ACTION ON FINITE MONOID

الفعل على المونويد المنتهي

Prof. Dr. Habeeb Kareem Abdulah
College of Education for girls, Department of Mathematics, University of Kufa, Najef , Iraq
E-mail: habeebk.abdullah@uokufa.edu.iq

Assist. Prof. Amal Khalaf Haydar
College of Education for girls, Department of Mathematics, University of Kufa , Najef , Iraq
E-mail:amalkh.hayder@uokufa.edu.iq

Yaqoob Ali Hussean
College of Education for girls, Department of Mathematics, University of Kufa, Najef , Iraq
E-mail: yaqoob.alshamarry@gmail.com

Abstract

In this paper we propose and explore a general notion of chaos in the abstract context of continuous actions of topological monoids and we introduce the notions of chaotic action and study these notions on a new types of actions. Also We stated and prove some theorems which determine the relationship between these notions and some types of actions . As we define new type of action be not chaotic always on any finite topological monoid.
Keywords : Active, Chaotic, Less chaotic, action by $*$, monoid by Z_P.

المستخلص

في هذا البحث قدمنا فكرة عامة عن الفوضى على الأفعال المستمرة المعرفة على المونويد التبلوجي وقدمنا مفاهيم الفعل الفوضوي وبعض الأنواع من الأفعال, وناقشنا هذه الأفكار على أنواع جديدة من الأفعال .كما ذكرنا، وأثبتنا بعض النظريات التي تحدد العلاقة بين هذه المفاهيم وعرفنا نوعا جديدا من الافعال الذي يكون دائما ليس فوضويا على أي مونويد تبلوجي منتهي.

Introduction

Chaos is one of the core phenomena studied in the theory of dynamical systems (see ,e.g., [1]). Devaney's well-recognized definition of chaos [10] involves the concept of sensitive dependence on initial conditions (see Section 4 from[6]) and therefore requires the respectively considered phase space to be metric, or uniform at least. However, a well-known result by Banks et al. [9], which has been modified and extended in various ways in [11, 5, 2,4, 3, 8], suggests that the notion of chaos may be reformulated entirely in terms of topology .The aim of the present paper is define action such that it is not chaotic always of all topological monoids .

1.PRELIMINARIES

The aim of this section gives some definitions and results on action which we use them in this work.

***Definition* (1.1) [6]:**

Let $(M,*)$ be monoid the mapping $\varphi: M \times X \longrightarrow X$ is called an action if it is satisfy

1- $\varphi(1,x) = x \, , \forall\, x \in X$

2- $\varphi(m, \varphi(n,x)) = \varphi(m*n,x)\ , \forall m,n \in M, x \in X.$

***Remark* (1.2) [6]:**

Let $\varphi: M \times X \longrightarrow X$ be action, for all $m \in M$ define the map $\varphi_m: X \longrightarrow X$ by, $\varphi_m(x) = \varphi(m\,,x)$

***Theorem* (1.3):**
Let$(M,*)$ be monoid. Then the map $\varphi: M \times X \longrightarrow X$ is action iff $(\varphi_m) = \{\varphi_m, m \in M\}$ is monoid with composition functions.

Proof: Let $(M,*)$ be monoid.

$(\Rightarrow)$ Let $\varphi: M \times X \longrightarrow X$ be action.

Since M is monoid then $\varphi_1 \in (\varphi_m)$, since for all $m \in M$

$$(\varphi_1 \circ \varphi_m)(x) = \varphi_1(\varphi_m(x))$$
$$= \varphi_1(\varphi(m,x)) = \varphi(1, \varphi(m,x))$$
$$= \varphi_m(x) = \varphi(m,x) \text{ and}$$
$$(\varphi_m \circ \varphi_1)(x) = \varphi_m(\varphi_1(x))$$
$$= \varphi_m(m, \varphi_1(x))$$
$$= \varphi(m, \varphi(1,x))$$
$$= \varphi(m * 1, x)$$
$$= \varphi(m,x)$$
$$= \varphi_m(x)$$

So φ_1 is identity element of (φ_m) .

Now $\varphi_m \circ (\varphi_n \circ \varphi_k)(x) = \varphi_m \circ \varphi_n(\varphi_k(x))$

$$= \varphi_m \circ \varphi_n(\varphi(k,x))$$
$$= \varphi_m(\varphi(n, \varphi(k,x)))$$
$$= \varphi_m(\varphi(n * k, x)) = \varphi(m, \varphi(n * k, x))$$
$$= \varphi(m * (n * k), x) = \varphi((m * n) * k, x)$$
$$= \varphi_{(m*n)*k}(x) = (\varphi_m \circ \varphi_n) \circ \varphi_k(x)$$

Then $((\varphi_m), \circ)$ is monoid.

$(\Leftarrow)$ $\varphi(1,x) = \varphi_1(x) = x, \forall x \in X$ [since φ_1 is identity element]

$$\varphi(m, \varphi(n,x)) = \varphi_m(\varphi(n,x)) = \varphi_m(\varphi_n(x))$$
$$= \varphi_{m*n}(x) = \varphi(m * n, x)$$

for all $m, n \in M, x \in X$. Then the map $\varphi: M \times X \longrightarrow X$ is action.

***Remark* (1.4):**
Let A be a non-empty finite set. We denoted of a number of elements of A by $\mathrm{ord}(A)$.

***Definition* (1.5):**
Let $(M,*)$ be monoid and let X be a non-empty set such that $\mathrm{ord}(M) = \mathrm{ord}(X) = k$. The action $T: M \times X \to X$, $T(m_i, x_j) = x_r$ where $m_i * m_j = m_r$, $1 \leq i,j,r \leq k$ is called action by $*$.

***Examples* (1.6):**
1) Take $M = \{m_1, m_2, m_3, m_4\}$ and $X = \{x_1, x_2, x_3, x_4\}$, with the following tables

$*$	m_1	m_2	m_3	m_4
m_1	m_1	m_2	m_3	m_4
m_2	m_2	m_4	m_1	m_3
m_3	m_3	m_1	m_4	m_2
m_4	m_4	m_3	m_2	m_1

T	x_1	x_2	x_3	x_4
m_1	x_1	x_2	x_3	x_4
m_2	x_2	x_4	x_1	x_3
m_3	x_3	x_1	x_4	x_2
m_4	x_4	x_3	x_2	x_1

Then it is clear T is action by $*$.

2) Let (M,$*$) be finite monoid .Then $*$ is action by $*$.

3) Take $M = \{ m_1 , m_2 , m_3 , m_4\} , (M,*)$ is monoid where

$*$	m_1	m_2	m_3	m_4
m_1	m_1	m_2	m_3	m_4
m_2	m_2	m_1	m_4	m_3
m_3	m_3	m_4	m_3	m_4
m_4	m_4	m_3	m_4	m_3

and let $X = \{x_1 , x_2 , x_3 , x_4\}$ then the trivial action $T: M \times X \to X, T(m , x) = x$ for all $m \in M$ and $x \in X$ is not action by $*$ since $m_2 * m_3 = m_4$ but $T(m_2 , x_3) = x_3$.

***Definition* (1.7) [7]:**
A topological monoid is monoid $(M,*)$ with a topology Ton M when the binary operation $*: M \times M \longrightarrow M$ is continuous.

***Examples* (1.8):**

1) Any monoid with discrete topology is topological monoid.
2) Any monoid with indiscrete topology is topological monoid.
3) Let $M = \{ m_1 , m_2 , m_3 , m_4\}$, defined the binary operation $*$ in the following table.

$*$	m_1	m_2	m_3	m_4
m_1	m_1	m_2	m_3	m_4
m_2	m_2	m_1	m_4	m_3
m_3	m_3	m_4	m_3	m_4
m_4	m_4	m_3	m_4	m_3

The monoid $(M,*)$ with topology $T = \{\phi, M , \{m_1\}\}$ is topological monoid.

***Definition* (1.9) [6]:**
Let M be monoid and let K and T are subset of M, we write

1) $m^{-1}\mathrm{T} = \{ \mathrm{n} \in M,\ mn \in \mathrm{T} \}$
2) $K^{-1}T = \bigcup_{m \in K} m^{-1}T$
3) $KT = \{mn / m \in K, n \in T\}$.

***Definition* (1.10) [6]:**
Let M be topological monoid. A subset $T \subseteq M$ is called right syndetic in M if there exists a compact subset $K \subseteq M$ such that $\mathrm{K}^{-1}T = KT = M$.

***Definition* (1.11)[6]:**
An action $\varphi: M \times X \to X$ is called continuous action if φ is continuous where M is topological monoid and X is topological space.

***Definition* (1.12) [6]:**
Let M be topological monoid and φ a continuous action of M on some topological space X. Then a point $x \in \mathrm{X}$ is said to be periodic with respect to φ if $M_x(\varphi) = \{m \in M$ such that $\varphi(m , x) = x\}$ is right syndetic in M and φ is said to be periodic if $M(\varphi) = \bigcap_{\forall x \in X} M_x(\varphi)$ is right syndetic in M.

***Remark* (1.13)[6]:**
Let M be topological monoid and φ a continuous action of M on some topological space X. Then all point $x \in X$ is periodic with respect to φ if φ is periodic.

***Examples* (1.14):**
1) Take the trivial action of any finite topological monoid M on any topological space X, since $M_x(\varphi) = M$ for all $x \in X$ then $\bigcap_{\forall x \in X} M_x(\varphi) = M$ and since M is compact, and $M^{-1}M = M\,M = M$ then $\bigcap_{\forall x \in X} M_x(\varphi)$ is right syndetic, so φ is periodic.
2) Take the monoid $M = \{1 , -1\}$ with usually product operation then M is topological monoid with discrete topology and take usually topology of set of real numbers , define the action $\varphi: M \times \mathbb{R} \to \mathbb{R}$, $\varphi(n,x) = \begin{cases} x & , \quad n = 1 \\ -x & , \quad n = -1 \end{cases}$
, then $x \in \mathbb{R}\backslash\{0\}$ is periodic with respect to φ since $M_x(\varphi) = \{1\}$ for all $x \in \mathbb{R}\backslash\{0\}$ and $\{1\}$ is right syndetic in M, 0 is periodic with respect to φ since $M_x(\varphi) = M$. So φ is periodic since $\cap_{x\in\mathbb{R}} M_x(\varphi) = \{1\}$.

***Definition* (1.15) [6]:**
Let M be topological monoid and φ a continuous action of M on some topological space X. We say that φ is topological transitive if for any two non-empty open set U, V, there exist $m \in M$ such that $\varphi(m , U) \cap V \neq \phi$.

***Definition* (1.16) [6]:**
Let M be topological monoid and φ a continuous action of M on some topological space X. We say that φ is algebraically transitive if for any point $x, y \in X$ there exists $m \in M$ such that $\varphi(m , x) = y$.

***Remark* (1.17) [6]:**
Let M be topological monoid and φ a continuous action of M on some topological space X. If φ is algebraically transitive then φ is topological transitive, but the convers is not necessary true as the following example.

***Example* (1.18):** Take the monoid $(\mathbb{R}, .)$ with discrete topology and take the continuous action $\varphi: \mathbb{R} \to \mathbb{R}$, $\varphi(n,x) = x^n$ for all $n, x \in \mathbb{R}$. Then is topological transitive but it is not algebraically transitive since, $\varphi(1, A) = A$, for all A is open set in $\mathbb{R}$ but $\varphi(n, 0) = 0 \neq 1$ for all $n \in \mathbb{R}$and $\varphi(m, 1) = 1 \neq 0$ for all $m \in \mathbb{R}$.

***Definition* (1.19) [6]:**
Let M be topological monoid and φ a continuous action of M on some topological space X. We say that φ is minimal if $\varphi(M , x)$ is dense in X for all $x \in X$.

2. THE MAIN RESULTS

In this section, we define the notions of a chaotic action and a less chaotic action and some properties of action. For our discussion, we shall link these notions with other notions which mentioned in preliminaries.

***Definition* (2.1):**
Let M be topological monoid and φ an action of M on some topological space X. Then φ is said to be active if for all open set U in X there exists non-empty open set V in M such that $\varphi(V, U)$ = { $\varphi(n, x)$: $n \in$ V and $x \in$ U } is open set in X.

***Example* (2.2):** Let $M = \{ m_1, m_2, m_3 \}$ and $X = \{x_1, x_2, x_3\}$ defined the binary operation $*$ in the following table

$*$	m_1	m_2	m_3
m_1	m_1	m_2	m_3
m_2	m_2	m_2	m_2
m_3	m_3	m_2	m_3

The action φ by $*$ is active with $T_X = \{\phi, X, \{x_2\}\}$ and T_M is indiscrete topology but it is not active with $T_X = \{\phi, X, \{x_3\}\}$ and $T_M = \{\phi, M, \{m_2\}\}$ or T_M is indiscrete topology.

***Proposition*(2.3):**
Let M be topological monoid and φ an action of M on some topological space X. If $\{1\}$ is open set in M then φ is active .
Proof: Clear since $\varphi(\{1\}, U) = U$ for all open set U in X

***Definition* (2.4) [6]:**
Let M be topological monoid and X be a topological space. A continuous action φ of M on X is called chaotic if
(1) φ is topological transitive.
(2) The set of points being periodic with respect to φ is dense in X.
(3) φ is not minimal.

***Example*(2.5):** Let $M = \{ m_1, m_2, m_3, m_4\}$ and $X = \{x_1, x_2, x_3, x_4\}$ defined the binary operation $*$ in the following table

$*$	m_1	m_2	m_3	m_4
m_1	m_1	m_2	m_3	m_4
m_2	m_2	m_2	m_3	m_4
m_3	m_3	m_2	m_3	m_4
m_4	m_4	m_2	m_3	m_4

The action φ by $*$ is chaotic with $T_X = \{\phi\} \cup \{A \subseteq X$ such that $\{x_2\} \subseteq A \}$ and T_M is discrete topology. Notice that φ is continuous and
(1) $\varphi(m_1, U) = U$ for all $U \in T_X \setminus\{\phi\}$ then $\varphi(m_1, U) \cap V \neq \phi$ for all $U, V \in T_X \setminus\{\phi\}$, so φ is topological transitive.
(2) $M_x(\varphi) = M$, $x \in X\setminus\{x_1\}$ and $M_{x_1}(\varphi) = \{m_1\}$
then the set of points being periodic with respect to $\varphi = X\setminus\{x_1\}$ is dense.
(3) Since $\varphi(M, x_3) = \{x_3\}$ *and* $\{x_3\} \cap \{x_2\} = \phi$ then $\varphi(M, x_3)$ is not dense. Hence φ is not minimal. So φ is chaotic.
If $T_X = \{\phi\} \cup \{A \subseteq X$ such that $\{x_1\} \subseteq A \}$ then φ is not chaotic [since the set of points being periodic with respect to φ is not dense]

Definition **(2.6):**
Let M be topological monoid and X a topological space. A continuous action φ of M on X is called less chaotic if

1) φ is topological transitive.
2) The intersection between the set of points being periodic with respect to φ and any open set A ≠ X is empty set.
3) There exists a unique point $x \in X$ such that $\varphi(M, x)$ is dense.

Example **(2.7):** Let $M = \{1, 2, 3\}$ and $X = \{x_1, x_2, x_3\}$ defined the binary operation $*$ in the following table

$*$	1	2	3
1	1	2	3
2	2	2	2
3	3	2	3

The action φ by $*$ is less chaotic with $T_X = \{\phi, X, \{x_1\}\}$ and $T_M = \{\phi, M, \{1\}\}$ since φ is continuous and,

(1) $\varphi(1, U) = U$ for all $U \in T_X \setminus \{\phi\}$ then $\varphi(1, U) \cap V \neq \phi$ for all $U, V \in T_X \setminus \{\phi\}$ so φ is topological transitive.

(2) Since $M_{x_1}(\varphi) = \{1\}$, $M_{x_2}(\varphi) = M$ and $M_{x_3}(\varphi) = \{1,3\}$ then the set of points being periodic with respect to $\varphi = \{x_2\}$ but $\{x_2\} \cap \{x_1\} = \phi$ so the intersection between the set of points being periodic with respect to φ and any open set $A \neq X$ is empty set.

(3) Since $\varphi(M, x_1) = X$ is dense, $\varphi(M, x_2) = \{x_2\}$ is not dense and $\varphi(M, x_3) = \{x_2, x_3\}$ is not dense. So φ is less chaotic.

Definition (2.8) :
Let $(\mathbb{Z}_p \setminus \{0\}, \cdot_p)$ be the group of integers modulo p, p is prime number, and let $M = \{m_1, m_2, m_3, \ldots, m_{p-1}\}$, define the binary operation $*$ of M by $\cdot_p$
$m_1 * m_k = m_k * m_1 = m_k$ and $m_i * m_j = m_{i \cdot_p j}$ for all $m_k, m_i, m_j \in M \setminus \{m_1\}$. Then $(M, *)$ is monoid and M is called monoid by $\mathbb{Z}_p$.

Example (2.9): Let $M = \{m_1, m_2, m_3, m_4\}$, where

$*$	m_1	m_2	m_3	m_4
m_1	m_1	m_2	m_3	m_4
m_2	m_2	m_4	m_1	m_3
m_3	m_3	m_1	m_4	m_2
m_4	m_4	m_3	m_2	m_1

clear $(M, *)$ is monoid by $\mathbb{Z}_5$.

Proposition **(2.10):**
Let $(M, *)$ be monoid by $\mathbb{Z}_p$ and let $X = \{x_1, x_2, x_3, \ldots, x_{p-1}\}$ then for any topology on X and any topology on M if the action φ by $*$ is continuous then it is minimal.
Proof: since $M * m_i = M$ for all $m_i \in M$ then $\varphi(M, x_i) = X$ for all $x_i \in X$ so $\varphi(M, x_i)$ is dense for all $x_i \in X$, this ends the proof.

***Corollary* (2.11):**
Let $(M,*)$ be monoid by $\mathbb{Z}_p$and let $X = \{ x_1 , x_2 , x_3 ,. . . , x_{p-1}\}$
then for any topology on X and any topology on M if the action φ by $*$ is continuous, then it is not chaotic and it is not less chaotic.
Proof: The poof is clear since φ is minimal by Proposition (2.10)**.**

***Theorem* (2.12):**
Let $(M,*)$ be monoid by $\mathbb{Z}_p$and let $X = \{ x_1 , x_2 , x_3 ,. . . , x_{p-1}\}$. Then for any topology on X and any topology on M if the action φ by $*$ is continuous then it is active.
Proof : since $\varphi(M , x_i) = X$ for all $x_i \in X$ then $\varphi(M , A) = X$ for all A is open set in X so φ is active.

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