# التأثيرات الفعلية المنتظمة القوة لبورباكي

## *المستخلص*

ان الهدف الرئيسي من هذا العمل هو تقديم نوع عام وجديد (حسب علمنا) من فضاءات $G-$ أسميناه فضاء G- السديد المنتظم القوة لبورباكي.وتضمن البحث بعض الامثلة والمبرهنات المهمة لفضاء G- السديد المنتظم القوة لبورباكي حيث تلك الفضاءات هاوزدورفية.

## *Strongly regular Bourbaki proper Actions*

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***Abstract***

The main goal of this work is to create a general type of proper $G$ – space , namely, strongly regular Bourbaki proper $G$ – space and to explain some of examples and theorems of st – r – Bourbaki proper action; where X and G are $T_2$ – spaces.

### Introduction

One of the very important concepts in topological groups is the concept of group actions and there are several types of these actions. This paper studies an important class of actions namely , strongly regular proper actions .Proper $G$ – spaces were studied by many mathematicians such as group , Bourbaki , Palais , Abels , Strantzalos , Dydo and others.

Let $B$ be a subset of a topological space $(X,T)$ . We denote the closure of $B$ and the interior of $B$ by $\overline{B}$ and $B^o$ , respectively .The subset $B$ of $(X, T)$ is called regular open (r – open) if $B = \overline{B}^o$ . The complement of a regular open set is defined to be a regular closed (r – closed). Then the family of all r – open sets in $(X,T)$forms a base of a smaller topology $T^r$ on $X$ ,called the semi – regularization of $T$. In section one of this work, we include some of results which then will needed in section two.

In section two, we deal with the definitions, examples, remarks, propositions, theorem and corollaries of strongly regular proper function. Section three recalls the definition of Bourbaki proper $G$ – space, gives a new type of Bourbaki proper $G$ – space (to the best of our Knowledge), namely, strongly regular proper $G$ – space and studies some of its properties, where $G$- space is meant $T_2$ – space topological $X$ on which an r – locally r – compact, non – compact, $T_2$ – topological group $G$ acts continuously on the left.

### 1. Preliminaries

**<u>1.1 Definition [3]:</u>**

A subset $B$ of a space $X$ is called regular open (R – open) set if $B = \overline{B}^o$ . The complement of a regular open set is defined to be regular closed (r – closed) set , then the family of all r – open sets in $(X,T)$forms a base of asmaller topology $T^r$on $X$ , called the semi – regularization of $T$ .

**1.2 Proposition [3,6]:**
Let $X$ be a space. Then
(i) If $A$ and B are r- open sets then $A \cap B$ is an r-open set.
(ii)If Ais an r-closed subset of $X$ and $B$ is a clopen set in $X$, then $A \cap B$ is an r – closed in $X$.

**1.3 Proposition [3]:** Let $X$ and $Y$ be two spaces. Then $A_1 \subseteq X$, $A_2 \subseteq Y$ be r – open(r – closed) sets in $X$ and $Y$ respectively if and only if $A_1 \times A_2$ is r – open(r – closed) in $X \times Y$.

**1.4 Definition:[3]** A subset $B$ of a space $X$ is called regular neighborhood (r – neighborhood) of $x \in X$ if there r- open subset $O$ of $X$ such that $x \in O \subseteq B$.

**1.5 Definition [3]:** Let $X$ and $Y$ be spaces and $f: X \rightarrow Y$ be a function. Then:
(i) $f$ is called regular continuous (r – continuous) function if $f^{-1}(A)$ is an r – open set in $X$ for every open set $A$ in $Y$.
(ii) $f$ is called regular irresolute (r – irresolute) function if $f^{-1}(A)$ is an r – open set in $X$ for every r- open set $A$ in $Y$.

**1.6 Proposition [3]:** Let $f: X \rightarrow Y$ be a function of spaces. Then $f$ is an r - continuous function if and only if $f^{-1}(A)$ is an r - closed set in $X$ for every closed set $A$ in $Y$.

**1.7 Proposition:** Let $X$ and $Y$ be spaces and let $f: X \rightarrow Y$ be a continuous, open function. Then $f$ is r – irresolute function.

. **Proof:**

Let A be an r-open set of Y, then $A = \overline{A}^{o}$ .Since f is continuous and open then

$f^{-1}(A) = f^{-1}(\overline{A}^{o}) = \left[f^{-1}(\overline{A})\right]^{\circ} = \left[\overline{f^{-1}(A)}\right]^{\circ}$ , $f^{-1}(A)$ is an r-open set of X.

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**1.8 Definition [3]:**
(i) A function $f: X \rightarrow Y$ is called strongly regular closed (st – r – closed) function if the image of each r – closed subset of $X$ is an r – closed set in $Y$.
(ii) A function $f: X \rightarrow Y$ is called strongly regular open (st – r – open) function if the image of each r – open subset of $X$ is an r – open set in $Y$.

**1.9 Definition [3]:** Let $X$ and $Y$ be spaces . Then a function $f: X \rightarrow Y$ is called a st –r– homeomorphism if:

(i) $f$ is bijective .

(ii) $f$ is continuous .

(iii) $f$ is st –r– closed (st – r– open).

**1.10 Proposition[3]:** Every r- homeomorphism is a st – r – homeomorphism.

**1.11 Definition [3]:** Let $(\chi_d)_{d\in D}$ be a net in a space $X$, $x\in X$. Then :

i) $(\chi_d)_{d\in D}$ is called r – converges to $x$ (written $\chi_d \xrightarrow{r} x$) if $(\chi_d)_{d\in D}$ is eventually in every r – neighborhood of $x$ . The point $x$ is called an r – limit point of $(\chi_d)_{d\in D}$, and the notation "$\chi_d \xrightarrow{r} \infty$" is mean that $(\chi_d)_{d\in D}$ has no r – convergent subnet.

ii) $(\chi_d)_{d\in D}$ is said to have $x$ as an r – cluster point [written $\chi_d \overset{r}{\alpha} x$] if $(\chi_d)_{d\in D}$ is frequently in every r - neighborhood of $x$ .

**1.12 Proposition:** Let $(\chi_d)_{d\in D}$ be a net in a space $(X, T)$ and $x_o$ in $X$. Then $\chi_d \overset{r}{\alpha} x_o$ if and only if there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \xrightarrow{r} x_o$.

**Proof:** $\Rightarrow$ Let $(\chi_d)_{d\in D}$ be a net in a space $(X, T)$ such that $\chi_d \overset{r}{\alpha} x_o$ in $(X, T)$ . Then $\chi_d \, \alpha \, x_o$ in $(X, T^r)$, so there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ such that $\chi_{dm} \to x_o$ in $(X, T^r)$. Then $\chi_d \xrightarrow{r} x_o$.

$\Leftarrow$ By same way we proof only if part.

**1.13 Remark:** Let $(\chi_d)_{d\in D}$ be a net in a space $(X, T)$ such that $\chi_d \overset{r}{\alpha} x$, $x\in X$ and let $A$ be an r – open set in $X$ which contains $x$. Then there exists a subnet $(\chi_{dm})_{dm\in D}$ of $(\chi_d)_{d\in D}$ in $A$ such that $\chi_{dm} \xrightarrow{r} x$.

**1.14 Definition [3]:** A subset $A$ of space $X$ is called r – compact set if every r – open cover of $A$ has a finite subcover. If $A=X$ then $X$ is called a r – compact space.

**1.15 Proposition [3]:** Let $X$ be a space and $F$ be an r – closed subset of $X$. Then $F\cap K$ is r – compact subset of $F$, for every r – compact set K in $X$.

**1.16 Proposition [3]:** Let $Y$ be an r – open subspace of space $X$ and $A\subseteq Y$. Then $A$ is an r – compact set in $Y$ if and only if $A$ is an r – compact set in $X$.

**1.17 Definition [3]:**

(i) A subset $A$ of space $X$ is called r - relative compact if $\overline{A}$ is r – compact.

(ii) A space $X$ is called r – locally r – compact if every point in $X$ has an r – relative compact r – neighborhood.

**1.18 Proposition[3]:** Let $X$ and $Y$ be spaces and $f: X\to Y$ be a function, then:
(i)If $f$ is continuous , then an image $f$(A)of any compact set A in X is a compact set in Y.
(ii)If $f$ is r-irresolute, then an image $f$(A) of any r- compact set A in X is an r-compact set in Y.

**1.19 Definition [3]:** Let $f: X \to Y$ be a function of spaces. Then:

(i) $f$ is called an regular compact (r – compact) function if $f^{-1}(A)$ is a compact set in $X$ for every r – compact set $A$ in $Y$.

(ii) $f$ is called a strongly regular compact (st – r – compact) function if $f^{-1}(A)$ is an r – compact set in $X$ for every r – compact set $A$ in $Y$.

**1.20 Proposition:** Let $X$, $Y$ be a spaces and $f: X \to Y$ be a st – r – compact function. If $F$ is an r – closed subset of $X$ and $B$ is an r – open set in $Y$, then $f_{/F}: F \to B$ is st – r– compact.

**Proof:** Let $K$ be an r – compact set in $B$ (To prove that $f_{/F}^{-1}(K)$ is r – compact in $F$). Since $f_{/F}^{-1}(K) = f^{-1}(K) \cap F$, then by Proposition (1.16) $K$ is an r – compact in $Y$. Since $f$ is st – r – compact, then $f^{-1}(K)$ is r – compact in $X$. Then by Proposition (1.15) $f^{-1}(K) \cap F$ is r – compact in $F$. Thus $f_{/F}^{-1}(K)$ is r – compact in $F$.

### *2 – Strongly Regular Proper Function*

**2.1 Definition [3]:** Let $X$ and $Y$ be two spaces. Then $f: X \to Y$ is called a strongly regular proper (st – r - proper) function if :

(i) $f$ is continuous function.

(ii) $f \times I_Z: X \times Z \to Y \times Z$ is a st – r – closed function, for every space $Z$.

**2.2 Proposition [3]:** Let $X$ and $Y$ be spaces and $f: X \to Y$ be a continuous function. If Y is a $T_2$ space, , then the following statements are equivalent:

(i) $f$ is a st – r – proper function.

(ii) $f$ is a st – r – closed function and $f^{-1}(\{y\})$ is an r – compact set, for each $y \in Y$.

(iii) If $(\chi_d)_{d \in D}$ is a net in $X$ and $y \in Y$ is an r – cluster point of $f(\chi_d)$, then there is an r – cluster point $x \in X$ of $(\chi_d)_{d \in D}$ such that $f(x) = y$.

**2.3 Proposition [3]:** Let $X$, $Y$ and $Z$ be spaces, $f: X \to Y$ and g: $Y \to Z$ be a st – r – proper functions . Then $g o f : X \to Z$ is a st – r – proper function.

**2.4 Proposition [3];** Let $f_1: X_1 \to Y_1$ and $f_2: X_2 \to Y_2$ be functions. Then $f_1 \times f_2: X_1 \times X_2 \to Y_1 \times Y_2$ is st – r- proper function if and only if $f_1$ and $f_2$ are st – r – proper functions.

**2.5 Proposition [3]:**

**(i)** Every st – r – proper function is st – r – closed.

**(ii)**Every r-proper function is st-r-proper..

**(iii)**Every st-r-homeomorphism is st-r-proper.

**2.6 Proposition[3]:** Let $f$: $X \rightarrow P = \{w\}$ be a function on a space $X$. Then $f$ is a st – r – proper function if and only if $X$ is an r – compact, where $w$ is any point which dose not belongs to $X$.

**2.7 Lemma:** Every r- continuous function from an r – compact space into a Hausdorff space is st -r- closed.

**Proof:** Let $f$: $X \rightarrow Y$ be an r – continuous function of a r – compact $X$ into a Hausdorff space $Y$ and let A be r – closed in $X$. Then $A$ is r – compact. Then by Proposition (1.19) $f(A)$ is r-compact in $Y$. Thus $f(A)$ is r – closed. Hence $f$ is st – r – closed.

**2.8 Remark:** If $X$ is a space, then the diagonal function $\Delta$: $X \rightarrow X \times X$ such that $\Delta(x) = (x, x)$ is continuous and r – irresolute function.

**2.9 Proposition :** If $X$ is a Hausdorff space, then the diagonal function $\Delta$: $X \rightarrow X \times X$ is st – r – proper function.

**Proof:** Since $\Delta$ is continuous and X is $T_2$ . Let $(\chi_d)_{d \in D}$ be a net in $X$ and $y = (x_1, x_2) \in X \times X$ be an r – cluster point of $\Delta(\chi_d)$. Then $\Delta(\chi_d) = (\chi_d, \chi_d) \overset{r}{\alpha} (x_1, x_2)$, so by Proposition (1.12) there exists a subnet of $(\chi_d, \chi_d)$ ,say itself such that $(\chi_d, \chi_d) \xrightarrow{r} (x_1, x_2)$, then $\chi_d \xrightarrow{r} x_1$ and $\chi_d \xrightarrow{r} x_2$, since $X$ is a $T_2$ – space , then $x_1 = x_2$ . Then there is $x_1 \in X$ such that $\chi_d \overset{r}{\alpha} x_1$ and $\Delta(x_1) = y$. Hence by Proposition (2.2.iii) $\Delta$ is a st – r – proper function.

**2.10 Proposition:** Let $f_1$: $X \rightarrow Y_1$ and $f_2$: $X \rightarrow Y_2$ be two st – r – proper functions. If $X$ is a Hausdorff space, then the function $f$: $X \rightarrow Y_1 \times Y_2$, $f(x) = (f_1(x), f_2(x))$ is a st – r- proper function.

**Proof:** Since $X$ is Hausdorff, then by Proposition (2.9) $\Delta$ is a st – r – proper function. Also by Proposition (2.4) $f_1 \times f_2$ is a st–r–proper function. Then by Proposition (2.3) $f = f_1 \times f_2 o \Delta$ is a st – r – proper function.

**2.11 Proposition [3]:** Let $X$ and $Y$ be a spaces, such that X is a compact , $T_2$ spaces and $f$: $X \rightarrow Y$ is a continuous function. Then the following statements are equivalent:

(i) $f$ is an r – proper function.

(ii) $f$ is a st – r – proper function.

**2.12 Proposition [3]:** Let $X$ and $Y$ be a spaces, such that $Y$ is a $T_2$ – space and $f$: $X \rightarrow Y$ be continuous, r– irresolute function. Then the following statements are equivalent:

(i) $f$ is a st – r– compact function.

(ii) $f$ is a st – r– proper function.

## *3 – Strongly Regular Bourbaki Proper G-Space.*

**3.1 Definition [5]:** A topological transformation group is a triple $(G,X,\varphi)$ where $G$ is a $T_2$ – topological group, $X$ is a $T_2$ – topological space and $\varphi: G\times X \to X$ is a continuous function such that:

(i) $\varphi(g_1, \varphi(g_2, x)) = \varphi(g_1 g_2, x)$ for all $g_1, g_2 \in G$, $x \in X$.

(ii) $\varphi(e, x) = x$ for all $x \in X$, where $e$ is the identity element of $G$.

In this case the space $X$ together with $\varphi$ is called a $G$ – space ( or more precisely left $G$ – space).

**3.2 Remark [4]:** Let $X$ be a $G$ – space and $x \in X$. Then :

(i) The function $\varphi$ is called an action of $G$ on $X$.

(ii) A set $A \subseteq X$ is said to be invariant under $G$ if $GA = A$.

**3.3 Definition:** A $G$ – space $X$ is called a strongly regular Bourbaki proper $G$ – space (st – r – proper $G$ – space) if the function $\theta: G\times X \to X\times X$ which is defined by $\theta(g, x) = (x, g.x)$ is a st – r – proper function.

**3.4 Example:** The topological group $Z_2 = \{-1, 1\}$ [as $Z_2$ with discrete topology] acts on the topological space $S^n$ [as a subspace of $R^{n+1}$ with usual topology] as follows:

$1. (x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_{n+1})$

$-1. (x_1, x_2, \ldots, x_{n+1}) = (-x_1, -x_2, \ldots, -x_{n+1})$

Since $Z_2$ is an compact, then by Proposition (2.6) the constant function $Z_2 \to P$ is an r – proper. Also the identity function is st- r – proper, then by Proposition (2.4) the function of $Z_2\times S^n$ into $P\times S^n$ is st- r – proper.

Since $P\times S^n$ is homeomorphic to $S^n$, the composition $Z_2\times S^n \to S^n$ is an r – proper function hence $Z_2\times S^n \to S^n$ is a st- r – proper function . Let $\varphi$ be the action of $Z_2$ on $S^n$. Then $\varphi$ continuous,. Since $S^n$ is $T_2$ – space. Then by Proposition (2.7) $\varphi$ is st r – proper function . Thus by Proposition (2.10) $Z_2\times S^n \to S^n\times S^n$ is a st – r – proper function ,therefore $S^n$ is st-r - proper $Z_2$- space.

**3.5 Lemma:** If $X$ is a $G$ – space then the function $\theta: G\times X \to X\times X$ which is defined by $\theta(g, x) = (x, g.x)$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $G\times X$ for every $(x, y) \in X\times X$.

**Proof:** $\theta: G\times X \xrightarrow{I_G\times\Delta} G\times X\times X \xrightarrow{\varphi\times I_X} X\times X \xrightarrow[\cong]{f} X\times X$, where $\varphi$ is action of $G$ on $X$. Then $\theta = f \circ \varphi \times I_X \circ I_G \times \Delta$ is continuous function and $\theta^{-1}(\{(x, y)\})$ is closed in $G\times X$ for every $(x, y) \in X\times X$.

**3.6 Theorem:** Let $X$ be an st – r – proper $G$ – space and $H$ be r – closed subset of $G$. If $Y$ is an r – open subset of $X$ which is invariant under $H$, then $Y$ is a st – r- proper $H$ – space.

**Proof:** Since $X$ is a st – r – proper $G$ – space, then the function $\theta: G\times X \to X\times X$ which is defined by $\theta(g, x) = (x, g.x)$ is a st – r – proper function. [To prove that $\omega: H\times Y \to Y\times Y$ is a st – r – proper function which is defined by $\omega(h, y) = \theta(h,y)$ for each $(h, y)\in H\times Y$.]

(1)Since $\theta: G\times X \to X\times X$ is continuous, then $\omega: H\times Y \to Y\times Y$ is continuous. .

(2) Let $(h_d, y_d)_{d\in D}$ be a net in $H\times Y$ such that $\omega((h_d, y_d)) \overset{r}{\alpha} (x, y)$ for some $(x, y)\in Y\times Y$. Then $(y_d, h_d y_d) \overset{r}{\alpha} (x, y)$ in $Y\times Y$. Let $A$ be an r – open subset of $X\times X$ such that $(x, y)\in A$. Since $Y$ is r – open in $X$, then $Y\times Y$ is an r – open set in $X\times X$. Then $A \cap (Y\times Y)$ is an r – open set in $X\times X$ . But $(x, y)\in A \cap (Y\times Y)$ and $(y_d, h_d y_d) \overset{r}{\alpha} (x, y)$, thus $(y_d, h_d y_d)$ is frequently in $A \cap (Y\times Y)$ and then $(y_d, h_d y_d)$ is frequently in $A$, thus $(y_d, h_d y_d) \overset{r}{\alpha} (x, y)$ in $X\times X$. since $\theta: G\times X \to X\times X$ is an st – r – proper function, then by Proposition (2.2) there exists $(h, x_1)\in G\times X$ such that $(h_d, y_d) \overset{r}{\alpha} (h, x_1)$ and $\theta((h, x_1))=(x, y)$, hence $(x_1, h x_1)=(x,y)$. Thus $x_1=x$ and therefore, $h_d \overset{r}{\alpha} h$. Since $(h_d)_{d\in D}$ is a net in $H$ , and $H$ is r – closed . Then there exists $(h, x) \in H\times Y$ such that $\omega(h, x) = \theta(h,x) = (x, y)$. Then from (1), (2) and by Proposition (2.2) the function $\omega: H\times Y \to Y\times Y$ is a st – r – proper function. Hence $Y$ is a st – r – proper $H$ – space.

**3.7 Corollary:** Let $X$ be a st – r – proper $G$ – space and $Y$ be an r – open subset of $X$ which is invariant under $G$. Then $Y$ is a st – r- proper $G$ – space.

**3.8 Corollary:** Let $X$ be a st – r – proper $G$ – space and let $H$ be r – closed subset of $G$. Then $X$ is a st – r- proper $H$ – space.

**3.9 Theorem :** Let $X$ be an st – r – proper $G$ – space, $x\in X$ such that $\{x\}$ is clopen and $T=\{x\}\times X$. Then the function $\theta_T: \theta^{-1}(T) \to T$ is an st – r – proper function, where $\theta: G\times X \to X\times X$ such that $\theta$ is an r – irresolute function and $\theta(g, x) = (x, g.x)$, $\forall (g, x)\in G\times X$.

**Proof:** Since $\{x\}$ is clopen in $X$ then $\{x\}$ is r-clopen set in $X$ . So each $G\times\{x\}$ and $\{x\}\times X$ are r – closed in $G\times X$ and $X\times X$ (respectively). Now, Let $F$ be an r – closed set in $\theta^{-1}(T) = G\times\{x\}$, then F is r-closed in $G\times X$.Since $F=F\cap(G\times\{x\})$, $\theta_T(F) = \theta(F)\cap(\{x\}\times X)$,since $\theta$ *is st* –r proper therefore , $\theta(F)$ is r– closed in $X\times X$ by Proposition (2.1) $\theta_T(F)$ is r – closed in $X\times X$. But $\theta_T(F) \subseteq \{x\}\times X$, then there exists a subset $V$ of $X$ such that $\theta_T(F) = \{x\}\times V$. Since $\theta_T(F)$ is r- – closed in $X\times X$, so $\{x\}\times V$ is an r – closed set in $\{x\}\times X$, hence $\theta_T(F) = \{x\}\times V$ is an r – closed set in $T=\{x\}\times X$ therefore, $\theta_T: \theta^{-1}(T) \to T$ is st – r – closed. Now, let $(x, y)\in \{x\}\times X$. Since $\theta$ is st – r – proper function, then by Proposition (2.12) $\theta$ is a st – r – compact function. Then $\theta^{-1}(\{(x, y)\})$

is r– compact in $G\times X$. Then $\theta_T^{-1}(\{(x, y)\})$ is r– compact set in $G\times\{x\} = \theta^{-1}(T)$. [By Lemma (3.5)] $\theta$ is continuous, then $\theta_T$: $\theta^{-1}(T)\to T$ is continuous. . Since $X$ is $T_2$ – space, then $\{x\}\times X$ is $T_2$ – space , Thus by Proposition (2.2) $\theta_T$ is an st – r – proper function.

Let $X$ be a $G$ – space and $A$ , $B$ be two subset of $X$ . We mean by $((A, B))$ the set $\{g\in G \,/\, gA\cap B\neq\phi\}$.

From now on, we will use $G$ – space, which satisfies the property if $(X,T)$ and $(Y,T')$ be two space and $\forall\ x_d \xrightarrow{r} x,\ y_d \xrightarrow{r} y$ in X and Y, respectively, then $(x_d\, y_d) \xrightarrow{r} (x,y)$.

**3.10 Theorem :** Let $X$ be a $G$ – space. If for every $x, y\in X$ there exists an r – open set $A_x$ of $X$ contains $x$ and an r – open set $A_y$ of $X$ contains $y$ such that $K=((A_x, A_y))$ is r – relatively compact in $G$, then $X$ is a st – r – proper $G$ – space.

**Proof:** We prove that $\theta : G\times X\to X\times X$, $\theta(g, x) = (x, gx)$ is a st – r – proper function. Let $(g_d, \chi_d)_{d\in D}$ be a net in $G\times X$ such that $\theta(g_d, \chi_d)=(\chi_d, g_d\chi_d) \overset{r}{\alpha} (x, y)$, where $(x, y)\in X\times X$. Now, since $x, y\in X$, then there exists an r – open set $A_x$ contains $x$ and an r – open set $A_y$ contains $y$ such that the set $K = ((A_x, A_y))$ is r – relatively compact in $G$. Thus $A_x\times A_y$ is an r – open set in $X\times X$ and $(x, y)\in A_x\times A_y$, so by Proposition (1.13) there exists a sub net $(\chi_{d_m}, g_{d_m}\chi_{d_m})_{d\in D}$ of $(\chi_d, g_d\chi_d)$ in $A_x\times A_y$ and $(\chi_{d_m}, g_{d_m}\chi_{d_m}) \xrightarrow{r} (x, y)$, hence $\chi_{d_m} \xrightarrow{r} x$ and $g_{d_m}\chi_{d_m} \xrightarrow{r} y$. Since $\chi_{d_m}\in A_x$, and $g_{d_m}\chi_{d_m}\in A_y$, Then $g_{d_m}.\ A_x\cap A_y \neq \phi$, $\forall\, d_m$, so $g_{d_m}\in K$, but $K$ is r – relatively compact in $G$, then by Proposition (1.12) $(g_{d_m})$ has an r – limit point, say $t\in G$. Since $\chi_{d_m} \xrightarrow{r} x$, then $(g_{d_m}, \chi_{d_m}) \xrightarrow{r} (t,x)$,so$\theta\ ((g_{d_m}, \chi_{d_m})) \xrightarrow{r} \theta\ ((t,x))$ ,i.e. ,$(\chi_{d_m}, g_{d_m}\chi_{d_m}) \xrightarrow{r} (x, tx)$, thus $g_{d_m}\chi_{d_m} \xrightarrow{r} tx$ but $g_{d_m}\chi_{d_m} \xrightarrow{r} y$ and since $X$ is a $T_2$ space, then $tx = y$. But $(\chi_{d_m}, g_{d_m}\chi_{d_m})_{d\in D}$ is a sub net of $(\chi_d, g_d\chi_d)$ and $(g_{d_m}, \chi_{d_m}) \xrightarrow{r} (t, x)$, then $(g_d, \chi_d) \overset{r}{\alpha} (t, x)$, thus $\theta((t, x)) = (x,y)$ and by Proposition (2.2) we have $\theta$ is a st – r – proper function. Hence $X$ is a st – r – proper $G$ – space.

**3.11 Corollary:** Let $X$ be a $G$ – space such that $G$ is discrete space. If for every $x, y\in X$ there is an r – open set $A_x$ in $X$ contains $x$ and an r – open set $A_y$ in $X$ contains $y$ such that the set $K = ((A_x, A_y))$ is finite, then $X$ is a st – r – proper $G$ – space.

Let $X$ be a $G$ – space and $x\in X$. The set $J^r(x)$={$y\in X$: there is a net $(g_d)_{d\in D}$ in $G$ and there is a net $(\chi_d)_{d\in D}$ in $X$ with $g_d \xrightarrow{r} \infty$ and $\chi_d \xrightarrow{r} x$ such that $g_d x \xrightarrow{r} y$} is called regular first prolongation limit set of $x$. $J^r(x)$ is a good tool to discover about the st – r – Bourbaki proper $G$ – space

**<u>3.12 Theorem :</u>** Let $X$ be a $G$ – space. Then $X$ is a st – r – Bourbaki proper $G$ – space if and only if $J^r(x)=\phi$ for each $x\in X$.

**<u>Proof:</u>** $\Rightarrow$ Suppose that $y\in J^f(x)$, then there is a net $(g_d)_{d\in D}$ in G with $g_d \xrightarrow{r} \infty$ and there is a net $(\chi_d)_{d\in D}$ in $X$ with $\chi_d \xrightarrow{r} x$ such that $g_d\chi_d \xrightarrow{r} y$, so $\theta((g_d,\chi_d))=(x_d, g_d\chi_d) \xrightarrow{r} (x, y)$. But $X$ is a st – r – Bourbaki proper, then by Proposition (2.2) there is $(g, x_1)\in G\times X$ such that $(g_d, x_d) \overset{r}{\propto} (g, x_1)$. Thus $(g_d)_{d\in D}$ has a sub net (say itself). such that $g_d \xrightarrow{r} g$, which is contradiction, thus $J^r(x)=\phi$.

$\Leftarrow$ Let $(g_d, \chi_d)_{d\in D}$ be a net in $G\times X$ and $(x, y)\in X\times X$ such that $\theta((g_d, \chi_d))=(\chi_d, g_d\chi_d) \overset{r}{\propto} (x, y)$, so $(\chi_d, g_d\chi_d)_{d\in D}$ has a sub net, say itself, such that $(\chi_d, g_d\chi_d) \xrightarrow{r} (x, y)$, then $\chi_d \xrightarrow{r} x$ and $g_d\chi_d \xrightarrow{r} y$. Suppose that $g_d \xrightarrow{r} \infty$ then $y\in J^r(x)$, which is contradiction . Then there is $g\in G$ such that $g_d \xrightarrow{r} g$, then $(g_d, \chi_d) \xrightarrow{r} (g, x)$ ,since $\theta$ is an r-irresolute then $\theta((g_d, \chi_d)) \xrightarrow{r} \theta(g, x)$ ,i.e $(\chi_d, g_d\chi_d) \xrightarrow{r}$(x,g.x), but $(\chi_d, g_d\chi_d) \xrightarrow{r}$(x,y),since $X\times X$ is$T_2$ space then (x,g.x)=(x,y) i.e $\theta(g, x) = (x, y)$. Thus by Proposition (2.2) $X$ is a st – r – Bourbaki proper $G$ – space.

**<u>3.13 Theorem :</u>** Let $X$ be a st – r – proper $G$ – space with the action $\varphi: G\times X\to X, \varphi(g, x) = g.x, \forall(g,x) \in G\times X$. Then for each $x\in X$,let x∈X such that $\{x\}$ is clopen set in X the function $\varphi_x: G\to X$, which is defined by: $\varphi_x(g)=\varphi(g,x)$ is a st – r – proper function.

**<u>Proof:</u>** Let T=$\{x\}\times X \subseteq X\times X$, then by Theorem (3.9) $\theta_T: \theta^{-1}(T)\to T$ is an st – r – proper function. But:

$$\varphi_x = G \overset{f}{\cong} G\times\{x\} \xrightarrow{\theta_T} \{x\}\times X \overset{h}{\cong} X,$$ such that $f$ and $h$ are homeomorphisms.

Now:

i) since each of these functions are continuous so $\varphi_x: G\to X$ is continuous.

ii) Let $F$ be an r – closed in $G$, then $F$ is closed in G thus f(F )is an r-closed in $G\times X$. Since $\theta_T$: $G\times\{x\}\rightarrow \{x\}\times X$ is a st – r – proper function, then by Proposition (2.5) $\theta_T(f(F))$ is r – closed, then by Proposition (1.11) $h(\theta_T(f(F)))$ is r– closed in $X$. Then $\varphi_x$: $G\rightarrow X$ is st – r – closed.

iii) Let $y\in X$, then $h^{-1}(\{y\}) = \{(x, y)\}$ such that $x\in X$, since $X$ is $T_2$ – space, then $\{(x, y)\}$ is an r – closed set in $\{x\}\times X$. Since $\theta_T$ is an r – continuous function, then $\theta_T^{-1}(\{(x, y)\})$ is r – closed in $G\times\{x\}$, so by Proposition (2.2,ii) $\theta_T^{-1}(h^{-1}(\{y\})) = \theta_T^{-1}\{(x, y)\}$ is r – compact. Since $f$ is homeomorphism, then $f^{-1}$ is an r – irresolute function, so its clear that $f^{-1}(\theta_T^{-1}\{(x, y)\})$ is r – compact in $G$. Then $f^{-1}(\theta_T^{-1}(h^{-1}(\{y\}))) = \varphi_x^{-1}(\{y\})$ is r – compact in $G$. Then by (i),(ii),(iii) and Proposition (2.2,ii) $\varphi_x$ is st – r – proper function.

**3.14 Theorem :** Let $X$ be a $G$ – space and $\theta$ : $G\times X\rightarrow X\times X$ be a function which is defined by $\theta(g, x) = (x, gx)$, $\forall(g, x)\in G\times X$. Then the following statements are equivalent:

(i) $X$ is a st – r – proper $G$ – space.

(ii) $\theta^{-1}(\{(x, y)\})$ is an r – compact set, $\forall(x, y)\in X\times X$ and for all $x, y\in X$ and for all $U\in N_r(((x, y)))$, $\exists V_x\in N_r(x)$ and $V\in N_r(y)$ such that $((V_x, V_y))\subseteq U$.

(iii) $\theta^{-1}(\{(x, y)\})$ is an r – compact set, $\forall(x, y)\in X\times X$ and for all $x, y\in X$ and for all $U\in N^r(\theta^{-1}(\{(x, y)\}))$, $\exists$ $V\in N_r((x, y))$ such that $\theta^{-1}(V)\subseteq U$.

**Proof:** i)→iii) Let $x, y\in X$ and $U$ be an r – open neighborhood of $\theta^{-1}(x, y)$. Since $\theta$ is a st – r – proper function, then by Proposition (2.2) $\theta$ is a st – r – closed function, so $V= (X\times X)\backslash\theta((G\times X)\backslash U)$ is an r – open neighborhood of $(x, y)$ with $\theta^{-1}(V)\subseteq U$. Since $\theta$ is continuous and $X\times X$ is $T_2$ space, so by Proposition (2.2) $\theta^{-1}(\{(x, y)\})$ is an r - compact set $\forall(x, y)\in X\times X$. Hence (iii), holds.

iii)→i) Let $F$ be an r – closed subset of $G\times X$ and let $(x,y)\in X\times X\backslash\theta(F)$, since$(G\times X)\backslash F$ r – open neighborhood of $\theta^{-1}(x, y)$, then by (iii) there is an r – neighborhood $V$ of $(x, y)$ such that $\theta^{-1}(V)\subseteq(G\times X)\backslash F$. Hence $V\cap\theta(F) = \phi$, so $(x, y)\notin \overline{\theta(F)}^r$, then $\overline{\theta(F)}^r = \theta(F)$. Hence $\theta$ is a st – r – closed function, since $\theta^{-1}(\{(x,$

$y)\}$) is an r – compact set for every $(x, y)\in X\times X$, therefore by Proposition (2.2)$\theta$ is an st – r – proper function. Hence $X$ is a st – r – proper $G$ – space.

ii)→iii) Let $x, y\in X$ and let $U$ be an r – neighborhood of $\theta^{-1}(\{(x, y)\}) = ((x, y))\times\{x\}$. Since $\theta^{-1}(\{(x, y)\})$ is r – compact, then there are r – neighborhood $U'$ of $((x, y))$ and $W$ of $\{x\}$ such that $U'\times W\subseteq U$, so by (ii) there are r – neighborhood $V_x$ of $x$ and $V_y$ of y such that $((V_x, V_y))\subseteq U'$. But $\theta^{-1}((V_x \cap W)\times V_y)\subseteq U'\times W\subseteq U$ . Hence (iii), hold.

iii) → ii) Let $x, y\in X$ and $U\in N_r((x, y))$. Then $U\times X\in N_r((x, y))\times\{x\}$. Thus $U\times X\in N_r(\theta^{-1}(x, y))$ so by (iii) there exists $V\in N_r(x, y)$ such that $\theta^{-1}(V)\subseteq U\times X$. Then there are r – neighborhood $V_x$ of $x$ and $V_y$ of $y$ such that $\theta^{-1}(V_x\times V_y)\subseteq U\times X$. Hence (ii), holds.

**3.15 Corollary:** Let $X$ be a st – r – proper $G$ – space , choose a point $x\in X$ and let $U$ be r – neighborhood of the stabilizer $G_x$ of $x$ , then $x$ has an r – neighborhood $V$ such that $U$ contains the stabilizer of all points in $V$.

**Proof:** Since $U$ is r – neighborhood of the stabilizer $G_x$ of $x$, then $U\in N_r(G_x)$. Since $G_x = ((x, x))$, then $U\in N_r(((x, x)))$. So by Theorem (3.14)there exist $V_x\in N^r(x, x)$ such that $((V_x, V_x))\subseteq U$. Let $y\in V_x$, then $G_y\subseteq ((V_x, V_x))\subseteq U$.

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