Exp-Function Method for Solving the Schrödinger and Improved Eckhaus Equations

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Abstract

In this research, the exp-function method has been used to find travelling wave solutions of Schrödinger and improved Eckhaus Equations. The solution procedure of this method, by the help of symbolic computation of Maple or Matlab, is of utter simplicity. The exp-function method is a powerful and straightforward mathematical tool to solve the nonlinear equations in mathematical physics.

Keywords: Exp-function method, Schrödinger equation , improved Eckhaus equation.

طريقة الدالة الأسية لحل معادلة شوردنجر ومعادلة Eckhaus المحسنة

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الخلاصة

استعملنا في هذا البحث طريقة الدالة الأسية لإيجاد حلول الموجة المتنقلة لمعادلتي شوردنجر و Eckhuas المحسنة. ويجرى الحل بهذه الطريقة بمساعدة الحساب الرمزي (المابل او الماتلاب) للسهولة. وتعد هذه الطريقة أداة قوية وبسيطة لحل المعادلات اللاخطية في الفيزياء الرياضياتية.

1. Introduction

Most scientific problems and phenomena in different fields of sciences and engineering occur nonlinearly. Except in a limited number of these problems are linear. This method has been effectively and accurately shown to solve a large class of nonlinear problems. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed . A variety of powerful methods ,such that, tanh-sech method[1-3] , extended tanh method [4-5], the modified tanh method [6], hyperbolic function method [7] , sine-cosine method [8], Jacobi elliptic function expansion method [9], and the first integer method [10]. Very recently, He and Wu [11] proposed a straightforward and concise method, called Exp-function method, to obtain generalized solitary solutions and periodic solutions, applications of the method can be found in [12- 18] for nonlinear evolution equations arising in mathematical physics. The solution procedure this method, with the aid of Maple, is of utter simplicity and this method can easily extended to other kinds of nonlinear evolution equations. In this research, we use the exp-function method to obtain new solitary wave solutions for the Schrödinger[19] and improved Eckhaus Equations[6,20].

2. Exp-Function Method

We consider a general nonlinear PDE in the form

$$P(u\,,u_t\,,u_x\,,u_{xx}\,,\ldots)=0. \tag{1}$$

Using a transformation

$$u(x\,,t)=U(\eta)\ \ ,\ \ \eta=x+\lambda t\,, \tag{2}$$

where $\alpha\,,\beta$ are constants, we can rewrite (1) in the following nonlinear ODE:

$$Q(U\,,U'\,,U''\,,U''',\ldots)=0, \tag{3}$$

where primes denote the ordinary derivative with respect to η.

we assume that the wave solution of equation (3) can be expressed in the form [11-18]:

$$U(\eta)=\frac{\sum_{n=-c}^{d}a_n\exp\,(n\eta)}{\sum_{m=-p}^{q}b_m\exp\,(m\eta)}=\frac{a_{-c}\exp\,(-c\eta)+\cdots+a_d\exp\,(d\eta)}{b_{-p}\exp\,(-p\eta)+\cdots+b_q\exp\,(q\eta)} \tag{4}$$

Where $c, d, p, and\ q$ are positive integers that could be determine subsequently, $a_n\ and\ b_m$ are unknown constants, equation (4) can be rewrite in the form:

$$U(\eta) = \frac{a_c \exp(c\eta) + \cdots + a_{-d} \exp(-d\eta)}{b_p \exp(p\eta) + \cdots + b_{-q} \exp(-q\eta)} \tag{5}$$

In order to determine the values of c and p, we balance the highest order linear term with the highest order nonlinear term in equation (3). Similarly, to determine the values of d and q we balance the lowest order linear term with the lowest order nonlinear term in equation (3). Substituted the values of c, d, p and q into equation (5), and then substituted equation (5) into equation (3) and simplifying , we obtain

$$\sum_k C_k \exp(\pm k\eta) = 0 \ , \qquad k = 0,1,2,3, \ldots \tag{6}$$

Then setting each coefficient $C_k = 0$, yields a set of algebraic equations for a_c's and b_p's. The unknown a_c's and b_p's can be obtained by solving the algebraic equations. Substituting these values into equation (5), we obtain traveling wave solutions of the equation (1).

3.Application of Nonlinear Equations

3.1 Schrödinger Equation

consider the following nonlinear Schrödinger equation[19]

$$iu_t + u_{xx} - 2u\,|u|^2 = 0 \tag{7}$$

We introduce the wave transformation

$$u(x,t) = e^{i\theta} U(\eta)\,, \qquad \theta = \alpha x + \beta t \ , \ \eta = x + \lambda t, \tag{8}$$

where α , β and λ are real constants and $U(\eta)$ is real function. Substituting Eq.(8) into Eq.(7) we find the relation $\lambda = -2\alpha$, then Eq.(7) is the following nonlinear ordinary differential equation

$$-(\beta + \alpha^2)U + U'' - 2U^3 = 0, \tag{9}$$

where primes denote the derivative with respect to η. The solution of Eq.(9) can be expressed in the form of Eq.(5). By balancing the term U'' with the term U^3 in Eq.(9), we obtain

$$U^3 = \frac{c_1 \, exp(3c + p)\eta + \cdots}{c_2 \, exp(4p\eta) + \cdots} \tag{10}$$

and

$$U'' = \frac{c_3 \, exp(c + 3p)\eta + \cdots}{c_4 \, exp(4p\eta) + \cdots} \tag{11}$$

Where $c_1, c_2, c_3 \; and \; c_4$ are coefficients for simplicity. By balancing the highest order of the exp-function in Eqs.(10) and (11), we obtain

$$3c + p = c + 3p$$

which in turn gives $p = c$.

To determine the value of d and q, we balance the linear term of lowest order of Eq.(9) with the lowest order non-linear term

$$U^3 = \frac{... + d_1 \, exp(-3d - q)\eta}{... + d_2 \, exp(-4q\eta)} \tag{12}$$

and

$$U'' = \frac{... + d_3 \, exp(-d - 3q)\eta}{... + d_4 \, exp(-4q\eta)} \tag{13}$$

Where $d_1, d_2, d_3 \; and \; d_4$ are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (12) and (13), we have

$$-3d - q = -d - 3q$$

which in turn gives $d = q$.

We can freely choose the values of c and d. But the final solutions of the Eq.(7) do not depend upon the choice of c and d.

We set $p = c = 1$ and $d = q = 1$. For this case , the trial solution in Eq.(5) reduces to

$$U(\eta)=\frac{a_1 e^{\eta}+a_0+a_{-1}e^{-\eta}}{b_1 e^{\eta}+b_0+b_{-1}e^{-\eta}} \tag{14}$$

By substituting Eq.(14) into Eq.(9) and equating the coefficients of $\exp(\pm n\eta)$, $n=0,1,2,3,\ldots$, with the aid of Maple 13, we have

$$\frac{1}{A}\left(C_3\, e^{3\eta}+C_2\, e^{2\eta}+C_1\, e^{\eta}+C_0\,+C_{-1}\, e^{-1\eta}+C_{-2}\, e^{-2\eta}+C_{-3}\, e^{-3\eta}\right)=0 \tag{15}$$

where

$$A=(b_1 e^{\eta}+b_0+b_{-1}e^{-\eta})^3,$$

$$C_3=-\alpha^2 b_1{}^2 a_1-\beta a_1 b_1{}^2-2a_1{}^3,$$

$$C_2=-\alpha^2 a_0 b_1{}^2-6a_1{}^2 a_0-a_1 b_1 b_0-\beta a_0 b_1{}^2+a_0 b_1{}^2-2\beta a_1 b_1 b_0-2\alpha^2 a_1 b_1 b_0,$$

$$C_1=-2\alpha^2 a_0 b_1 b_0-2\alpha^2 a_1 b_1 b_{-1}-2\beta a_1 b_1 b_{-1}-2\beta a_0 b_1 b_0-\alpha^2 a_1 b_0{}^2-\beta a_1 b_0{}^2 -\alpha^2 a_{-1} b_1{}^2-a_0 b_1 b_0+a_1 b_0{}^2-6a_1{}^2 a_{-1}-4a_1 b_1 b_{-1}-6b_{-1}a_0{}^2 -\beta a_{-1} b_1{}^2+4a_{-1} b_1{}^2,$$

$$C_0=3a_1 b_0 b_{-1}-2\beta a_1 b_0 b_{-1}-2\alpha^2 a_1 b_0 b_{-1}-\beta a_0 b_0{}^2-\alpha^2 a_0 b_0{}^2-12a_1 a_0 a_{-1} -2a_0{}^3+3a_{-1} b_1 b_0-2\alpha^2 a_0 b_1 b_{-1}-2\beta a_0 b_1 b_{-1}-2\alpha^2 a_{-1} b_1 b_0 -2\beta a_{-1} b_1 b_0-6a_0 b_1 b_{-1},$$

$$C_{-1}=4a_1 b_{-1}{}^2-4a_{-1} b_1 b_{-1}-6a_0{}^2 a_{-1}-\beta a_1 b_{-1}{}^2-\beta a_{-1} b_0{}^2-\alpha^2 a_1 b_{-1}{}^2 -\alpha^2 a_{-1} b_0{}^2-a_0 b_{-1} b_0-2\alpha^2 a_{-1} b_1 b_{-1}-2\,\beta a_{-1} b_1 b_{-1}-2\,\beta a_0 b_0 b_{-1} -2\alpha^2 a_0 b_0 b_{-1}+a_{-1} b_0{}^2-6a_1 a_{-1}{}^2-\alpha^2 a_0 b_{-1}{}^2,$$

$$C_{-2}=-\beta a_0 b_{-1}{}^2-\alpha^2 a_0 b_{-1}{}^2-a_{-1} b_0 b_{-1}-\beta a_{-1} b_0 b_{-1}-2\alpha^2 a_{-1} b_0 b_{-1}-6a_0 a_{-1}{}^2 +a_0 b_{-1}{}^2,$$

$$C_{-3}=-\beta a_{-1} b_{-1}{}^2-\alpha^2 a_{-1} b_{-1}{}^2-2a_{-1}{}^3,$$

Setting each coefficient of $\exp(\pm n\eta)$, $n=0,1,2,3$ to zero, we obtain

$$C_5=0, C_3=0, C_2=0, C_1=0, C_0=0, C_{-1}=0, C_{-2}=0, C_{-3}=0 \tag{16}$$

Solving the system of algebraic equations with the aid of Maple 13 , we obtain:

Case 1.

$$a_0 = a_0\,,\ a_1 = 0\,,\ a_{-1} = 0\,, \alpha = \alpha\,, \beta = -\alpha^2 + 1,\ b_0 = 0,\ b_{-1} = b_{-1},\ b_1 = -\frac{1}{4}\frac{{a_0}^2}{b_{-1}} \tag{17}$$

Case 2.

$$a_0 = 0\,,\ a_1 = 0,\ a_{-1} = a_{-1}, \alpha = \alpha\,, \beta = -\frac{\alpha^2\,{b_{-1}}^2 + 2\,{a_{-1}}^2}{{b_{-1}}^2},\ b_0 = 0,\ b_1 = 0, b_{-1} = b_{-1} \tag{18}$$

Case 3.

$$a_0 = \frac{1}{2}\ b_0,\ a_1 = 0\,,\ a_{-1} = -\frac{1}{2}\ b_{-1}\,, \alpha = \alpha\,, \beta = -\alpha^2 - \frac{1}{2}\,,\ b_0 = b_0,\ b_1 = 0, b_{-1} = b_{-1} \tag{19}$$

Case 4.

$$a_0 = a_0,\ a_1 = 0\,,\ a_{-1} = \frac{a_0\,b_{-1}}{b_0}, \alpha = \alpha\,, \beta = -\frac{\alpha^2\,{b_0}^2 + 2\,{a_0}^2}{{b_0}^2}\,,\ b_0 = b_0,\ b_1 = 0, b_{-1} = b_{-1} \tag{20}$$

Case 5.

$$a_0 = -\frac{1}{2}\ b_0,\ a_1 = 0\,,\ a_{-1} = \frac{1}{2}\ b_{-1}\,, \alpha = \alpha\,, \beta = -\alpha^2 - \frac{1}{2}\,,\ b_0 = b_0,\ b_1 = 0, b_{-1} = b_{-1} \tag{21}$$

Case 6.

$$a_0 = a_0\,,\ a_1 = \frac{1}{2}\ b_1,\ a_{-1} = \frac{1}{8}\frac{4\,{a_0}^2 - {b_0}^2}{b_1}, \alpha = \alpha\,, \beta = -\alpha^2 - \frac{1}{2},\ b_0 = b_0,\ b_1 = b_1, b_{-1} = -\frac{1}{4}\frac{4\,{a_0}^2 - {b_0}^2}{b_1} \tag{22}$$

Case 7.

$$a_0 = \frac{1}{2}\ b_0,\ a_1 = \frac{1}{2}\ b_1,\ a_{-1} = a_{-1}, \alpha = \alpha\,, \beta = -\alpha^2 - \frac{1}{2},\ b_0 = b_0,\ b_1 = b_1, b_{-1} = 2\,a_{-1} \tag{23}$$

Case 8.

$$a_0 = a_0\,,\ a_1 = a_1,\ a_{-1} = -\frac{1}{8}\frac{4\,a_0{}^2 - b_0{}^2}{b_1},\alpha = \alpha\,,\beta = -\,\alpha^2 - \frac{1}{2},\ b_0 = b_0,\ b_1 = b_1,\ \ b_{-1} = -\frac{1}{4}\frac{4\,a_0{}^2 - b_0{}^2}{b_1} \qquad (24)$$

Case 9.

$$\alpha = \alpha\,,\beta = -\,\alpha^2 - \frac{1}{2},\ a_{-1} = a_{-1},\ a_0 = -\frac{1}{2}b_0,\ a_1 = -\frac{1}{2}b_1,\ b_{-1} = -2\,a_{-1},\ b_1 = b_1, b_0 = b_0 \qquad (25)$$

Case 10.

$$\alpha = \alpha\,,\beta = -\,\alpha^2 - 2,\ a_{-1} = -\,b_{-1},\ a_0 = 0,\ a_1 = b_1,\ b_{-1} = b_{-1},\ b_1 = b_1, b_0 = 0 \qquad (26)$$

Case 11.

$$\alpha = \alpha\,,\beta = -\,\alpha^2 - 2,\ a_{-1} = b_{-1},\ a_0 = 0,\ a_1 = -\,b_1,\ b_{-1} = b_{-1},\ b_1 = b_1, b_0 = 0 \qquad (27)$$

Case12.

$$a_0 = \frac{a_1\,b_0}{b_1},\ a_1 = a_1\,,\ a_{-1} = \frac{a_1\,b_{-1}}{b_1},\alpha = \alpha\,,\beta = -\,\frac{\alpha^2\,b_1{}^2 + 2\,a_1{}^2}{b_1{}^2}\,,\ b_0 = b_0,\ b_1 = b_1, b_{-1} = b_{-1} \qquad (28)$$

Substituting Eq.(17-28) into Eq.(15) and Eq.(8) yields

$$u_1(x,t) = \frac{a_0}{-\frac{1}{4}\frac{a_0{}^2}{b_{-1}}\mathrm{e}^{\eta} + b_{-1}\,\mathrm{e}^{-\eta}}\,\mathrm{e}^{i(\alpha x + (-\,\alpha^2 + 1)t)} \qquad (29)$$

$$u_2(x,t) = \frac{a_{-1}\,\mathrm{e}^{-\eta}}{b_{-1}\,\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x - \left(\frac{\alpha^2\,b_{-1}{}^2 + 2\,a_{-1}{}^2}{b_{-1}{}^2}\right)t\right)} \qquad (30)$$

$$u_3(x,t) = \frac{\frac{1}{2}\,b_0 - \frac{1}{2}\,b_{-1}\mathrm{e}^{-\eta}}{b_0 + b_{-1}\,\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x + \left(-\,\alpha^2 - \frac{1}{2}\right)t\right)} \qquad (31)$$

$$u_4(x,t) = \frac{a_0 + \frac{a_0\,b_{-1}}{b_0}\mathrm{e}^{-\eta}}{b_0 + b_{-1}\,\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x - \left(\frac{\alpha^2\,b_0{}^2 + 2\,a_0{}^2}{b_0{}^2}\right)t\right)} \qquad (32)$$

$$u_5(x,t) = \frac{-\frac{1}{2}\ b_0 + \frac{1}{2}\ b_{-1}\mathrm{e}^{-\eta}}{b_0 + b_{-1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x + \left(-\alpha^2 - \frac{1}{2}\right)t\right)} \tag{33}$$

$$u_6(x,t) = \frac{\frac{1}{2}\ b_1\mathrm{e}^{\eta} + a_0 + \frac{1}{8}\frac{4a_0{}^2 - b_0{}^2}{b_1}\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_0 - \frac{1}{4}\frac{4a_0{}^2 - b_0{}^2}{b_1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x + \left(-\alpha^2 - \frac{1}{2}\right)t\right)} \tag{34}$$

$$u_7(x,t) = \frac{\frac{1}{2}\ b_1\mathrm{e}^{\eta} + \frac{1}{2}\ b_0 + a_{-1}\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_0 + 2a_{-1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x + \left(-\alpha^2 - \frac{1}{2}\right)t\right)} \tag{35}$$

$$u_8(x,t) = \frac{a_1\mathrm{e}^{\eta} + a_0 - \frac{1}{8}\frac{4a_0{}^2 - b_0{}^2}{b_1}\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_0 - \frac{1}{4}\frac{4a_0{}^2 - b_0{}^2}{b_1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x + \left(-\alpha^2 - \frac{1}{2}\right)t\right)} \tag{36}$$

$$u_9(x,t) = \frac{-\frac{1}{2}b_1\mathrm{e}^{\eta} - \frac{1}{2}b_0 + a_{-1}\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_0 - 2\,a_{-1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x + \left(-\alpha^2 - \frac{1}{2}\right)t\right)} \tag{37}$$

$$u_{10}(x,t) = \frac{b_1\mathrm{e}^{\eta}{}_0 - b_{-1}\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_{-1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i(\alpha x + (-\alpha^2 - 2)t)} \tag{38}$$

$$u_{11}(x,t) = \frac{-b_1\mathrm{e}^{\eta} - b_{-1}\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_{-1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i(\alpha x + (-\alpha^2 - 2)t)} \tag{39}$$

$$u_{12}(x,t) = \frac{a_1\mathrm{e}^{\eta} + \frac{a_1 b_0}{b_1}{}_0 + a_1\mathrm{e}^{-\eta}}{b_1\mathrm{e}^{\eta} + b_0 + b_{-1}\mathrm{e}^{-\eta}}\,\mathrm{e}^{i\left(\alpha x - \left(\frac{\alpha^2 b_1{}^2 + 2a_1{}^2}{b_1{}^2}\right)t\right)} \tag{40}$$

To compare some of our results with those obtained in [19], we set $\alpha = a_{-1} = b_{-1} = 1$ in Eq.(30), $\alpha = a_0 = b_{-1} = 1$ in Eq.(32) , $\alpha = b_1 = 1, b_{-1}=0$ in Eq.(38) and we set $\alpha = a_1 = b_1 = b_{-1} = 1$ in Eq.(40) becomes

$$u_2(x,t) = u_4(x,t) = u_{10}(x,t) = u_{12}(x,t) = \mathrm{e}^{i(x-3t)}.$$

3.2 Improved Eckhaus Equation

In this section, we apply the exp-function method to construct the traveling wave solutions of the improved Eckhaus equation.

We consider the improved Eckhaus equation[6,20]:

$$iu_t + u_{xx} + 2(|u|^2)_{xx}u + |u|^4 u = 0 \tag{41}$$

This equation is of nonlinear Schrödinger type. We introduce the wave transformation Eq.(8), then Eq.(41) is following nonlinear ordinary differential equation

$$-(\beta+\alpha^2)U+U''+4U''U^2+4(U')^2U+U^5=0, \tag{42}$$

where primes denote the derivative with respect to η. The solution of Eq.(42) can be expressed in the form of Eq.(5). By balancing the term $U''U^2$ with the term U^5 in Eq.(42), we obtain

$$U^5=\frac{c_1\,exp(p+5c)\eta+\cdots}{c_2\,exp(6p\eta)+\cdots} \tag{43}$$

and

$$U''U^2=\frac{c_3\,exp(3p+3c)\eta+\cdots}{c_4\,exp(6p\eta)+\cdots} \tag{44}$$

Where $c_1, c_2, c_3\ and\ c_4$ are coefficients for simplicity. By balancing the highest order of the exp-function in Eqs.(43) and (44), we obtain

$$p+5c=3p+3c$$

which in turn gives $p=c$.

To determine the value of d and q, we balance the linear term of lowest order of Eq.(42) with the lowest order non-linear term

$$U^5=\frac{\ldots+d_1\,exp(-5d-q)\eta}{\ldots+d_2\,exp(-6q\eta)} \tag{45}$$

and

$$U''U^2=\frac{\ldots+d_3\,exp(-3d-3q)\eta}{\ldots+d_4\,exp(-6q\eta)} \tag{46}$$

Where $d_1, d_2, d_3\ and\ d_4$ are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (45) and (46), we have

$$-5d-q=-3d-3q$$

Which in turn gives $d=q$.

We can freely choose the values of c and d. But the final solutions of the Eq.(41) do not depend upon the choice of c and d.

We set $p = c = 1$ and $d = q = 1$. For this case , the trial solution Eq.(5) reduces to

$$U(\eta) = \frac{a_1 e^{\eta} + a_0 + a_{-1} e^{-\eta}}{b_1 e^{\eta} + b_0 + b_{-1} e^{-\eta}} \quad (47)$$

Since, $b_1 \neq 0$, Eq.(47) can be simplified:

$$U(\eta) = \frac{a_1 e^{\eta} + a_0 + a_{-1} e^{-\eta}}{e^{\eta} + b_0 + b_{-1} e^{-\eta}} \quad (48)$$

By substituting Eq.(48) into Eq.(8) and equating the coefficients of $\exp(\pm n\eta)$,

$n = 0,1,2,3, \ldots$, with the aid of Maple 13, we have

$$\frac{1}{A}\,(C_5\, e^{5\eta} + C_4\, e^{4\eta} + C_3\, e^{3\eta} + C_2\, e^{2\eta} + C_1\, e^{\eta} + C_0 + C_{-1}\, e^{-\eta} + C_{-2}\, e^{-2\eta} + C_{-3}\, e^{-3\eta} + C_{-4}\, e^{-4\eta} + C_{-5}\, e^{-5\eta}) = 0 \quad (49)$$

Where

$A = (e^{\eta} + b_0 + b_{-1} e^{-\eta})^5$,

$$C_5 = -\alpha^2 a_1 - \beta a_1 + 1\,,$$

$$C_4 = -4{a_1}^3 b_0 + 4{a_1}^2 a_0 - a_1 b_0 - \alpha^2 a_0 - \beta a_0 - 4\beta b_0 a_1 - 4\alpha^2 b_0 a_1 + a_0 + 5b_0\,,$$

$$\begin{aligned} C_3 = {} & -4\beta b_{-1} a_1 - 4\beta b_0 a_0 - 4\alpha^2 b_0 a_0 - 6\alpha^2 {b_0}^2 a_1 - 4\alpha^2 b_{-1} a_1 - 6\beta {b_0}^2 a_1 - 20{a_1}^2 b_0 a_0 \\ & + a_0 b_0 - 4a_1 b_{-1} - 16\,{a_1}^3 b_{-1} - \alpha^2 a_{-1} - \beta a_{-1} + 8\,{a_1}^3 {b_0}^2 - a_1 {b_0}^2 \\ & + 12\,{a_0}^2 a_1 + 16\,{a_1}^2 a_{-1} + 4a_{-1} + 5b_{-1} + 10{b_0}^2\,, \end{aligned}$$

$$\begin{aligned} C_2 = {} & -{b_0}^2 a_0 - 4a_0 b_{-1} + 11\, b_0 b_0 + 20 b_0 b_{-1} + {b_0}^3 a_1 - 12\,\alpha^2 b_0 b_{-1} a_1 \\ & - 12\,\beta b_0 b_{-1} a_1 - 4\alpha^2 {b_0}^3 a_1 - 4\beta b_{-1} a_0 \\ & - 4\beta b_0\, a_{-1} - 4\alpha^2 b_0 a_{-1} - 6\alpha^2 {b_0}^2 a_0 - 4\alpha^2 b_{-1} a_0 - 4\beta {b_0}^3 a_1 - 6\beta {b_0}^2 a_0 \\ & + 28{a_1}^3 b_0 b_{-1} - 12\,{a_1}^2 b_0 a_{-1} - 20\, a_1 b_0\, {a_0}^2 - 72\, {a_1}^2 b_{-1} a_0 + 56\, a_{-1} a_0 a_1 \\ & + 12{a_1}^2\, {b_0}^2 a_0 - 7a_1 b_0 b_{-1} + 8{a_0}^3 + 10{b_0}^3\,, \end{aligned}$$

$$\begin{aligned} C_1 = {} & 30\, {b_0}^2 b_{-1} + 10\, {b_{-1}}^2 + 5\, {b_0}^4 - {b_0}^3 a_0 + 4a_{-1} b_{-1} + {b_0}^4\, a_1 + 11\, a_{-1}\, {b_0}^2 \\ & - 4\, a_1\, {b_{-1}}^2 - 4\, {a_0}^3\, b_0 + 44a_{-1}\, {a_0}^2 + 48{a_{-1}}^2 a_1 + 32\, {a_1}^3\, {b_{-1}}^2 \\ & - 12\alpha^2\, b_0\, b_{-1} a_0\,, \end{aligned}$$

$$
\begin{aligned}
C_0 = {} & -4\,\alpha^2\, b_0{}^3 a_{-1} - 4\beta\, b_0{}^3 a_{-1} - 12\beta\, b_0{}^2 b_{-1} a_0 - 12\alpha^2 b_0{}^2\, b_{-1} a_0 + 68\, a_{-1}{}^2 a_0 \\
& - 12\alpha^2\, b_0\, b_{-1}{}^2 a_1 - 12\alpha^2\, b_0\, b_{-1} a_{-1} - 4\alpha^2\, b_0{}^3\, b_{-1} a_1 - 12\beta\, b_0\, b_{-1}{}^2 a_1 \\
& - 12\beta\, b_0 b_{-1} a_{-1} - 4\beta\, b_0{}^3 b_{-1}\, a_1 - 6\alpha^2\, b_{-1}{}^2 a_0 - \alpha^2\, b_0{}^4 a_0 - 6\beta\, b_{-1}{}^2 a_0 \\
& - \beta\, b_0{}^4 a_0 + 20 a_1\, b_0 a_{-1}{}^2 + 8 a_1\, b_0{}^2\, a_{-1}\, a_0 + 68\, a_1{}^2\, b_{-1}{}^2\, a_0 \\
& + 12\, a_1 b_0\, a_0{}^2\, b_{-1} + 5\, a_{-1}\, b_0\, b_{-1} - 10\, b_0{}^2 a_0\, b_{-1} + 5\, b_0\, a_1\, b_{-1}{}^2 \\
& + 12\, a_{-1}\, b_0 a_0{}^2 + 20 a_1{}^2\, b_0\, b_{-1}\, a_{-1} - 176 a_1\, b_{-1} a_0 a_{-1} - 32\, a_0{}^3\, b_{-1} \\
& - 10 a_0 b_{-1}{}^2 + 5\, b_0{}^3 a_{-1} + 5\, b_0{}^3 a_1\, b_{-1} + 30\, b_0\, b_{-1}{}^2 + 20\, b_0{}^3 b_{-1} + \ b_0{}^5\,,
\end{aligned}
$$

$$
\begin{aligned}
C_{-1} = {} & -8 a_1\, b_0 a_0\, b_{-1} a_{-1} + 5 b_{-1} + \ b_0{}^4 a_{-1} + 4 a_1\, b_{-1}{}^3 + 30\, b_0{}^2\, b_{-1}{}^2 - 4 a_{-1}\, b_{-1}{}^2 \\
& - 6\beta\, b_0{}^2 b_{-1}{}^2\, a_1 - 12\alpha^2\, b_0\, b_{-1}{}^2 a_0 - 4\alpha^2\, b_0{}^3\, b_{-1} a_0 - 6\alpha^2\, b_0{}^2\, b_{-1}{}^2\, a_1 \\
& - 12\beta\, b_0 b_{-1}{}^2\, b_0 - 4\beta\, b_0{}^3\, b_{-1} a_0 - 12\beta\, b_0{}^2 b_{-1} a_{-1} - 12\alpha^2\, b_0{}^2 b_{-1} a_{-1} \\
& - 4\beta\, b_{-1}{}^3 a_1 - 4\alpha^2\, b_{-1}{}^3 a_1 - 6\alpha^2\, b_{-1}{}^2 a_{-1} - \alpha^2\, b_0{}^4 a_{-1} - 6\beta\, b_{-1}{}^2 a_{-1} \\
& - \beta\, b_0{}^4 a_{-1} + 8 a_1\, b_0{}^2\, a_{-1}{}^2 + 44 a_1\, b_{-1}{}^2\, a_0{}^2 - 88\, a_{-1}\, a_0{}^2\, b_{-1} \\
& + 44\, a_{-1}{}^2\, b_0\, a_0 - 80\, a_1 b_{-1} a_{-1}{}^2 + 48\, a_1{}^2\, b_{-1}{}^2\, a_{-1} - 4\, a_0{}^3\, b_{-1}\, b_0 \\
& + 4\, a_0{}^2\, a_{-1}\, b_0{}^2 + 2\, b_0{}^2\, a_{-1}\, b_{-1} - 13\, b_0\, a_0\, b_{-1}{}^2 - \ b_0{}^3\, a_0\, b_{-1} \\
& + 11\, b_0{}^2\, a_1\, b_{-1}{}^2 + 32\, a_{-1}{}^3 + 10\, b_{-1}{}^3\,,
\end{aligned}
$$

$$
\begin{aligned}
C_{-2} = {} & -4\beta\, b_{-1}{}^3\, a_0 - 4\alpha^2\, b_{-1}{}^3\, a_0 + 10\, b_0{}^3\, b_{-1}{}^2 - 4\, a_0\, b_{-1}{}^3 + 20\, b_0\, b_{-1}{}^3 \\
& - 6\beta\, b_0{}^2\, b_{-1}{}^2\, a_0 - 4\beta\, b_0\, b_{-1}{}^3\, a_1 - 12\,\alpha^2\, b_0\, b_{-1}{}^2\, a_{-1} - 4\alpha^2\, b_0{}^3\, b_{-1}\, a_{-1} \\
& - 6\,\alpha^2\, b_0{}^2\, b_{-1}{}^2\, a_0 - 4\alpha^2\ \, b_0\, b_{-1}{}^3\, a_1 - 12\beta\, b_0\, b_{-1}{}^2\, a_{-1} - 4\beta\, b_0{}^3\, b_{-1}\, a_{-1} \\
& + 56\, a_1\, b_{-1}{}^2\, a_0\, a_{-1} - 12\, a_1\, b_{-1}\, a_{-1}{}^2\, b_0 - 20\, a_{-1}\, b_0\, a_0{}^2\, b_{-1} + 28\, a_{-1}{}^3\, b_0 \\
& + 8\, a_0{}^3\, b_{-1}{}^2 - 72\, a_{-1}{}^2\, a_0\, b_{-1} + 12\, a_{-1}{}^2\, b_0{}^2\, a_0 - 7\, a_{-1}\, b_0\, b_{-1}{}^2 \\
& + \ b_0{}^3\, a_{-1}\, b_{-1} - \ b_0{}^3\, a_{-1}\, b_{-1} - \ b_0{}^2\, a_0\, b_{-1}{}^2 + 11\, b_0\, b_{-1}{}^3\, a_1
\end{aligned}
$$

$$
\begin{aligned}
C_{-3} = {} & -4\, b_{-1}{}^3\, a_{-1} + 10\, b_0{}^2\, b_{-1}{}^3 - 16\, a_{-1}{}^3\, b_{-1} + 4\, b_{-1}{}^4\, a_1 + 8\, a_{-1}{}^3\, b_0{}^2 \\
& - 6\beta\, b_0{}^2\, b_{-1}{}^2\, a_{-1} - 4\beta\, b_0\, b_{-1}{}^3\, a_0 - 6\alpha^2\, b_0{}^2\, b_{-1}{}^2\, a_{-1} - 4\alpha^2\, b_0\, b_{-1}{}^3\, a_0 \\
& - 20\, a_{-1}{}^2\, b_0\, a_0\, b_{-1} - 4\beta\, b_{-1}{}^3\, a_{-1} - \alpha^2\, b_{-1}{}^4\, a_1 - 4\alpha^2\, b_{-1}{}^3\, a_{-1} \\
& - \beta\, b_{-1}{}^4\, a_1 + 12\, a_0{}^2\, b_{-1}{}^2\, a_{-1} + 16\, a_{-1}{}^2\, a_1\, b_{-1}{}^2 - \ b_0{}^2\, b_{-1}{}^2\, a_{-1} \\
& + \ b_0\, b_{-1}{}^3\, a_0 + 5\, b_{-1}{}^4,
\end{aligned}
$$

$$
\begin{aligned}
C_{-4} = {} & 4\, a_{-1}{}^2\, a_0\, b_{-1}{}^2 - 4\, a_{-1}{}^3\, b_0\, b_{-1} - \ b_{-1}{}^3\, a_{-1}\, b_0 - \beta\, b_{-1}{}^4\, a_0 - \ \alpha^2\, b_{-1}{}^4\, a_0 \\
& + \ b_{-1}{}^4\, a_0 + 5\ \, b_0 b_{-1}{}^4 - -4\beta\, b_0\, b_{-1}{}^3\, a_{-1} - 4\,\alpha^2\, b_0 b_{-1}{}^3\, a_{-1},
\end{aligned}
$$

$$
C_{-5} = \ b_{-1}{}^5 - \ \alpha^2\, b_{-1}{}^4\, a_{-1} - \beta\, b_{-1}{}^4\, a_{-1},
$$

Setting each coefficient of $\exp(\pm n\eta)$, $n = 0,1,2,3, \ldots$, to zero, we obtain

$$C_5 = 0, C_4 = 0, C_3 = 0, C_2 = 0, C_1 = 0, C_0 = 0, C_{-1} = 0, C_{-2} = 0, C_{-3} = 0, C_{-4} = 0. \qquad (50)$$

Solving the system of algebraic equations, we obtain three different cases namely,

Case 1.

$$a_0 = a_0 \, , a_1 = a_1 \, , a_{-1} = 0 \, , \alpha = \alpha \, , \beta = -\frac{\alpha^2 \; a_1 - 1}{a_1} \, , \; b_0 = \frac{a_0}{a_1}, \; b_{-1} = 0 \qquad (51)$$

Case 2.

$$a_0 = a_1 \, b_0 \, , \, a_1 = a_1 \, , \, a_{-1} = a_{-1} \, , \alpha = \alpha \, , \beta = -\frac{\alpha^2 \; a_1 - 1}{a_1} \, , \; b_0 = b_0 \, , \; b_{-1} = \frac{a_{-1}}{a_1} \qquad (52)$$

Case 3.

$$a_0 = \left(-\frac{1}{4} \mp \frac{1}{4}\sqrt{7}i\right) b_0, \; a_1 = -\frac{1}{4} \mp \frac{1}{4}\sqrt{7}i \, , \; a_{-1} = a_{-1}, \alpha = \alpha \, , \beta = -\frac{3}{2} \mp \frac{1}{2}\sqrt{7}i - \alpha^2$$

$$, b_0 = b_0, \; b_{-1} = - a_{-1} + \left(\frac{1}{2} \mp \frac{1}{2}\sqrt{7}i\right) a_{-1} \qquad (53)$$

Substituting Eq.(51-53) into Eq.(48) and Eq.(8), we have the following generalized solitary solutions of Eq.(41)

$$u_1(x,t) = u_2(x,t) = a_1 e^{i\left(\alpha x + \left(\frac{1-\alpha^2 a_1}{a_1}\right)t\right)} \qquad (54)$$

$$u_3(x,t) = e^{i\left(\alpha x + \left(-\frac{3}{2} \mp \frac{1}{2}\sqrt{7}i - \alpha^2\right)t\right)} \frac{\left(\begin{array}{c}\left(-\frac{1}{4} \mp \frac{1}{4}\sqrt{7}i\right) e^{x-2\alpha t} + \left(-\frac{1}{4} \mp \frac{1}{4}\sqrt{7}i\right) b_0 \\ + a_{-1} e^{-(x-2\alpha t)}\end{array}\right)}{e^{x-2\alpha t} + b_0 + \left(-a_{-1} + \left(\frac{1}{2} \mp \frac{1}{2}\sqrt{7}i\right) a_{-1}\right) e^{-(x-2\alpha t)}}$$

$$(55)$$

Conclusion

Schrődinger and improved Eckhaus equations is investigated by exp-function method. The generalized travelling wave solutions of these equations are obtained with the help of symbolic computation. The main idea of this method is to take full advantage of the generalized auxiliary equation which has more new solutions. It seems that the exp-function method is more effective and simple than other methods and a lot of solutions can be obtained in the same time.

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