About affine- connection spaces With $F^{P+1}=0$

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ABSTRACT

In this paper considered generalization pF- planar mappings of affine-connected spaces $A_n, \overline{A}_n$ with a torsion – free affine connection of affinor structure F, with $F^{P+1}=0$.Finding general aspects of pF- planar mappings , these results generalize obtained for geodesic,holomorphically projective and special $F\&2F$-planar mappings of Riemannian and Kählerian spaces by N.S.Sinyukov , J. Mikes , I.N.Kurbatova ,Raad J. K. Al lamy.

1. INTRODUCTION

This paper is concerned with certain questions of $F\&2F$-planar mappings, quasigeodesic and holomorphically projective. The analysis is carried out in tensor form, locally is a class of sufficiently smooth real functions.

Let us consider the space of a torsion – free affine –connection A_n with a coordinate system $x^1, x^2, \ldots, x^n$, where is defined an affinor structure $F_i^h(x)$

Definition 1 (Raad J. K. , I. N. Kurbatova [5]) The curve L, which is given by the equations

$$x^h = x^h(t)\,, \quad \lambda^h \overset{def}{=} dx^h/dt \quad (\neq o) \qquad (1)$$

$(h=1,2,\ldots,n)$, where t is parameter , is called pF- planar mapping if its tangent vector λ^h under the parallel transformation along L remains in the plane formed by the tangent vector λ^h and the vectors

$$\lambda^\alpha \overset{1}{F}{}_\alpha^h\,,\ \lambda^\alpha \overset{2}{F}{}_\alpha^h\,,\ \ldots\,,\ \lambda^\alpha \overset{P}{F}{}_\alpha^h\,,$$

where

$$\overset{s+1}{F}{}_i^h \overset{def}{=} F_\alpha^h \overset{s}{F}{}_i^\alpha\,,\ \overset{0}{F}{}_\alpha^h \overset{def}{=} \delta_i^h\,, \qquad \delta_i^h \text{ is Kronecker delta .}$$

In correspondence with this definition , L is pF-planar if and only if the following conditions hold

$$\lambda^h{}_{,\alpha}\lambda^\alpha = \sum_{\sigma=0}^{P} \rho_\sigma(t) \overset{\sigma}{F}{}^h_\alpha \lambda^\alpha \qquad (2)$$

Where the comma denotes the covariant derivative with respect to the affine connection space A_n, $\rho_\sigma,...,\rho_P$ are some (arbitrary) functions of the parameter t .

We note that the class of pF -planar curves of the space A_n is very broad.

Obviously through any point in each direction there passes a set of pF -planar curves depending on p arbitrary functions $\rho_1,...,\rho_P$. The class of pF -planar curves contains : geodesic line , F-planar curve , quasigeodesic curves , and almost geodesic of type of π_2 curves, analytic planar curves ([6 , 7 , 8 , 9 , 10]) .

In the following we are going to consider the special class of pF -planar curves which distinguished :

An pF -planar curves is called $\left(\overset{1}{\varphi}_i, \overset{2}{\varphi}_i, ..., \overset{P}{\varphi}_i, F\right)$ - *planar* if the following condition hold for its tangent vector :

$$\lambda^h{}_{,\alpha}\lambda^\alpha = \rho(t)\lambda^h + 2\sum_{\sigma=1}^{P} \overset{\sigma}{\varphi}_\beta \lambda^\beta \overset{\sigma}{F}{}^h_\alpha \lambda^\alpha \qquad (3)$$

where $\overset{\sigma}{\varphi}_i(x)$ are some vector fields which are defined above .

For given p vector fields $\overset{\sigma}{\varphi}_i(x)$ and an affinor $F_i^h(x)$ through any point of A_n in each direction there passes a unique pF -planar curve.

2 . pF - planar mappings between two spaces with affine connection

We suppose two spaces $A_n, \overline{A}_n$ with the affine connections Γ and $\overline{\Gamma}$ respectively. The affinor structure F_i^h and the vector fields $\left(\overset{1}{\varphi}_i(x), \overset{2}{\varphi}_i(x), ..., \overset{P}{\varphi}_i(x)\right)$ are define on $\overline{A}_n$.

Definition 2 A mapping from A_n onto $\overline{A}_n$ is called pF -planar if any geodesic of A_n is mapped on a $\left(\overset{1}{\varphi}_i, \overset{2}{\varphi}_i, ..., \overset{P}{\varphi}_i, F\right)$ - plane curve $\overline{A}_n$.

Theorem 1 A pF -planar mapping from A_n onto $\overline{A}_n$ is characterized by the following conditions

$$\overline{\Gamma}_{ij}^{h}(x) = \Gamma_{ij}^{h}(x) + \sum_{\sigma=0}^{p} \overset{\sigma}{\varphi}_{(i} \overset{\sigma}{F}{}_{j)}^{h}, \qquad (4)$$

where $\overset{0}{\varphi}_i(x)$ is some vector , $\overline{\Gamma}_{ij}^{h}, \Gamma_{ij}^{h}$ are the objects of affine connections of $\overline{A}_n$ and A_n respectively , $x = (x^1, x^2, ..., x^n)$ is general coordinates system with respect to a pF -planar mapping and (ij) denoted a symmetrization of indices .

Proof : If a geodesic in a A_n , which satisfies the equations (1) and

$\lambda^h{}_{,\alpha}\lambda^\alpha = \rho(t)\lambda^h$, is mapped on a $\left(\overset{1}{\varphi}_i, \overset{2}{\varphi}_i, ..., \overset{P}{\varphi}_i, F\right)$ -planar curve $\overline{A}_n$, therefore hold

$$(\overline{\Gamma}_{\alpha\beta}^{h}(x) - \Gamma_{\alpha\beta}^{h}(x) - 2\sum_{\sigma=1}^{p} \overset{\sigma}{\varphi}_{\alpha} \overset{\sigma}{F}{}_{\beta}^{h})\lambda^\alpha \lambda^\beta = \left(\overset{0}{\sigma}(t) - \rho(t)\right)\lambda^h \qquad (5)$$

Because (5) hold for any tangent vector λ^h therefore the validity (4) implies from (5) . The sufficiency of formulas (4) is obvious .

Remark : For this pF -planar mappings hold that :

a) $\left(\overset{1}{\varphi}_i, \overset{2}{\varphi}_i, ..., \overset{P}{\varphi}_i, F\right)$ -planar curve A_n is mapped on $\left(\overset{1}{\varphi}_i, \overset{2}{\varphi}_i, ..., \overset{P}{\varphi}_i, F\right)$ -planar curve $\overline{A}_n$.

b) pF -planar curves A_n is mapped on pF -planar curves $\overline{A}_n$.

Generally from the condition , that pF -planar curves A_n is mapped on pF -planar curves $\overline{A}_n$, the equations (4) fail to satisfy , thus this mapping is not pF -planar in our conception .

3 . pF -planar mappings with structure $F^{P+1} = 0$

We have following

Theorem 2 Suppose $f : A_n \to \overline{A}_n$ is a diffeomorphism and F is affine structure on A_n for which is satisfied

$$F^{P+1} = 0 , \qquad Rank\left\|F^p\right\| > 1 \qquad (6)$$

If any pF -planar curves A_n is mapped on a pF -planar curves $\overline{A}_n$, then f is a pF -planar mapping .

At first we prove

Lemma 1 If for any vector λ^h holds

$$Q^h_{\alpha\beta}\,\lambda^\alpha\lambda^\beta = \rho\, B^h_\alpha\,\lambda^\alpha \quad , \qquad (7)$$

where $Q^h_{ij}\left(\equiv Q^h_{ji}\right), B^h_i \ \left(Rank\left\|B^h_i\right\| > 1\right)$ do not depend on λ^h , therefore exist ρ_i so that

$$Q^h_{ij} = \rho_i\, B^h_j + \rho_j B^h_i \quad \text{and} \quad \rho = 2\rho_\alpha\lambda^\alpha \ , \qquad (8)$$

Proof. Let condition (7) hold for any vector . We multiply (7) with $B^i_\alpha\lambda^\alpha$ and then we make the alternation through indices h and i . After the adoption we obtain

$$(Q^h_{\alpha\beta}B^i_\gamma - Q^i_{\alpha\beta}B^h_\gamma)\,\lambda^\alpha\lambda^\beta\lambda^\gamma = 0 \quad . \qquad (9)$$

The term in brackets dose not depend on λ^h and (9) holds for any vector λ^h , therefore

$$Q^h_{(\alpha\beta}B^i_{\gamma)} - Q^i_{(\alpha\beta}B^h_{\gamma)} = 0 \quad , \qquad (10)$$

have to hold , where the round brackets denote symmetrization of indices. Because $B^h_i \neq 0$, it means , that there exist vectors ε^i and υ_h such that $B^h_i\varepsilon^i\upsilon_h = 1$.

From the beginning we contract the formula (10) with $\varepsilon^\alpha\varepsilon^\beta\varepsilon^\gamma\upsilon_i$. we make sure of

$$Q^h_{\alpha\beta}\varepsilon^\alpha\lambda^\beta = aB^h \ ,$$

where a is a function and $B^h \overset{def}{=} B^h_\alpha\varepsilon^\alpha$.

After all we contract (10) with $\varepsilon^\beta\varepsilon^\gamma\upsilon_i$ and once more make sure of

$$Q^h_{\alpha\beta}\ \lambda^\beta = bB^h_\alpha + B^h a_\alpha \quad ,$$

where b is a function and a_α is a vector .

Lastly we contract (10) with $\varepsilon^\gamma\upsilon_i$ and get

$$Q^h_{\alpha\beta} = \rho_\alpha\, B^h_\beta + \rho_\beta B^h_\alpha + B^h a_{\alpha\beta} \ , \qquad (11)$$

Where $a_{\alpha\beta}$ is a tensor .

Now we substitute (11) to (10) and obtain

$$B^h a_{(\alpha\beta}B^i_{\gamma)} - B^i a_{(\alpha\beta}B^h_{\gamma)} = 0 \quad .$$

It is easy to see , from the previous formula , if $a_{\alpha\beta} \neq 0$ it implies $Rank\left\|B^h_i\right\| \leq 1$,but it is a contradictory proposition .

Then $a_{\alpha\beta} = 0$ and the formula (11) has the first expression (8) .

If we institute this formula to (7) , we will see , that $\rho = 2\rho_\alpha\lambda^\alpha$.

Now we start the proof of the Theorem 2 .

It is obvious , that geodesics are the special case of pF -planar curves .

Let a geodesic of A_n , which satisfies the equation (1) and $\lambda^h{}_{,\alpha}\lambda^\alpha = \rho(t)\lambda^h$, is mapped on pF -planar curve $\overline{A}_n$, which satisfies the equation (1) and

$$\lambda^h_{/\alpha}\lambda^\alpha = \overline{\rho}(t)\lambda^h + \sum_{\sigma=1}^{P} \rho_\sigma(t)\overset{\sigma}{F}{}^h_\alpha\lambda^\alpha \ ,$$

Where "$/$" is the covariant derivative on $\overline{A}_n$.

Because $\lambda^h_{/\alpha}\lambda^\alpha - \lambda^h{}_{,\alpha}\lambda^\alpha = P^h_{\alpha\beta}\lambda^\alpha\beta\lambda^\beta$, where $P^h_{ij}(x) \overset{def}{=} \overline{\Gamma}^h_{ij}(x) - \Gamma^h_{ij}(x)$ is the deformation tensor of the mapping $\boldsymbol{f : A_n \to \overline{A}_n}$, then we have

$$P^h_{\alpha\beta}(x)\lambda^\alpha\beta\lambda^\beta = \sum_{\sigma=0}^{p}\rho_\sigma(t)\overset{\sigma}{F}{}^h_\alpha\lambda^\alpha \qquad , \qquad (12)$$

Where $\underset{0}{\rho} = \overline{\rho} - \rho$.

We contract (12) with $\overset{p}{F}{}^k_h$:

$$\overset{p}{F}{}^k_h \, P^h_{\alpha\beta}(x)\lambda^\alpha\lambda^\beta = \underset{0}{\rho}\,\overset{p}{F}{}^h_\alpha\lambda^\alpha \ . \qquad (13)$$

Based on Lemma 1 it follows , that there exists a vector field $\overset{0}{\varphi}_i(x)$, for which

$$\underset{0}{\rho} = 2\overset{0}{\varphi}_\alpha\,\lambda^\alpha \ . \qquad (14)$$

After that we contract the term (12) with $\overset{p-1}{F}{}^k_h$ and accept (14) . Thus

$$(\overset{p}{F}{}^k_h\,P^h_{\alpha\beta} - 2\overset{0}{\varphi}_\alpha\,\delta^h_\beta)\lambda^\alpha\lambda^\beta = \underset{1}{\rho}\,\overset{p}{F}{}^h_\alpha\lambda^\alpha \ .$$

Based on Lemma 1 we obtain by analogy , that

$$\underset{1}{\rho} = 2\overset{1}{\varphi}_\alpha\,\lambda^\alpha$$

Below we make the contraction of the formula (12) with

$$\overset{p-2}{F}{}^k_h\ , \quad \overset{p-3}{F}{}^k_h\ , \ \ldots\ , \ F^k_h \ ,$$

gradually and we make sure of

$$\underset{2}{\rho} = 2\overset{2}{\varphi}_\alpha\,\lambda^\alpha \ , \ \ldots\ , \ \underset{p-1}{\rho} = 2\overset{p-1}{\varphi}_\alpha\,\lambda^\alpha$$

The formula (12) will be in the following form

$$(\,P^h_{\alpha\beta}(x) - 2\sum_{\sigma=0}^{p-1}\overset{\sigma}{\varphi}_\alpha\,\overset{\sigma}{F}{}^h_\beta)\,\lambda^\alpha\,\lambda^\beta = \underset{p}{\rho}(t)\overset{p}{F}{}^h_\alpha\,\lambda^\alpha \ .$$

Hence it follows , that $\underset{p}{\rho} = 2\overset{p}{\varphi}_\alpha\,\lambda^\alpha$ and formula (4) .

Clearly the mapping $f : A_n \to \overline{A}_n$ is pF -planar .

The Theorem was proved completely .

4. pF -planar mappings with structure $F^{P+1}=0$ and with preserve covariantly derivation of structure .

Finally we will ponder over the problem pF -planar mapping , in which the conditions (6) hold and

$$F^h_{i/j} = F^h_{i,j} \ , \qquad (15)$$

i.e. the covariant derivative of the structure F is preserved .
These conditions are hold , for example , if the structure is covariantly constant as in A_n so in $\overline{A}_n$.

In respect of the definition of the covariant derivative

$$F^h_{i,j} \overset{def}{=} \partial_i F^h_i + F^\alpha_i \Gamma^h_{\alpha j} - F^h_\alpha \Gamma^\alpha_{ij} \quad \text{and}$$

$$F^h_{i/j} \overset{def}{=} \partial_i F^h_i + F^\alpha_i \overline{\Gamma}^h_{\alpha j} - F^h_\alpha \overline{\Gamma}^\alpha_{ij}$$

And based on formulas (4) we have

$$F^h_{i/j} = F^h_{i,j} + \sum_{\sigma=0}^{p} (\overset{\sigma}{\varphi}_\alpha F^\alpha_i \overset{\sigma}{F}{}^h_j - \overset{\sigma}{\varphi}_i \overset{\sigma+1}{F}{}^h_j \quad . \qquad (16)$$

It means , that on conditions (15) from (16) follows :

$$\sum_{\sigma=0}^{p} (\overset{\sigma}{\varphi}_\alpha F^\alpha_i \overset{\sigma}{F}{}^h_j - \overset{\sigma}{\varphi}_i \overset{\sigma+1}{F}{}^h_j) = 0 \ . \qquad (17)$$

An elementary analysis of the formulas (17) in the conditions

$$F^{P+1}=0$$

and $F^P \neq 0$ follows :

$$\overset{\sigma}{\varphi}_i = \overset{\sigma+1}{\varphi}_\alpha F^\alpha_i \quad , \quad \sigma = 0,1,\ldots,p-1 \ .$$

If we denote $\varphi_i \overset{def}{=} \overset{p}{\varphi}_i$, then clearly

$$\overset{\sigma}{\varphi}_i = \varphi_\alpha \overset{p-\sigma}{F}{}^\alpha_i \quad , \quad \sigma = 0,1,\ldots,p \ . \qquad (18)$$

On the other hand from (18) follows (17) . We can formulate :

Theorem 3 Suppose $f : A_n \to \overline{A}_n$ is a pF -planar mapping and F is an affine structure on A_n for which is satisfied (6) . The covariant derivation F is preserved (i . e . hold (15)) , if and only if for vector $\overset{\sigma}{\varphi}_i$, in formulas (4) hold

$$\overset{\sigma}{\varphi}_i = \varphi_\alpha \overset{p-\sigma}{F}{}^{\alpha}_{i} \quad , \quad \sigma = 0,1,\ldots,p \quad .$$

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