DOI: http://doi.org/10.32792/utq.jceps.11.01.12

Numerical Solution For Time-Delay Burgers Equation By Homotopy Analysis Method

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Received 16/01/2011 Accepted 02/03/2011 Published 10/06/2021

Abstract:

In this paper, the homotopy analysis method(HAM) has been applied for solving time-delay Burgers equation. HAM contains the auxiliary parameter ħ, which provides us with a simple way to control the convergence region of solution series. In this study, we compare obtained results through HAM with the exact solutions to show the accuracy of HAM

1-Introduction

In most cases nonlinear equations do not have exact analytic solutions, so approximation and numerical techniques must be used. the adomian decomposition method(ADM)[8], the homotopy perturbation method (HPM)[22,23,24], the homotopy analysis method (HAM), the variational iteration method (VIM)[8] and other methods have been used to provide analytical approximation to linear and nonlinear problems. Time-delay Burgers equation phenomena in physical processes and do not have a precise analytical solution so this nonlinear equation be solved using approximate methods. E.S.Fahmy[9] solved the generalized Time-delay Burgers-Fisher equation by using factorization method to find explicit particular travelling wave solutions and yields to the general solution. While H.Kim and R.Sakthivel[16] used the $\frac{G'}{G}$- expansion method which is implemented to establish travelling wave solutions for time-delay Burgers-Fisher equation are expressed by hyperbolic functions and trigonometric functions. And by using the first–integral which is based on the ring theory of commutative algebra, E. S. Fahmy and his co-authors [8] used the improved tanh function method to construct exact multiple soliton and triangular periodic solutions and calculated the numerical solutions of this equation by using the adomian decomposition method and variational iteration method to the boundary value problem.

The HAM is proposed first by Liao[33,34]for solving linear and nonlinear differential and integral equations, such as n th-order initial value problems(IVPs)[1], K(2,2),Burgers and coupled Burgers equations[2], parabolic equations[14], nonlinear Gas dynamic equation[17], fuzzy system of linear equations[18], linear integral equation[3], differential algebraic equations(DAEs)[3], singular two point boundary value problems(BVPs)[4], Thomas-Fermi equation[5] and Time-fractional differential partial equations(FPDEs)[32]. Recently, Ahmed and his co-authors[6] constructed analytical solutions of fractional differential equations using HAM, Zulkifly and his co-authors[36] are used the HAM to find the numerical solution for linear integro-differential equation, Hossein J. and M.A. Firoozjaee[19] are used the HAM for solving Korteweg-de Vries (KdV) and Korteweg-de Vries Burgers (KdVB) equations, H.Hossein and his co-authors[20] solved the integeral and integro differential equations by HAM, and M. Matinfar and M. Saeidy [27] used HAM for solving fourth order parabolic partial differential equation. The HAM offers certain advantages over routine numerical methods. This method is better since it does not involve discretization of the variables. Hence, is free from rounding off errors and does not require large computer memory or time [27].

The generalized time-delayed Burgers-Fisher equation takes in the following form[8]:

$$\tau u_{tt} + [1\text{-}\tau f_u]u_t = u_{xx}\text{-}pu^s u_x + f(u), \qquad f(u) = \delta u(1\text{-}u) \qquad (1)$$

Where τ, p are any real numbers and $s \in N$.
When $\delta = \tau = 0$ and $p = s = 1$, Eq.(1) assumes the form of the classical Burgers equation:

$$u_t\text{-}u_{xx} + puu_x = 0 \qquad (2)$$

In this work, we use the homotopy analysis method to find the numerical solution for time-delay Burgers equation. The improved tanh function method is used by Fahmy and co-authors [8] to find exact travelling wave solution for Eq.(1) for $\delta = 0$ and different values of τ, p, s; i.e., we solve the equation:

$$\tau u_{tt} + u_t = u_{xx}\text{-}pu^s u_x \qquad (3)$$

The paper is organized as follows: in the next section, the improved tanh function method is presented. In section 3,the Homotopy analysis method is introduced. The application of Homotopy analysis method for solving time-delay Burgers equation is introduced in section 4 in three examples. Section 5 ends this paper with a brief conclusion.

2-The improved tanh function method[8]

The improved tanh function method is used to construct exact multiple soliton and triangular periodic solutions to the time-delay Burgers equation in the form:

$$u_1(x,t) = [Q \mp Q(\tanh\zeta \pm I\,\text{sech}\zeta)]^{1/s}, \zeta = k_1(x\text{-}\omega t). \qquad (12)$$
$$u_2(x,t) = [Q \mp QI(\coth\zeta \pm \cosh\zeta)]^{1/s}, \zeta = k_2(x\text{-}\omega t). \qquad (13)$$
$$u_3(x,t) = [Q \mp QI(\tanh\zeta)]^{1/s}, \zeta = k_3(x\text{-}\omega t). \qquad (14)$$
$$u_4(x,t) = [Q \mp Q(\tanh\zeta)]^{1/s}, \zeta = k_4(x\text{-}\omega t). \qquad (15)$$
$$u_5(x,t) = [Q \mp Q(\tan\zeta)]^{1/s}, \zeta = k_5(x\text{-}\omega t). \qquad (16)$$
$$u_6(x,t) = [Q \mp QI(\cot\zeta)]^{1/s}, \zeta = k_6(x\text{-}\omega t). \qquad (17)$$

where

$I = \sqrt{\text{-}1}$
$Q = (1 + s)\omega/2p$
$k_1 = \mp s\omega/(\tau\omega^2\text{-}1)$
$k_2 = \pm s\omega I/(\tau\omega^2\text{-}1)$
$k_3 = \mp s\omega I/(\tau\omega^2\text{-}1)$
$k_4 = \pm s\omega/(\tau\omega^2\text{-}1)$
$k_5 = \pm s\omega I/(\tau\omega^2\text{-}1)$
$k_6 = \pm s\omega I/(\tau\omega^2\text{-}1)$

When $\tau \to 0$ the solution of the equation (2) in the following form:

$$u_1(x,t) = [Q \mp Q(\tanh\zeta \pm I\,\text{sech}\zeta)]^{1/s}, \zeta = \pm s\omega(x\text{-}\omega t). \qquad (18)$$
$$u_2(x,t) = [Q \mp QI(\coth\zeta \pm \cosh\zeta)]^{1/s}, \zeta = \mp s\omega I(x\text{-}\omega t). \qquad (19)$$
$$u_3(x,t) = [Q \mp QI(\tanh\zeta)]^{1/s}, \zeta = \pm s\omega I(x\text{-}\omega t). \qquad (20)$$
$$u_4(x,t) = [Q \mp Q(\tanh\zeta)]^{1/s}, \zeta = \mp s\omega(x\text{-}\omega t). \qquad (21)$$

$u_5(x,t) = [Q \mp Q(\tan\zeta)]^{1/s}, \zeta = \mp s\omega I(x\text{-}\omega t).$ (22)

$u_6(x,t) = [Q \mp QI(\cot\zeta)]^{1/s}, \zeta = \pm s\omega I(x\text{-}\omega t).$ (23)

Such that $u_4(x,t)$ is the real solution but the others are the complex solutions for time-delay Burgers equation.

3-Basic idea of Homotopy analysis method[17,19]

To show its basic idea let us consider a differential equation

$$N[u(\xi)] = 0 \quad (24)$$

where N is a nonlinear operator, ξ denotes independent variables, $u(\xi)$ is an unknown function. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [33] constructs the so called zero-order deformation equation.

$$(1\text{-}q)L[\phi(\xi;q)\text{-}u_0(\xi)] = q\hbar H(\xi)N[\phi(\xi;q)] \quad (25)$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is anon zero parameter, $H(\xi) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\xi)$ is an initial guess of $u(\xi)$, $\phi(\xi;q)$ is an unknown function. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $q = 0$ and $q = 1$, it holds:

$$\phi(\xi;0) = u_0(\xi) \quad , \quad \phi(\xi;1) = u(\xi) \quad (26)$$

Respectively. Thus, as p increases from 0 to 1, the solution $\phi(\xi;q)$ varies from the initial guesses $u_0(\xi)$ to the solution $u(\xi)$. Expanding $\phi(\xi;q)$ in Taylor series with respect to p, we have

$$\phi(\xi;q) = u_0(\xi) + \sum_{m=1}^{+\infty} u_m(\xi)\, q^m \quad (27)$$

where

$$u_m(\xi) = \frac{1}{m!}\frac{\partial^m \phi(\xi;q)}{\partial q^m}\Big|_{q=0} \quad (28)$$

If the auxiliary linear operator, the initial guess, the auxiliary $\hbar$, and the auxiliary function are so properly chosen, the series (27) converges at $q = 1$, then we have

$$u(\xi) = u_0(\xi) + \sum_{m=1}^{+\infty} u_m(\xi) \quad (29)$$

Define the vector

$$\vec{u_n} = \{u_0(\xi), u_1(\xi), \ldots, u_n(\xi)\} \quad (30)$$

Differentiating equation (25) m times with respect to the embedding parameter q and then setting $q = 0$ and then finally dividing them by m! , we obtain the mth-order deformation equation

$$L[u_m(\xi)\text{-}\chi_m u_{m\text{-}1}(\xi)] = \hbar H(\xi) R_m(\vec{u}_{m\text{-}1}) \quad (31)$$

where

$$R_m(\vec{u}_{m\text{-}1}) = \frac{1}{(m\text{-}1)!}\frac{\partial^{m-1} N[\phi(\xi;q)]}{\partial q^{m-1}}\Big|_{q=0} \quad (32)$$

and

$$\chi_m = \begin{cases} 0 , & m \leq 1 \\ 1 , & m > 1 \end{cases} \quad (33)$$

Applying L^{-1} on both of equation (31), we get

$$u_m(\xi) = \chi_m u_{m\text{-}1}(\xi) + \hbar L^{-1}[H(\xi) R_m(\vec{u}_{m\text{-}1})] \quad (34)$$

In this way, it is easily to obtain u_m for $m \leq 1$, at mth-order, we have

$$u(\xi) = \sum_{m=0}^{M} u_m(\xi) \tag{35}$$

When $M \to +\infty$, we get an accurate approximation of the original (24). If equation (24) admits unique solution, then this method will produce the unique solution. If equation (24) does not possess unique solution, the HAM will give a solution among many other (possible) solutions.

4- Application of homotopy analysis method

In this section, this method for solving time-delay Burgers equation.
the linear operator should be:

$$L[\emptyset(x,t;q)] = \frac{\partial \emptyset(x,t;q)}{\partial t} \tag{36}$$

Example1

By taking $s = 1$, we will solve the equation:

$$\tau u_{tt} + u_t = u_{xx} - pu\, u_x \tag{37}$$

the nonlinear operator is:

$$N[u(x,t)] = \tau \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + pu(x,t)\frac{\partial u(x,t)}{\partial x} \tag{38}$$

So that

$$N[\emptyset(x,t;q)] = \tau \frac{\partial^2 \emptyset(x,t;q)}{\partial t^2} + \frac{\partial \emptyset(x,t;q)}{\partial t} - \frac{\partial^2 \emptyset(x,t;q)}{\partial x^2} + p\emptyset(x,t;q)\frac{\partial \emptyset(x,t;q)}{\partial x} \tag{39}$$

Thus, by equation (32), we have

$$R_m(\vec{u}_{m-1}) = \tau \frac{\partial^2 u_{m-1}(x,t)}{\partial t^2} + \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} + p\sum_{i=0}^{m-1} u_i(x,t)\frac{\partial u_{m-i-1}(x,t)}{\partial x} \tag{40}$$

Substitute equation (40) in (34) with assumption $H(x,t) = 1$, we obtain the solution of equation (37).
We will take $p = 0.1, \tau = 0.5, k = -0.05, \omega = 0.1, t = 0.1$ [8] with initial conditions
$u_0(x,t) = 1 + 0.00502513\, t\, \mathrm{sech}^2(0.05025x) - \tanh(0.05025x)$.

We obtain

$$u_1(x,t) = \hbar\Big[-8.459\times 10^{-8}\operatorname{sech}^4(0.05025x)\tanh(0.05025x)t^3 + 0.5\Big(-0.00005\operatorname{sech}^2(0.05025x)\tanh^2(0.05025x) + 0.0005\operatorname{sech}^2(0.05025x)\big(0.05025-0.05025\tanh^2(0.05025x)\big)-0.05025\big(1-\tanh(0.05025x)\big)\operatorname{sech}^2(0.05025x) + 0.0005\operatorname{sech}^2(0.05025x)\big(-0.05025+0.05025\tanh^2(0.05025x)\big)\Big)t^2 + \Big(0.005025\operatorname{sech}^2(0.05025x)-0.1005\tanh(0.05025x)\big(0.05025-0.05025\tanh^2(0.05025x)\big) + \big(0.1-0.1\tanh(0.05025x)\big)\big(-0.05025+0.05025\tanh^2(0.05025x)\big)\Big)t\Big].$$

$$u_2(x,t) = u_1(x,t) + \hbar[0.2\hbar(0.0005\operatorname{sech}^2(0.05025x)(1.7\times 10^{-8}\operatorname{sech}^4(0.05025x)\tanh^2(0.05025x)-8.46\times 10^{-8}\operatorname{sech}^4(0.05025x)\big(0.05025-0.05025\tanh^2(0.05025x)\big)) + 4.2722\times 10^{-12}\hbar\operatorname{sech}^6(0.05025x)\tanh^2(0.05025x))\,t^5 + \cdots$$

$\vdots$

The solutions $u_2(x,t), u_3(x,t), \ldots$ are very large equations.

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t)$$

Solutions of equation (37) by tanh function method and Homotopy analysis method for some values of x in the interval [-100,100] will be showing in (table 1).

Table1: The results represent the exact solution $u_E(x,t)$by tanh function method and numerical

x_i	u_E	u_{HAM}			
		ħ = -2	ħ = -1	ħ = 0.6	ħ = 2
-100	1.99991374	1.99991372	1.99991372	1.99991372	1.99991374
-90	1.99976436	1.99976430	1.99976430	1.99976431	1.99976437
-80	1.99935637	1.99935622	1.99935624	1.99935625	1.99935642
-70	1.99824259	1.99824222	1.99824228	1.99824232	1.99824278
-60	1.99520609	1.99520522	1.99520537	1.99520549	1.99520673
-50	1.98695729	1.98695527	1.98695566	1.98695599	1.98695933
-40	1.96476561	1.96476109	1.96476212	1.96476297	1.96477169
-30	1.90659064	1.90658142	1.90658393	1.90658598	1.90660705
-20	1.76390587	1.76390455	1.76389539	1.76389944	1.76394101
-10	1.46448505	1.46446952	1.46447519	1.46447983	1.46452759
0	1.00050251	1.00050252	1.00050251	1.00050256	1.00050296
10	0.53630351	0.53631903	0.53631337	0.53630881	0.53626157
20	0.23651299	0.23652838	0.23652346	0.23651947	0.23647807
30	0.09358851	0.09359771	0.09359523	0.09359320	0.09357217
40	0.03530404	0.03530854	0.03530752	0.03530668	0.03529798
50	0.01306878	0.01307080	0.01307041	0.01307009	0.01306675
60	0.00480353	0.00480440	0.00480426	0.00480414	0.00480290
70	0.00176094	0.00176131	0.00176125	0.00176121	0.00176075
80	0.00064493	0.00064508	0.00064506	0.00064504	0.00064487
90	0.00023611	0.00023617	0.00023617	0.00023616	0.00023610

solution $u_{HAM}(x,t)$ by HAM for $s = 1, p = 0.1, \tau = 0.5, k = -0.05, \omega = 0.1, t = 0.1$.

Example2

By taking $s = 2$, we will solve the equation

$$\tau u_{tt} + u_t = u_{xx} - pu^2 u_x \qquad (41)$$

The nonlinear operator is

$$N[u(x,t)] = \tau \frac{\partial^2 u(x,t)}{\partial t^2} + \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + pu^2(x,t)\frac{\partial u(x,t)}{\partial x} \qquad (42)$$

So that

$$N[\emptyset(x,t;q)] = \tau \frac{\partial^2 \emptyset(x,t;q)}{\partial t^2} + \frac{\partial \emptyset(x,t;q)}{\partial t} - \frac{\partial^2 \emptyset(x,t;q)}{\partial x^2} + p\emptyset^2(x,t;q)\frac{\partial \emptyset(x,t;q)}{\partial x} \qquad (43)$$

and thus, by equation (32), we have

$$R_m(\vec{u}_{m-1}) = \tau \frac{\partial^2 u_{m-1}(x,t)}{\partial t^2} + \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2}$$

$$+p\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} u_i(x,t)u_k(x,t)\frac{\partial u_{j-k-i-1}(x,t)}{\partial x} \qquad (44)$$

Substitute equation (44) in (34) with assumption $H(x,t) = 1$, we obtain the solution of equation (41) We will take $p = 1, \tau = 0.2, k = -0.1002, \omega = 0.1, t = 1$ [8] with initial conditions

$$u_0(x,t) = \frac{0.000751503\, t\, \mathrm{sech}^2(0.1002x)}{\sqrt{0.15-0.15\tanh(0.1002x)}} - \sqrt{0.15-0.15\tanh(0.1002x)}$$

We obtain

$$u_1(x,t) = \hbar[(0.00025\mathrm{sech}^2(0.1002x)\left(-\frac{0.00015\mathrm{sech}^2(0.1002x)\tanh(0.1002x)}{\sqrt{0.15-0.15\tanh(0.1002x)}} - \frac{0.000375\mathrm{sech}^2(0.1002x)(-0.15 + 0}{\sqrt{(0.15-0.15\tanh(0.10}}\right.$$

$$/\sqrt{0.15-0.15\tanh(0.1002x)} + \cdots].$$

$$u_2(x,t) = \cdots$$

$$\vdots$$

The solutions $u_1(x,t), u_2(x,t), \ldots$ are very large equations.

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t)$$

Solutions of equation (41) by tanh function method and Homotopy analysis method for some values of x in the interval [-100,100] will be showing in (table 2).

Table 2: The results represent the exact solution $u_E(x,t)$by tanh function method and numerical

x_i	u_E	u_{HAM}			
		ħ = -1.8	ħ = -0.2	ħ = 0.4	ħ = 1.6
-100	0.54772256	0.54772256	0.54772256	0.54772256	0.54772256
-90	0.54772255	0.54772255	0.54772255	0.54772255	0.54772255
-80	0.54772253	0.54772253	0.54772253	0.54772253	0.54772250
-70	0.54772234	0.54772235	0.54772235	0.54772232	0.54772216
-60	0.54772095	0.54772106	0.54772099	0.54772081	0.54771965
-50	0.54771061	0.54771146	0.54771091	0.54770959	0.54770101
-40	0.54763396	0.54764024	0.54763614	0.54762635	0.54756273
-30	0.54706632	0.54711276	0.54708247	0.54701000	0.54653928
-20	0.54290958	0.54324798	0.54302678	0.54250037	0.54290958
-10	0.51477148	0.51697449	0.51551499	0.51215902	0.49015584
0	0.38923380	0.39635525	0.39147501	0.38120940	0.31248456
10	0.19044408	0.19431185	0.19169693	0.18594552	0.14774323
20	0.07389145	0.07461344	0.07413192	0.07302380	0.06572760
30	0.02734487	0.02745129	0.02737995	0.02721540	0.02613046
40	0.01005032	0.01006565	0.01005517	0.01003145	0.00987369
50	0.00369045	0.00369279	0.00369111	0.00368751	0.00366300
60	0.00135495	0.00135536	0.00135504	0.00135441	0.00134996
70	0.00049746	0.00049755	0.00049748	0.00049734	0.00049632
80	0.00018264	0.00018267	0.00018264	0.00018260	0.00018231
90	0.00006705	0.00006706	0.00006706	0.00006704	0.00006695
100	0.00002462	0.00002462	0.00002462	0.00002461	0.00002458

solution $u_{HAM}(x,t)$ by HAM for $s = 2, p = 1, \tau = 0.2, k = -0.1002, \omega = 0.1, t = 1$.

Example3

By taking $s = 2$ and $\tau = 0$ we will solve the equation

$$u_t = u_{xx} - pu^2\, u_x \tag{45}$$

The nonlinear operator is

$$N[u(x,t)] = \frac{\partial u(x,t)}{\partial t} - \frac{\partial^2 u(x,t)}{\partial x^2} + pu^2(x,t)\frac{\partial u(x,t)}{\partial x} \tag{46}$$

So that

$$N[\emptyset(x,t;q)] = \frac{\partial \emptyset(x,t;q)}{\partial t} - \frac{\partial^2 \emptyset(x,t;q)}{\partial x^2} + p\emptyset^2(x,t;q)\frac{\partial \emptyset(x,t;q)}{\partial x} \tag{47}$$

and thus by equation (32), we have

$$R_m(\vec{u}_{m-1}) = \frac{\partial u_{m-1}(x,t)}{\partial t} - \frac{\partial^2 u_{m-1}(x,t)}{\partial x^2} + p\sum_{i=0}^{j-1}\sum_{k=0}^{j-i-1} u_i(x,t)u_k(x,t)\frac{\partial u_{j-k-i-1}(x,t)}{\partial x} \qquad (48)$$

Substitute equation (48) in (34) with assumption $H(x,t) = 1$, we obtain the solution of equation (45) We will $p = 0.1, k = -0.1, \omega = 0.1, t = 1, \lambda = -1$ [8] with initial conditions

$u_0(x,t) = \sqrt{1.5-1.5\tanh(0.1x)}$

We obtain

$$u_1(x,t) = \hbar\left[\left(-0.0075 + \frac{0.25\left(-0.15 + 0.15\tanh^2(0.1x)\right)^2}{\sqrt{(1.5-1.5\tanh(0.1x))^3}} - (0.15\frac{\tanh(0.1x)\left(0.1-0.1\tanh^2(0.1x)\right)}{\sqrt{1.5-1.5\tanh(0.1x)}} + 0.0075\tanh^2(0.1x)\right)t\right].$$

$$u_2(x,t) = u_1(x,t) + \hbar[-0.5\hbar(\frac{0.9375\left(-0.15 + 0.15\tanh^2(0.1x)\right)^4}{\sqrt{(1.5-1.5\tanh(0.1x))^7}} - \left(0.675\left(-0.15 + 0.15\tanh^2(0.1x)\right)^2\tanh(0.1x) + \cdots\right].$$

⋮

The solutions $u_2(x,t), u_3(x,t), \ldots$ are very large equations.

$$u(x,t) = \sum_{m=0}^{\infty} u_m(x,t)$$

Solutions of equation (45) by tanh function method and Homotopy analysis method for some values of x in the interval [-100,100] will be showing in (table 3).

Table 3: The results represent the exact solution $u_E(x,t)$ by tanh function method and numerical

x_i	u_E	u_{HAM}			
		ħ = -1.2	ħ = -0.4	ħ = 0.8	ħ = 1.4
-100	1.73205080	1.73205081	1.73205081	1.73205081	1.73205081
-90	1.73205079	1.73205079	1.73205079	1.73205079	1.73205079
-80	1.73205071	1.73205071	1.73205071	1.73205071	1.73205072
-70	1.73205010	1.73205008	1.73205008	1.73205011	1.73205014
-60	1.73204559	1.73204546	1.73204546	1.73204562	1.73204585
-50	1.73201228	1.73201127	1.73201132	1.73201250	1.73201414
-40	1.73176611	1.73175877	1.73175913	1.73176777	1.73177991
-30	1.72995048	1.72989662	1.72989924	1.72996180	1.73004985
-20	1.71670935	1.71633166	1.71634781	1.71674430	1.71730550
-10	1.62746521	1.62552815	1.62552475	1.62575909	1.62618563
0	1.23085316	1.22920941	1.22822085	1.22920941	1.16771807
10	0.60328808	0.60456553	0.60311666	0.56635544	0.51391447
20	0.23458191	0.23501080	0.23444020	0.21934137	0.19797237
30	0.08699047	0.08706588	0.08685552	0.08166465	0.07430293
40	0.03203696	0.03204926	0.03197514	0.03014432	0.02754727
50	0.01178748	0.01178972	0.01176296	0.01110161	0.01016335
60	0.00433646	0.00433696	0.00432719	0.00408561	0.00374287
70	0.00159530	0.00159544	0.00159185	0.00150322	0.00137746
80	0.00058688	0.00058692	0.00058561	0.00055303	0.00050681
90	0.00021590	0.00021592	0.00021543	0.00020345	0.00018646
100	0.00007942	0.00007943	0.00007925	0.00007485	0.00006859

solution $u_{HAM}(x,t)$ by HAM for $s = 2, \tau = 0, p = 0.1, k = -0.1, \omega = 0.1, t = 1, \lambda = -1$.

5-Conclusion:

In this paper, the Homotopy analysis method has been successfully applied to find the solution of time-delay Burgers equation. All the examples show that the results of HAM are excellent agree for different values of ħ with those obtained by tanh function method. We notice from Tables 1, 2 and 3 that the HAM is very accurate method and afew terms are required to obtain an accurate solution in comparison with the results of VIM and ADM, see[8]. So that the HAM is remarkably effective for solve the time-delay Burgers equation. In our work, we use the Maple13 to calculate the results obtained from the iteration method HAM.

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