After that, we prove \widehat{W} to be a 2-normed space. Then, we construct an isometric by defining the function from W to \widehat{W}_0 ; thus \widehat{W}_0 and W are isometric, where \widehat{W}_0 is the subset of \widehat{W} composed of the equivalence classes that contains constant b-Cauchy sequences. Finally, we prove that \widehat{W}_0 is dense in \widehat{W} , \widehat{W} is complete and the uniqueness of \widehat{W} is up to isometrics.

Keywords: b-Cauchy Sequence, Equivalent Class, Metric space, Completion Generalized 2-Inner Product Space.

1. Introduction

Cho and Freese [3-4] introduced 2-normed space by: Let W be a real linear space with a dimension greater than 1. Suppose that $\|\cdot, \cdot\|$ is a real-valued function on $W \times W$ for all w, y, z in W and $\alpha \in \mathbb{R}$ satisfying the following requirements:

1. $\|w, y\| = 0$ if and only if w and y are linearly dependent.
2. $\|w, y\| = \|y, w\|$
3. $\|\alpha w, y\| = |\alpha| \|w, y\|$
4. $\|w + y, z\| \leq \|w, z\| + \|y, z\|$

Then $\|\cdot, \cdot\|$ is called a 2-norm on W and the pair $(W, \|\cdot, \cdot\|)$ is called a linear 2-normed space or 2-normed space. For more details, see [11-12]

Riys and Ravinderan [10] defines the generalized 2-inner product space as a complex vector space W , called a generalized 2-inner product space if there exists a complex valued function $\langle\langle \cdot, \cdot \rangle\rangle$ on $W^2 \times W^2$ such that $a, b, c, d \in W$, and $\alpha \in \mathbb{C}$, as the following:

1. $\langle\langle a, b \rangle\rangle, \langle\langle c, d \rangle\rangle = \overline{\langle\langle c, d \rangle\rangle, \langle\langle a, b \rangle\rangle}$
2. If a and b are linear independent in W , then $\langle\langle a, b \rangle\rangle, \langle\langle c, d \rangle\rangle > 0$.
3. $\langle\langle a, b \rangle\rangle, \langle\langle c, d \rangle\rangle = -\langle\langle b, a \rangle\rangle, \langle\langle c, d \rangle\rangle$.
4. $\langle\langle \alpha a + e, b \rangle\rangle, \langle\langle c, d \rangle\rangle = \alpha \langle\langle a, b \rangle\rangle, \langle\langle c, d \rangle\rangle + \langle\langle e, b \rangle\rangle, \langle\langle c, d \rangle\rangle$. For more details, see [8][2][5].

Ghafoor [6], shows that the generalized 2-inner product space is a 2-normed space with $\|w, y\| = \langle\langle w, y \rangle\rangle, \langle\langle w, y \rangle\rangle^{\frac{1}{2}}$.

After many failures to define orthogonal vectors in 2-normed space, Riyas and Ravindran, in 2014, [10] defined orthogonal vectors in a 2-normed space by restriction space $W \times W$ to $W \times \langle b \rangle$, where $\langle b \rangle$ is a non-zero subspace in W . Thus, the domain of the generalized 2-inner product restriction with space $W^2 \times \langle b \rangle^2$.

It is well-known that there are incomplete metric spaces. Kreyszig, in 1978 [7] discussed the strategy for completing every metric space by defining an equivalent relation on Cauchy sequences and the metric on it. In 2001, Cho and Freese [3] used the same strategy for the completion of a 2-normed space, but some difficulties appeared when defining a metric on it, and then, he had to give another condition on the space.

Despite that all generalized 2-inner product space is 2-normed space, but in this paper, we give a completion of the generalized 2-inner product space without need any condition using the b -Cauchy sequences.

This paper includes two sections. The first section discusses some of the properties of the b -Cauchy sequence in a generalized 2-inner product space. The second section proves the completion of the generalized 2-inner product.

We will abbreviate $\|w, b\|$ by $\|w\|_b$ in the sequel.

2. b -Cauchy sequences.

This section discusses some of the properties of the b -Cauchy sequence in a generalized 2-inner product space. Mazaheri and Kazemi in [9] introduce a b -Cauchy sequence concept as follows

Definition (2.1)[9]: Let W be a generalized 2-inner product space, $0 \neq b \in W$, $\{w_n\}$ be a sequence in W , then, it is called a b -Cauchy sequence if $\lim_{n,m \rightarrow \infty} \|w_n - w_m, b\| = 0$.

The following definition has been devised from [9]

Definition (2.2): An open ball of radius r and centered at y in a generalized 2-inner product space W is defined as: $B_r(y) := \{w \in W : \|w - y\|_b < r\}$.

The following proposition is a characterization of b -Cauchy sequences in a generalized 2-inner product space. But first, we define a neighborhood of 0.

Definition (2.3): If w is a point in a generalized 2-inner product space W , then a neighborhood U of w is a set containing $B_r(w)$ for some $r > 0$, i.e, there exists $r > 0$, such that $w \in B_r(w) \subset U$.

Proposition (2.4): Let W be a generalized 2-inner product space, $\{w_n\}$ is a b-Cauchy sequence in W if and only if for any neighborhood U of 0 ; there is an integer $M(U)$ such that for all $n, m \geq M(U)$ implies that $w_n - w_m \in U$.

Proof: Let U be a neighborhood of 0 , then there exists $\varepsilon > 0$ such that $B_0(\varepsilon) \subseteq U$. Since $\{w_n\}$ is a b-Cauchy sequence in W , thus $\lim_{n,m \rightarrow \infty} \|w_n - w_m\|_b = 0$. It implies that there exists $M(\varepsilon) > 0$ such that $\|w_n - w_m\|_b < \varepsilon$; $n, m \geq M(\varepsilon)$. Then, $w_n - w_m \in B_0(\varepsilon) \subseteq U$.

Conversely, let $\{w_n\}$ be a sequence in W such that for every neighborhood U of 0 there is an integer $M(U) > 0$; $w_n - w_m \in U$ where $n, m \geq M(U)$. Then, there exists $\delta(U) > 0$ such that $\|w_n - w_m\|_b < \delta$, where $n, m \geq M(U)$. It implies that for all $\varepsilon > 0$, there exists $M(\varepsilon)$ such that $\|w_n - w_m\|_b < \varepsilon$ for $n, m \geq M(\varepsilon)$, then $\lim_{n,m \rightarrow \infty} \|w_n - w_m\|_b = 0$. Thus, $\{w_n\}$ is a b-Cauchy sequence in W . ■

3. Completion of the generalized 2-inner product spaces.

Kreyszig [1] states few steps to prove that an arbitrary incomplete metric space can be completed. In this section, we follow Kreyszig strategy to prove the completion of the generalized 2-inner product space:

Step1: Forming \widehat{W} is the set of all b-Cauchy sequence equivalence classes.

Definition (3.1): Two b-Cauchy sequences $\{w_n\}$ and $\{y_n\}$ in a generalized 2-inner product space W have a relation denoted by $\{w_n\} \sim \{y_n\}$, if for every neighborhood U of 0 there is an integer $M(U)$ such that $n \geq M(U)$ implies that $w_n - y_n \in U$.

It is clear that \sim is a reflexive and symmetric relation. The following proposition shows that this relation is equivalent.

Proposition (3.2): The relation \sim on the set of b-Cauchy sequences in W is an equivalent relation on W .

Proof: Let $\{w_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$. Let, U is an arbitrary neighborhood of 0 . There exists a neighborhood V of 0 such that $V + V \subset U$. By Definition (2.1) and for this V , there exists an integer M such that $w_n - y_n, y_n - z_n \in V$ for $n \geq M$. Hence, $w_n - z_n = (w_n - y_n) + (y_n - z_n)$ is an element of U for $n \geq M$. Therefore, $\{w_n\} \sim \{z_n\}$. ■

Define $\widehat{W} := \{\widehat{w} : \widehat{w} \text{ is equivalent class of b-Cauchy sequences}\}$.

Step 2: Proof \widehat{W} vector space.

Let \widehat{w}, \widehat{y} in \widehat{W} . Define the terms addition and scalar multiplication. On \widehat{W} where $\{w_n\} \in \widehat{w}$ and $\{y_n\} \in \widehat{y}$, as shown below:

- $\widehat{w} + \widehat{y} = \{w_n + y_n\}$
- $\alpha \widehat{w} = \{\alpha w_n\}$

The following proposition explains that the two operations defined on \widehat{W} are well-defined because they are unaffected by the elements chosen from $\{\widehat{w}_n\}$ and $\{\widehat{y}_n\}$. But first, we need the following proposition:

Proposition (3.3): A b -Cauchy sequence $\{w_n\}$ is equivalent to $\{a_n\}$ in a generalized 2-inner product space W if and only if $\lim_{n \rightarrow \infty} \|w_n - a_n\|_b = 0$.

Proof: Let U be a neighborhood of zero, then, there exists $M(U) > 0$, such that $w_n - a_n \in U$ for $n \geq M(U)$. Hence, for all neighborhood U of 0, there exists $\varepsilon = \varepsilon(U) > 0$ such that $\|w_n - a_n\|_b < \varepsilon$; $n \geq M(U)$. Then, for every $\delta > 0$, there exists $M(\delta) > 0$ such that $\|w_n - a_n\|_b < \delta$ for all $n \geq M(\delta)$, therefore, $\lim_{n \rightarrow \infty} \|w_n - a_n\|_b = 0$ for $n \geq M(\delta)$.

Conversely, let $\{w_n\}, \{a_n\}$ be b -Cauchy sequences in W such that for every neighborhood U of 0, there exists $\varepsilon > 0$ such that $B_\varepsilon(0) \subset U$. By our hypothesis $\lim_{n \rightarrow \infty} \|w_n - a_n\|_b = 0$, then there exists $M(\varepsilon) > 0$ such that $\|w_n - a_n\|_b < \varepsilon$ for $n \geq M(\varepsilon)$. Hence, $w_n - a_n \in B_\varepsilon(0) \subset U$ for $n \geq M(\varepsilon)$, then $\{w_n\} \sim \{a_n\}$. ■

Proposition (3.4): If $\{a_n\}$ and $\{b_n\}$ are equivalent to $\{w_n\}$ and $\{y_n\}$ in a generalized 2-inner product space W . Then, $\{a_n + b_n\}$ is equivalent to $\{w_n + y_n\}$ and $\{\alpha a_n\}$ is equivalent to $\{\alpha w_n\}$. Moreover, \widehat{W} is a linear space.

Proof: Since $\{a_n\} \sim \{w_n\}$ and $\{b_n\} \sim \{y_n\}$ thus we get

$$\|(w_n + y_n) - (a_n + b_n)\|_b \leq \|w_n - a_n\|_b + \|y_n - b_n\|_b$$

Then

$$\lim_{n, m \rightarrow \infty} \|(w_n + y_n) - (a_n + b_n)\|_b = 0 \dots (1)$$

On the other hand,

$$\lim_{n \rightarrow \infty} \|\alpha w_n - \alpha a_n\|_b = 0 \dots (2)$$

It implies that $\{a_n + b_n\} \sim \{w_n + y_n\}$ and $\{\alpha a_n\} \sim \{\alpha w_n\}$. Therefore, from (1), (2) and Proposition (2.3), \widehat{W} is a linear space. ■

Step3: Proof \widehat{W} is a 2-normed space.

We will define a 2-norm function on the space \widehat{W} . as:

$$\|\cdot\|_{\widehat{b}}: \widehat{W} \times \widehat{b} \rightarrow \mathbb{R}^+$$

is defined as:

$$\|\widehat{w} - \widehat{y}\|_{\widehat{b}} = \lim_{n \rightarrow \infty} \|w_n - y_n\|_b \dots (3)$$

where $\{w_n\} \in \widehat{w}, \{y_n\} \in \widehat{y}$.

The function is well-defined as follows:

Proposition (3.5): If W is a generalized 2-inner product space, then for any two b -Cauchy sequences $\{w_n\}$ and $\{y_n\}$ in W :

1. $\lim_{n \rightarrow \infty} \|w_n - y_n\|_b$ exists.
2. For pairs of equivalent b-Cauchy sequences $\{a_n\} \sim \{w_n\}$ and $\{b_n\} \sim \{y_n\}$, $\lim_{n \rightarrow \infty} \|w_n - y_n\|_b = \lim_{n \rightarrow \infty} \|a_n - b_n\|_b$.

Proof:

1. Let $\{w_n\} \in \widehat{w}, \{y_n\} \in \widehat{y}$ be any two b-Cauchy sequences, then

$$\|w_n - y_n\|_b \leq \|w_n - w_m\|_b + \|w_m - y_m\|_b + \|y_m - y_n\|_b$$

Hence, $\|w_n - y_n\|_b - \|w_m - y_m\|_b \leq \|w_n - w_m\|_b + \|y_m - y_n\|_b$

On the other hand, if we change m by n,

$$\|w_m - y_m\|_b - \|w_n - y_n\|_b \leq \|w_m - w_n\|_b + \|y_n - y_m\|_b$$

It implies that

$$|\|w_n - y_n\|_b - \|w_m - y_m\|_b| \leq \|w_n - w_m\|_b + \|y_n - y_m\|_b \dots (4)$$

Thus, by taking $n, m \rightarrow \infty$ and Definition (1.1), it follows that

$$\lim_{n \rightarrow \infty} |\|w_n - y_n\|_b - \|w_m - y_m\|_b| = 0$$

Then, $\{\|w_n - y_n\|_b\}$ is a Cauchy sequence in R. But, R is complete, thus $\lim_{n \rightarrow \infty} \|w_n - y_n\|_b$ exists.

2. Let $\{a_n\} \sim \{w_n\}$ and $\{b_n\} \sim \{y_n\}$. By the same argument of (4), it implies that

$$|\|w_n - y_n\|_b - \|a_n - b_n\|_b| \leq \|w_n - a_n\|_b + \|y_n - b_n\|_b$$

By taking $n, m \rightarrow \infty$ and proposition (2.3), we get

$$\lim_{n \rightarrow \infty} \|w_n - y_n\|_b = \lim_{n \rightarrow \infty} \|a_n - b_n\|_b . \blacksquare$$

From equation (3) and Proposition (2.3), the conditions (1-3) of a 2-normed space are done.

Proposition (3.6): $(\widehat{W}, \|\cdot\|_{\widehat{b}})$ is a 2-normed space.

Proof: since $\|\widehat{w} - \widehat{z}\|_{\widehat{b}} = \lim_{n \rightarrow \infty} \|w_n - z_n\|_b \leq \lim_{n \rightarrow \infty} \|w_n - y_n\|_b + \lim_{n \rightarrow \infty} \|y_n - z_n\|_b = \|\widehat{w} - \widehat{y}\|_{\widehat{b}} + \|\widehat{y} - \widehat{z}\|_{\widehat{b}}$, then $(\widehat{W}, \|\cdot\|_{\widehat{b}})$ is a 2-norm. \blacksquare

Step4: Construction of an isometric $T: W \rightarrow \widehat{W}_0 \subset \widehat{W}$.

Let \widehat{W}_0 be the subset of \widehat{W} composed of the equivalence classes containing constant b-Cauchy sequences.

Define a function $T: W \rightarrow \widehat{W}_0 \subset \widehat{W}$ by $T(w) = \widehat{w} = (w, w, \dots)$. It is clearly that T is a well-defined onto and one to one. In fact,

$$\|Tw - Ty\|_b = \|\widehat{w} - \widehat{y}\|_{\widehat{b}} = \lim_{n \rightarrow \infty} \|w - y\|_b = \|w - y\|_b,$$

Thus, \widehat{W}_0 and W are isometric.

Step 5: Proof \widehat{W}_0 is dense in \widehat{W} .

Proposition (3.7): If W is a generalized 2-inner product space, then, \widehat{W}_0 is dense in \widehat{W} .

Proof: Let $\widehat{w} \in \widehat{W} - \widehat{W}_0$, then, there exists a b-Cauchy sequence $\{w_n\} \in \widehat{w}$ where $\{w_n\} = \{w_1, w_2, \dots\}$. Define $\widehat{w}^m = \{w_m, w_m, \dots\}$ for all $m \in \mathbb{N}$, thus $\widehat{w}^m \in \widehat{X}_0$. Hence, by Definition (1.1)

$$\|\widehat{w}^m - \widehat{w}\|_{\widehat{b}} = \lim_{n,m \rightarrow \infty} \|w_n - w_m\|_b = 0.$$

Then, \widehat{W}_0 is dense in \widehat{W} . ■

Step 6: Proof completeness of \widehat{W} .

Theorem (3.8): If W is a generalized 2-inner product space, then \widehat{W} is complete.

Proof: Let $\{\widehat{w}_n\}$ be a b-Cauchy sequence in \widehat{W} . Since \widehat{W}_0 is dense in \widehat{W} , thus there exists $\{\widehat{z}_n\} \in \widehat{W}_0$ such that $\|\widehat{w}_n - \widehat{z}_n\|_{\widehat{b}} = 0$. But

$$\|\widehat{z}_n - \widehat{z}_m\|_{\widehat{b}} \leq \|\widehat{z}_n - \widehat{w}_n\|_{\widehat{b}} + \|\widehat{w}_n - \widehat{w}_m\|_{\widehat{b}} + \|\widehat{w}_m - \widehat{z}_m\|_{\widehat{b}}.$$

Then, by equation (3) and Definition (1.1) and if we take $n, m \rightarrow \infty$, we get

$\lim_{n,m \rightarrow \infty} \|\widehat{z}_n - \widehat{z}_m\|_{\widehat{b}} = 0$, it implies that $\{\widehat{z}_n\}$ is a b-Cauchy sequence in \widehat{W}_0 . But W and \widehat{W} are isometric. Thus, there exists a b-Cauchy sequence $\{z_n\}$ in W which is contained in an equivalent class in \widehat{W} , say \widehat{w} .

Note that, $\|\widehat{w}_n - \widehat{w}\|_{\widehat{b}} \leq \|\widehat{w}_n - \widehat{z}_n\|_{\widehat{b}} + \|\widehat{z}_n - \widehat{w}\|_{\widehat{b}} = \|\widehat{w}_n - \widehat{z}_n\|_{\widehat{b}} + \|\widehat{z}_n - \widehat{z}_m\|_{\widehat{b}}$. Thus, $\lim_{n \rightarrow \infty} \|\widehat{w}_n - \widehat{w}\|_{\widehat{b}} = 0$. Therefore, \widehat{W} is complete. ■

Step7: Proof uniqueness of \widehat{W} up to isometrics.

Theorem (3.9): The space \widehat{W} is unique up to isometrics.

Proof: Let \widehat{Y} be another completion to W with a dense subset \widehat{Y}_0 in \widehat{Y} . Then, there exists $S: W \rightarrow \widehat{Y}_0$ is isometric by step 4 defined by $S(w) = \widehat{y} = (y, y, \dots)$.

We will define $h: \widehat{W}_0 \rightarrow \widehat{Y}_0$ by $h(\widehat{w}) = ST^{-1}(\widehat{w})$. It implies that \widehat{W}_0 isometric to \widehat{Y}_0 . For $\widehat{y}_1, \widehat{y}_2$ in \widehat{Y} there exists b-Cauchy sequences $\{\widehat{y}_{1n}\}, \{\widehat{y}_{2n}\}$ in \widehat{Y}_0 such that $\widehat{y}_{1n} \rightarrow \widehat{y}_1$ and $\widehat{y}_{2n} \rightarrow \widehat{y}_2$. Thus, by equation (4)

$$|\|\widehat{y}_1 - \widehat{y}_2\|_{\widehat{b}} - \|\widehat{y}_{1n} - \widehat{y}_{2n}\|_b| \leq \|\widehat{y}_1 - \widehat{y}_{1n}\|_b + \|\widehat{y}_2 - \widehat{y}_{2n}\|_b \rightarrow 0$$

By taking $n \rightarrow \infty$

$$\|\widehat{y}_1 - \widehat{y}_2\|_{\widehat{b}} = \lim_{n \rightarrow \infty} \|\widehat{y}_{1n} - \widehat{y}_{2n}\|_b \quad (5)$$

by the same argument

$$\|\widehat{w}_1 - \widehat{w}_2\|_{\widehat{b}} = \lim_{n \rightarrow \infty} \|\widehat{w}_{1n} - \widehat{w}_{2n}\|_b \quad (6)$$

Thus, by (5) and (6) we get

$$\|\hat{y}_1 - \hat{y}_2\|_{\hat{b}} = \lim_{n \rightarrow \infty} \|\hat{y}_{1n} - \hat{y}_{2n}\|_b = \lim_{n \rightarrow \infty} \|\hat{w}_{1n} - \hat{w}_{2n}\|_b = \|\hat{w}_1 - \hat{w}_2\|_{\hat{b}}$$

It implies that \hat{W} is isometric to \hat{Y} . ■

4. Discussion and Conclusion

A complete metric space is a well-known concept. Every non-complete metric space W can be built into a complete metric space \hat{W} , which is known as a completion of W . In this paper, we construct equivalent classes of b -Cauchy sequences to complete a generalized 2-inner product space.

References

1. Anshul, R.; Ravinder, K., S., ; Sumit, C., Stability of Complex Functional Equations in 2-Banach Spaces. *Journal of Mathematical Physics, Analysis, Geometry* . **2021**, *17*, 3, 341–368.
2. Bahram, D. ; Mohammad, J., Atomic Systems in 2-inner Product Spaces, *Iranian Journal of Mathematical Sciences and Informatics*. **2018**, *13*, 1, 103-110.
3. Cho, Y., J ;Freese, R. W., Geometry of Linear 2-Normed Spaces, Nova Science Publishers, New York, **2001**.
4. Y. J. Cho, M. Matic, J. E., Pecaric, On Gram's Determinant in 2-Inner Product Spaces, *J. Korean Math. Soc.*, **2001**, *38*(4), 1125–1156.
5. Debnath, P., Saha, M., Categorization of n -inner product space. *Asian Res. J. Math.* **2018**, *11*(4), 1–10.
6. Ghafoor, G., R. ; Jamil, Z., Z., Study of b -Hilbert Spaces and Some Classes of Operators, University of Baghdad, Baghdad, **2018**, 27-28.
7. Kreyszig, Erwin, Introductory Functional Analysis with Applications, *John Wiley and Sons, New York*, **1978**.
8. Prasenjit, G., Frame operator for K -frame in 2-inner product space, *International Journal of Mathematics Trends and Technology*. **2021**, *67*.
9. Mazaheri, H.; Kazemi, R., Some Results on 2-Inner Product Spaces, *Nove Sad. J. Math.* **2007**, *37*, 35-40.
10. Riyas, P., ; Ravindran, K., T., Riesz Theorems and \sim -Adjoint Operators On Generalized 2-Inner Product Spaces, *Global Journal mathematics*, **2015**, *3*, 1, May 18, 244-254.
11. Sibel, E. Ideal Strong Lacunary Quasi Cauchy Sequences in 2-normed spaces. AIP Conference Proceedings 2334, 040004 .**2021**.
12. Vijayakumar, S.; Baskaran B., A Characterization of 2-Inner Product Spaces. AIP Conference Proceedings 2282, 020040 .**2020**