Journal of Education for Pure Science- University of Thi-Qar
Vol.10, No.2 (June, 2020)
Website: jceps.utq.edu.iq Email: jceps@eps.utq.edu.iq

DOI: http://doi.org/10.32792/utq.jceps.10.02.015

# Existence Of Solutions for The Systems of Non-Linear Hemiequilibrim Broblem

**Alaa Mijd Jaber and Ayed E. Hashoosh**

Department of Mathematics, University of Thi-Qar, Iraq

**Received 9/08/2020 Accepted 28/09/2020 Published 30/11/2020**



**Abstract:**
In this article, we present existence results for a general class of systems of nonlinear hemi-equilibrium problems by using a fixed point theorem. Our parse comprises both the statuses of bounded and unbounded closed convex subset in real reflexive Banach space.

**Keywords:** Nonlinear hemi-equilibrium problems; Clarkes generalized gradient; locally Lipschitz functional; set-valued operator; Nonsmooth functions.

## Introduction:

In the last few years the theory hemivariational inequalities acquired private interests as many articles were consecrated to the study of existence and plurality of solutions for this type of inequalities **see[ 2,3,5,6,7,9,10,11,19,23].**The idea of hemivariation inequality was presented by **P. D.** Panagiotopoulos at the beginning of the 1980 **see [27,28 ]** as a variational forging for numerous classes of unilateral mechanical problems with non-smooth and nonconvex energy functional .The hemivariational inequality problem trivializes to a variational inequality problem ,if we delimit the involved functionals to be convex which were studied earlier by many authors (see Fichera [13 ] , Lions and Stampacchia [22 ], Hartman and Stampacchia [ 18]) .In the last decades the theory of hemivariational inequalities has outputted numerousness significant results both in Pure and applied mathematics besides as in other scopes such as mechanics and engineering sciences as it permitted mathematical formulation for new classes of invitingly problems **see [ 14,19,24, 25,26 ,29]) .**The objective of this article is to proof the existence of at least one solution for a general class of systems of nonlinear hemivariational on bounded or unbounded closed and convex subset by using a fixed point theorem including set valued mapping **see [21] .** In order to accomplishes the objective, the article is divided to the following sections. In section 2 we present some definitions and results that help us in the study. In section 3 we formulate the system of generalized hemivariational problem and the main results are proofed

## 2. Preliminaries with basic assumptions:

We introduce in this section some ideas and results that need to be imposed in order to demonstrate our main results from non-smooth analysis that will be used throughout the article. We suppose that $\mathbf{S_i}$ is a Banach space and $\mathbf{S_i^*}$ is the topological dual space of the Banach space $\mathbf{S_i}$ , with $\langle\cdot,\cdot\rangle_{\mathbf{i}}$ and $\|\cdot\|_{\mathrm{i}}$ denote the duality pairing between $\mathbf{S_i}$ and $\mathbf{S_i^*}$ ,respectively for every $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$.

We resummons that $\mathbf{A}: \mathbf{S} \rightarrow \mathbf{R}$ is said to be locally Lipschitz function if for every $\mathbf{x} \in \mathbf{S}$ there exists a neighborhood $\mathbf{Y}$ of $\mathbf{x}$ and a constant $\mathbf{L_x} > \mathbf{0}$ in which

$$|\mathbf{A}(\mathbf{w}) - \mathbf{A}(\mathbf{z})| \leq \mathbf{L_x} \, \|\mathbf{w} - \mathbf{z}\|_{\mathbf{S}} \;, \forall \, \mathbf{w}, \mathbf{z} \in \mathbf{Y}$$

**Definition 2.1.[8]** Suppose that $\mathbf{A} : \mathbf{S} \rightarrow \mathbf{R}$ is a locally Lipschitz.The generalized derivative of $\mathbf{A}$ at $\mathbf{x} \in \mathbf{S}$ in the direction $\mathbf{y} \in \mathbf{S}$ is denoted by $\mathbf{A^o}(\mathbf{x}; \mathbf{y})$ . The following define

$$\mathbf{A^o}(\mathbf{x}; \mathbf{y}) = \limsup_{\mathbf{w} \rightarrow \mathbf{x} \quad \alpha \downarrow 0} \frac{\mathbf{A}(\mathbf{w}+\alpha\mathbf{y}) - \mathbf{A}(\mathbf{w})}{\alpha} \;.$$

In the same way , one can predefine the partial generalized derivative and partial generalized gradient of locally Lipschitz function in the $\mathrm{i^{th}}$ variable.

**Definition 2.2:**[8] Let $\mathbf{A} : \mathbf{S_1} \times \ldots \times \mathbf{S_i} \times \ldots \times \mathbf{S_n} \rightarrow \mathbb{R}$ be a locally Lipschitz function in the $\mathbf{i^{th}}$. The partial generalized derivative of $\mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n})$ at the point $\mathbf{x_i} \in \mathbf{S_i}$ in the direction $\mathbf{y_i} \in \mathbf{S_i}$, denoted by $\mathbf{A^o_i}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}; \mathbf{y_i})$ is

$$\mathbf{A^o_i}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}; \mathbf{y_i}) = \limsup_{\mathbf{w_i} \rightarrow \mathbf{x_i} \quad \alpha \downarrow 0} \frac{\mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{w_i} + \alpha \mathbf{y_i}, \ldots, \mathbf{x_n}) - \mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{w_i}, \ldots, \mathbf{x_n})}{\alpha} \;.$$

**Lemma 2.3.**[8]. Suppose that $\mathbf{A}: \mathbf{S} \rightarrow \mathbf{R}$ is a locally Lipschitz function of rank $\mathbf{L_x}$ nigh the point $\mathbf{x} \in \mathbf{S}$ . Then

**1.** $\mathbf{y} \rightarrow \mathbf{A^o}(\mathbf{x}; \mathbf{y})$ the function is positively homogeneous, subadditive,finite and holds

$$|\mathbf{A}^{o}(\boldsymbol{x}; \boldsymbol{y})| \leq \mathbf{L_x} \|\boldsymbol{y}\|_{\boldsymbol{s}};$$

**2.** $\mathbf{A^o}(\mathbf{x}; \mathbf{y})$ **i**s upper semicontinuous as a function of $(\mathrm{x}, \mathrm{y})$.

**Definition 2.4.**[8] Let $\mathbf{A}: \mathbf{S} \rightarrow \mathbf{R}$ be generalized gradient of a locally Lipschitz function at point $\mathbf{x} \in \mathbf{S}$ a subset
of a dual space $\mathbf{S}^*$,is defined by

$$\boldsymbol{\partial}\mathbf{A}(\mathbf{x}) = \{\boldsymbol{\varrho} \in \mathbf{S}^* : \langle \boldsymbol{\varrho}, \mathbf{y} \rangle_{\mathbf{s}} \leq \mathbf{A^o}(\mathbf{x}; \mathbf{y}), \textbf{for all } \mathbf{y} \in \mathbf{S}\}.$$

For a function $\mathbf{A}: \mathbf{S_1} \times \ldots \times \mathbf{S_i} \times \ldots \times \mathbf{S_n} \rightarrow \mathbf{R}$ is locally Lipschitz in the $\mathbf{i^{th}}$variable. The partial generalized gradient
of the mapping $\mathbf{x_i} \rightarrow \mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n})$ we denote by $\partial_i \mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n})$,for all $\mathbf{y_i} \in \mathbf{S_i}$ that is

$$\boldsymbol{\partial}_{\mathbf{i}} \, \mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}) = \{\boldsymbol{\varrho}_{\mathbf{i}} \in \boldsymbol{S}_{\boldsymbol{i}}^* : \langle \boldsymbol{\varrho}_{\mathbf{i}}, \mathbf{y_i} \rangle_{\mathbf{S_i}} \leq \mathbf{A^o_{,i}}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}; \mathbf{y_i}), \forall \mathbf{y_i} \in \mathbf{S_i}\}.$$

**Definition 2.5.**[8] Suppose that $\mathbf{S}$ is a Banach space and $\mathbf{A}: \mathbf{S} \rightarrow \mathbb{R}$ be a locally Lipschitz function .We say that
$\mathbf{A}$ is regular at $\mathbf{x} \in \mathbf{S}$, if every $\mathbf{y} \in \mathbf{S}$ the one-sided directional derivative $\acute{\mathbf{A}}(\boldsymbol{x}, \boldsymbol{y})$exists and $\acute{\mathbf{A}}(\mathbf{x}; \mathbf{y}) = \mathbf{A^o}(\mathbf{x}, \mathbf{y})$.If this right for each $\mathbf{y} \in \mathbf{S}$, we say that $\mathbf{A}$ is regular.

**Definition 2.6** Suppose that $\mathbf{S}$ is a Banach space. A mapping $\mathbf{f} : \mathbf{S} \rightarrow \mathbb{R}$ is said to be

(i) lower sem continuous (for short,(l.s.c)) at $\mathbf{X_o} \in \mathbf{S}$ if
$\mathrm{f}(\mathrm{X_o}) \leq \lim_{\mathrm{n}} \inf \, \mathrm{f}(\mathrm{X_n})$

(ii) upper sem continuous (for short, (u.s.c)) at $\mathrm{X_o} \in \mathrm{S}$ if
$\mathrm{f}(\mathrm{X_o}) \geq \lim_{\mathrm{n}} \sup \mathrm{f}(\mathrm{X_n})$

for any sequence $X_n$ of X such that $X_n \rightarrow X_o$ .

**Lemma 2.7 [ 8]** Suppose that $\mathbf{A}$:$\mathbf{S_1} \times \ldots \times \mathbf{S_i} \times \ldots \times \mathbf{S_n} \rightarrow \mathbb{R}$ **is** a reqular , locally Lipschitz function . Then
following hypotheses are fulfilled;

**i.** $\boldsymbol{\partial}\mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}) \subseteq \boldsymbol{\partial}_{\mathbf{1}}\mathbf{A}(\mathbf{x_1}, \ldots \mathbf{x_i}, \ldots, \mathbf{x_n}) \times \ldots \times \boldsymbol{\partial}_{\mathbf{i}}\mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}) \times \ldots \times \boldsymbol{\partial}_{\mathbf{n}}\mathbf{A}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n})$;

**ii.** $\mathbf{A^o}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}; \mathbf{y_1}, \ldots, \mathbf{y_i}, \ldots, \mathbf{y_n}) \leq \sum_{\mathbf{i}=\mathbf{1}}^{\mathbf{n}} \mathbf{A^o_i} \, (\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}; \mathbf{y_i})$;

**iii.** $\mathbf{A^o}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}; \mathbf{0}, \ldots, \mathbf{y_i}, \ldots, \mathbf{0}) \leq \mathbf{A^o_i}(\mathbf{x_1}, \ldots, \mathbf{x_i}, \ldots, \mathbf{x_n}; \mathbf{y_i})$.

This section will be ended with very important theorem (Lin fixed point theorem) for set-valued mapping which is
key to prove the main results of the study. See[**19**]

**Theorem (2.8) [19].** Let $\mathbf{K}$ be a nonempty convex of a Hausdorff topological vector space. Assume that $\mathrm{F} \subseteq \mathbf{K} \times \mathbf{K}$ is subset such that:

**i.** The set $\mathbf{H}(\mathbf{r}) = \{\mathbf{t} \in \mathbf{K} : (\mathbf{r}, \mathbf{t}) \in \mathbf{F}\}$ is closed in $\mathbf{K}$,for all $\mathbf{r} \in \mathbf{K}$;

**ii.**The set $\mathbf{Q}(\mathbf{t}) = \{\mathbf{r} \in \mathbf{K} : (\mathbf{r}, \mathbf{t}) \notin \mathbf{F}\}$ is either convex or empty, for all $\mathbf{t} \in \mathbf{K}$;

**iii.**For every $\mathbf{r} \in \mathbf{K}$ then $(\mathbf{r}, \mathbf{r}) \in \mathbf{F}$;

**iv.K** has a nonempty compact convex subset of $\mathbf{K_o}$,such that the set $\mathbf{D} = \{\mathbf{y} \in \mathbf{K} : (\mathbf{x}, \mathbf{y}) \in \mathbf{F}, \forall \mathbf{x} \in \mathbf{K_o}\}$ is compact.

Then there exists a point $\mathbf{y_o} \in \mathbf{D}$ in which that $\mathbf{K} \times \{\mathbf{y_o}\} \subset \mathbf{F}$.

**3. Main Results.**

In the section, we introduced formulation of the problem and main results. Let $\mathbf{X_1}, \dots, \mathbf{X_n}$ be real reflexive Banach spaces and let $\mathbf{Y_1}, \dots, \mathbf{Y_n}$ be real Banach spaces ,where $\mathbf{n}$ is a positive integer ,and let $\mathbf{K_i}$ be a nonempty closed and convex subset of a real reflexive Banach space $\mathbf{X_i}$ ,for $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$. We a assume that for $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$ there exist linear compact operators $\mathbf{Z_i} : \mathbf{X_i} \to \mathbf{Y_i}$ ,the single-valued functions
$\boldsymbol{\eta_i} : \boldsymbol{X_i} \times \boldsymbol{X_i} \to \boldsymbol{X_i}$ and the non-linear functional $\mathbf{B_i} : \mathbf{X_1} \times \dots \times \mathbf{X_i} \times \dots \times \mathbf{X_n} \times \mathbf{X_i} \to \mathbb{R}$ . We also suppose that
$\boldsymbol{J} : \boldsymbol{Y_1} \times \dots \times \boldsymbol{Y_n} \to \boldsymbol{R}$ is a reqular locally Lipschitz functional. We present the following notations:

$\mathbf{X} = \mathbf{X_1} \times \dots \times \mathbf{X_n}$ ,$\mathbf{Y} = \mathbf{Y_1} \times \dots \times \mathbf{Y_n}$ and $\mathbf{K} = \mathbf{K_1} \times \dots \times \mathbf{K_n}$ ;

$\bar{\boldsymbol{u}}_i = \boldsymbol{Z_i}(\boldsymbol{u_i})$ ,$\bar{\boldsymbol{\eta}}_i(\boldsymbol{u_i}, \boldsymbol{v_i}) = \boldsymbol{Z_i}(\boldsymbol{\eta_i}(\boldsymbol{u_i}, \boldsymbol{v_i}))$ ,$\forall\, \boldsymbol{i} \in \{\mathbf{1}, \dots, \boldsymbol{n}\}$ ;

$\boldsymbol{u} = (\boldsymbol{u_1}, \dots, \boldsymbol{u_n})$ , $\bar{\boldsymbol{u}} = (\bar{\boldsymbol{u}}_1, \dots, \bar{\boldsymbol{u}}_n)$;

$\boldsymbol{\eta}(\boldsymbol{u}, \boldsymbol{v}) = \big(\boldsymbol{\eta_1}(\boldsymbol{u_1}, \boldsymbol{v_1}), \dots, \boldsymbol{\eta_n}(\boldsymbol{u_n}, \boldsymbol{v_n})\big)$ and $\bar{\eta}(\boldsymbol{u}, \boldsymbol{v}) = \big(\bar{\boldsymbol{\eta}}_1(\boldsymbol{u}, \boldsymbol{v}), \dots, \bar{\boldsymbol{\eta}}_n(\boldsymbol{u}, \boldsymbol{v})\big)$;

$\mathbf{B} : \mathbf{X} \times \mathbf{X} \to \mathbb{R}$, $\mathbf{B}(\mathbf{u}, \mathbf{v}) = \sum_{\mathbf{i}=1}^{\mathbf{n}} \mathbf{B_i}\big(\mathbf{u_1}, \dots, \mathbf{u_n}, \boldsymbol{\eta}_\mathbf{i}(\mathbf{u_i}, \mathbf{v_i})\big)$.

**Now,** we present the formulation of the problem, and our objective is study the following system of nonlinear
hemiequilibrium problems (**SNHEP**) .
Find $(\mathbf{u_1}, \dots . \mathbf{u_n}) \in \mathbf{K_1} \times \dots \times \mathbf{K_n}$ such that for all $(\mathbf{v_1}, \dots, \mathbf{v_n}) \in \mathbf{K_1} \times \dots \times \mathbf{K_n}$

$$\begin{cases} \langle \mathbf{T_1}(\mathbf{u_1}, \dots, \mathbf{u_n}), \boldsymbol{\eta}_1(\mathbf{u_1}, \mathbf{v_1}) \rangle + \mathbf{B_1}\big(\mathbf{u_1}, \dots, \mathbf{u_n}, \boldsymbol{\eta}_1(\mathbf{u_1}, \mathbf{v_1})\big) + \mathbf{J}^{\mathbf{o}}_{,1}\big(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n, \bar{\boldsymbol{\eta}}_1(\mathbf{u_1}, \mathbf{v_1})\big) \geq \mathbf{0} \\ \dots \dots \dots \quad \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \\ \langle \mathbf{T_n}(\mathbf{u_1}, \dots, \mathbf{u_n}), \boldsymbol{\eta}_\mathbf{n}(\mathbf{u_n}, \mathbf{v_n}) \rangle + \mathbf{B_n}\big(\mathbf{u_1}, \dots, \mathbf{u_n}, \boldsymbol{\eta}_\mathbf{n}(\mathbf{u_n}, \mathbf{v_n})\big) + \mathbf{J}^{\mathbf{o}}_{,\mathbf{n}}(\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_n; \bar{\boldsymbol{\eta}}_\mathbf{n}(\mathbf{u_n}, \mathbf{v_n}))) \geq \mathbf{0} \end{cases}$$

Where for each $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$.

Now, we recall special case, if $\mathrm{T} = 0$ reduces to nonlinear hemivariational-like inequality systems (**NHLIS**) .(see [4]).

In order to introduce the existence of at one solution for (**SNHEP**) we shall suppose the following conditions are satisfied.

$(\mathbf{R_1})$ The functional $\mathbf{B_i} : \mathbf{X_1} \times \dots \times \mathbf{X_i} \times \dots \times \mathbf{X_n} \times \mathbf{X_i} \to \mathbb{R}$ , $\forall \mathbf{i} \in \{\mathbf{1}, \dots \mathbf{n}\}$ holds.

**(i)** $\mathbf{B_i}(\mathbf{u_1}, \dots, \mathbf{u_i}, \dots, \mathbf{u_n}, \mathbf{0}) = \mathbf{0}$ , $\forall\, \mathbf{u_i} \in \mathbf{X_i}$;

**(ii)** $\forall\, \mathbf{v_i} \in \mathbf{X_i}$ the mapping $(\mathbf{u_1}, \dots, \mathbf{u_n}) \to \mathbf{B_i}\big(\mathbf{u_1}, \dots, \mathbf{u_n}, \boldsymbol{\eta}_\mathbf{i}(\mathbf{u_i}, \mathbf{v_i})\big)$ is weakly upper semi continuous ;

**(iii)**$\forall (\mathbf{u_1}, \dots, \mathbf{u_n}) \in \mathbf{X_1} \times \dots \times \mathbf{X_n}$ the mapping $\mathbf{v_i} \to \mathbf{B_i}\big(\mathbf{u_1}, \dots, \mathbf{u_n}, \boldsymbol{\eta}_\mathbf{i}(\mathbf{u_i}, \mathbf{v_i})\big)$ is convex.

$(\mathbf{R_2})$ The mapping $\eta_i(\cdot,\cdot) : \boldsymbol{X_i} \times \boldsymbol{X_i} \to \boldsymbol{X_i}$ , for each $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$ satisfied the following conditions ;

$(i)$ $\eta_i(u_i, u_i) = 0$, $\forall\, u_i \in X_i$
$(ii)$ $\eta_i(u_i, \cdot)$ is linear operator ,for ever $u_i \in X_i$ ;
$(iii)$ $\eta_i(u_i^m, v_i) \to \eta(u_i, v_i)$ $\forall\, v_i \in X_i$ ,whenever $u_i^m \to u_i$
$(R_3)$
$(i)$ $\limsup_m \langle T_i(u_i^m, \dots, u_n^m), \eta_i(u_i^m, v_i)\rangle_{X_i} \le \langle T_i(u_1, \dots, u_n), \eta_i(u_i, v_i)\rangle_{X_i}$ where $(u_1^m, \dots, u_n^m) \to (u_1, \dots, u_n)$ as $m \to \infty$ and $v_i \in X_i$ is fixed.
$(ii)$ $v_i \to \sum_{i=1}^{n} \langle T_i(u_1, \dots, u_n), \eta_i(u_i, v_i)\rangle$ is a convex for every $u_i \in X_i$ .

The first major result of this article refers to the state when the sets $K_i$ are bounded, closed and convex it is given by
the following theorem.

**Theorem 3.1.** Assume that $K_i \subset X_i$ be a nonempty, bounded closed and convex set for every $i \in \{1, \dots, n\}$ .If the
condition $R_1$, $R_2$ and $R_3$ are satisfied ,then the system of non-linear hemiequilibrium problems (**SNHEP**) admits at least one solution.

In what follows, we are going to introduce formulation of the following vector hemiequilibrium problems:

(**VHEP**) Find $u \in K$ such that for all $v \in K$

$$\langle T_u, \eta(u, v)\rangle + B(u, v) + J^o\big(\bar{u}, \bar{\eta}(u, v)\big) \ge 0 .$$

**Remark 3.1.**Suppose that the conditions $(R_1)\_(i)$ , and $(R_2)\_(i)$ are satisfied, then any solution $u^o = (u_1^o, \dots, u_n^o) \in K_1 \times \dots \times K_n$ of the vector hemiequilibrium problems (**VHEP**) **,** then $u^o$is a solution of the system (**SNHEP**).

**Proof:** In work, if for an $i \in \{1, \dots, n\}$ we fix a point $v_i \in K_i$and for $j \neq i$ we suppose that $v_j = u_j^o$ ,by Lemma **2.7** and the fact that $u^o$ is a solution (**VHEP**) we get that

$$\begin{aligned} 0 &\le \langle Tu^o, \eta(u^o, v)\rangle_X + B(u^o, v) + J^o\big(\bar{u}^o; \bar{\eta}(u^o, v)\big) \\ &\le \sum_{j=1}^{n} \langle T_j(u_1^o, \dots, u_n^o), \eta_j(u_j^o, v_j)\rangle + \sum_{j=1}^{n} B_j\Big(u_1^o, \dots, u_j^o, \dots, u_n^o, \eta_j(u_j^o, v_j)\Big) \\ &\quad + \sum_{j=1}^{n} J_{,j}^o\,\big(\bar{u}_1^o, \dots, \bar{u}_n^o; \bar{\eta}_j(u_j^o, v_j)\big) \\ &= \langle T_i(u_1^o, \dots, u_n^o), \eta_i(u_i^o, v_i)\rangle + B_i\big(u_1^o, \dots, u_n^o, \eta_i(u_i^o, v_i)\big) + J_{,i}^o\big(\bar{u}_1^o, \dots, \bar{u}_n^o, \bar{\eta}_i(u_i^o, v_i)\big) \end{aligned}$$

$\forall\, i \in \{1, \dots, n\}$ and $v_i \in K_i$ **a** fix point implies that $(u_1^o, \dots, u_n^o) \in K_1 \times \dots \times K_n$ is a solution of the problem (**SNHEP**).

**Remark 3.2. Since** $J_i^o(u_1, \dots, u_n; v_i)$ is convex and $\eta_i(u_i, .)$ is linear for each $i \in \{1, \dots, n\}$and for each $(u_1, \dots, u_n) \in X_1 \times \dots \times X_n$ ,it follows that the mapping $v_i \to J_i^o\big(u_1, \dots, u_n; \eta_i(u_i, v_i)\big)$ is convex.

**Proof of Theorem 3.1.**According to Remark **3.1** it is adequately to prove that problem (**VHEP**) admits a solution. We deem the set $F \subset K \times K$ as follows

$$F = \big\{(v, u) \in K \times K \colon \langle Tu, \eta(u, v)\rangle_X + B(u, v) + J^o\big(\bar{u}; \bar{\eta}(u, , v)\big) \ge 0\big\}.$$

Now, we shall prove the set **F** holds the assumptions needed in Theorem **2.8** for the weak topology of the space **X.**

**Dunning 1.** The set $\mathbf{H}(\mathbf{v}) = \{\mathbf{u} \in \mathbf{K}: (\mathbf{v}, \mathbf{u}) \in \mathbf{F}\}$ is weakly closed in K ,for every $\mathbf{v} \in \mathbf{K}$ .

To prove the relation above, for a fixed $\mathbf{v} \in \mathbf{K}$ we let us consider the functional $\boldsymbol{\delta}: \mathbf{K} \to \mathbf{R}$ defined by

$$\boldsymbol{\delta}(\mathbf{u}) = \langle Tu, \eta(u, v)\rangle_X + \mathbf{B}(\mathbf{u}, \mathbf{v}) + \mathbf{J}^o(\bar{u}; \bar{\eta}(u, v)) .$$

And we shall prove that it is weakly upper semi continuous .Assume that $\{\mathbf{u}^m\} \subset \mathbf{K}$ be a sequence such that $\mathbf{u}^m \to \mathbf{u}$ as m→ ∞. Using $(\mathbf{R}_1)$_(ii) we get that

$$\limsup_{m\to\infty} \mathbf{B}(\mathbf{u}^m, \mathbf{v}) = \limsup_{m\to\infty} \sum_{i=1}^{n} \mathbf{B}_i\left(\mathbf{u}_1^m, \dots, \mathbf{u}_n^m, \boldsymbol{\eta}_i(u_i^m, v_i)\right)$$
$$\leq \sum_{i=1}^{n} \limsup_{m\to\infty} \mathbf{B}_i\left(\mathbf{u}_1^m, \dots, \mathbf{u}_n^m, \boldsymbol{\eta}_i(u_i^m, v_i)\right)$$
$$\leq \sum_{i=1}^{n} B_i\left(\mathbf{u}_1, \dots, \mathbf{u}_n, \boldsymbol{\eta}_i(u_i, v_i)\right)$$
$$= \mathbf{B}(\mathbf{u}, \mathbf{v}).$$

Now, the other hand ,we suppose that $\mathbf{Z}_i$ is a compact operator, for each $\mathbf{i} \in \{1, \dots, n\}$.So, we get that $\bar{\mathbf{u}}^m$ converges strongly to some $\bar{\mathbf{u}} \in \mathbf{K}$ **for** $i \in \{1, \dots, n\}$ and$\mathbf{v}_i \in \mathbf{K}_i$ ,$\bar{\boldsymbol{\eta}}_i(\mathbf{u}_i^m, \mathbf{v}_i)$ converges strongly to $\bar{\boldsymbol{\eta}}_i(\mathbf{u}_i, \mathbf{v}_i)$,hence $\bar{\boldsymbol{\eta}}(\mathbf{u}^m, \mathbf{v})$ converges

Strongly to $\bar{\eta}(u, v)$, for each $\mathbf{v} \in K$ .Implementing this verity, together with Lemma( 2.3)-(ii) we get that

$$\limsup_{m \to \infty} \mathbf{J}^o\left(\bar{\mathbf{u}}^m; \bar{\boldsymbol{\eta}}(\mathbf{u}^m, \mathbf{v})\right) \leq \mathbf{J}^o\left(\bar{\mathbf{u}}; \bar{\boldsymbol{\eta}}(\mathbf{u}, \mathbf{v})\right).$$

Finally, using $\mathbf{R}_3$_(i) we obtain

$$\limsup_{m \to \infty} \langle \mathbf{T}(\mathbf{u}^m), \boldsymbol{\eta}(\mathbf{u}^m, \mathbf{v})\rangle_X = \limsup_{m \to \infty} \sum_{i=1}^{n} \langle \mathbf{T}_i(\mathbf{u}_1^m, \dots, \mathbf{u}_n^m), \boldsymbol{\eta}_i(\mathbf{u}_i^m, \mathbf{v}_i)\rangle_{X_i}$$
$$\leq \sum_{i=1}^{n} \limsup_{m \to \infty} \langle \mathbf{T}_i(\mathbf{u}_1^m, \dots, \mathbf{u}_n^m), \boldsymbol{\eta}_i(\mathbf{u}_i^m, \mathbf{v}_i)\rangle_{X_i}$$
$$\leq \sum_{i=1}^{n} \langle \mathbf{T}_i(\mathbf{u}_1, \dots, \mathbf{u}_n), \boldsymbol{\eta}_i(\mathbf{u}_i, \mathbf{v}_i)\rangle_{X_i}$$
$$= \langle \mathbf{T}(\mathbf{u}), \boldsymbol{\eta}(\mathbf{u}, \mathbf{v})\rangle_X .$$

Consequently ,$\boldsymbol{\delta}$ is weakly **u.s.c.** So,the set $[\boldsymbol{\delta} \geq \boldsymbol{\beta}] = \{\mathbf{u} \in \mathbf{K}: \boldsymbol{\delta}(\mathbf{u}) \geq \boldsymbol{\beta}\}$ is weakly closed for any $\boldsymbol{\beta} \in \mathbf{R}$. Take $\beta = \mathbf{0}$ .We get that the set $\mathbf{H}(\mathbf{v})$ is weakly closed.

**Dunning.2.** $\mathbf{Q}(\mathbf{u}) = \{\mathbf{v} \in \mathbf{K}: (\mathbf{v}, \mathbf{u}) \notin \mathbf{F}\}$ is either convex or empty, for each $\mathbf{u} \in \mathbf{K}$ .

Assume that fix $\mathbf{u} \in \mathbf{K}$ and itis clear $\mathbf{Q}(\mathbf{u})$ is a nonempty for $\mathbf{u} \in \mathbf{K}$. Let us choose $\mathbf{v}^1, \mathbf{v}^2 \in \mathbf{Q}(\mathbf{u}), \mathbf{t} \in (\mathbf{0}, \mathbf{1})$ and $v^t = \mathbf{t}\mathbf{v}^1 + (\mathbf{1} - \mathbf{t})\mathbf{v}^2$. By $(R_1) - (\mathbf{iii})$ , we get that

$$\mathbf{B}(\mathbf{u}, \mathbf{v}^t) = \sum_{i=1}^{n} \mathbf{B}_i\left(\mathbf{u}_1, \dots, \mathbf{u}_n, \boldsymbol{\eta}_i(\mathbf{u}_i, \mathbf{v}_i^t)\right)$$
$$\leq \sum_{i=1}^{n} \mathbf{B}_i\left(\mathbf{u}_1, \dots, \mathbf{u}_n, \boldsymbol{\eta}_i\left(\mathbf{u}_i, \mathbf{t}\mathbf{v}_i^1 + (\mathbf{1} - \mathbf{t})\mathbf{v}_i^2\right)\right)$$
$$\leq \mathbf{t}\sum_{i=1}^{n} \mathbf{B}_i\left(\mathbf{u}_1, \dots, \mathbf{u}_n, \boldsymbol{\eta}_i\left(\mathbf{u}_i, \mathbf{v}_i^1\right)\right) + (\mathbf{1} -$$
$$\mathbf{t})\sum_{i=1}^{n} \mathbf{B}_i\left(\mathbf{u}_1, \dots, \mathbf{u}_n, \boldsymbol{\eta}_i\left(\mathbf{u}_i, \mathbf{v}_i^2\right)\right),$$
$$= \mathbf{t}\,\mathbf{B}(\mathbf{u}, \mathbf{v}^1) + (\mathbf{1} - \mathbf{t})\mathbf{B}(\mathbf{u}, \mathbf{v}^2) \quad ,\forall t \in (\mathbf{0}, \mathbf{1}) .$$

This means that $\mathbf{v} \to \mathbf{B}(\mathbf{u}, \mathbf{v})$ is convex ,$\forall\ \mathbf{t} \in (\mathbf{0}, \mathbf{1})$ .On the other hand side ,from Remark **3.2** we deduce that the mapping

$\mathbf{v} \to \mathbf{J}^o\left(\bar{\mathbf{u}}, \bar{\boldsymbol{\eta}}_i(\mathbf{u}, \mathbf{v}_i)\right)$ is convex .Then $\mathbf{Q}(\mathbf{u})$ is a convex set for the fixed $u \in K$.

**Dunning .3.** For every $\mathbf{u} \in \mathbf{K}$ then $(\mathbf{u}, \mathbf{u}) \in \mathbf{F}$ .

Assume that $\mathbf{u} \in \mathbf{K}$ is a fixed point .From $(\mathbf{R}_1) - (\mathbf{i})$ and $(\mathbf{R}_2) - (\mathbf{i})$ we get that

$$\langle \mathbf{T}\mathbf{u}, \boldsymbol{\eta}(\mathbf{u}, \mathbf{u})\rangle + \mathbf{B}(\mathbf{u}, \mathbf{u}) + \mathbf{J}^o\left(\bar{\mathbf{u}}; \bar{\boldsymbol{\eta}}(\mathbf{u}, \mathbf{u})\right) =$$
$$\sum_{i=1}^{n} [\langle \mathbf{T}_i(\mathbf{u}_1, \dots, \mathbf{u}_n), \boldsymbol{\eta}_i(\mathbf{u}_i, \mathbf{u}_i)\rangle + \mathbf{B}_i(\mathbf{u}_1, \dots, \mathbf{u}_i, \dots, \mathbf{u}_n, \eta_i(\mathbf{u}_i, u_i)) .$$

$$+J_{,i}^{o}(\bar{u}_1, \dots, \bar{u}_n, \bar{\eta}_i(u_i, u_i))$$
$$= 0$$

This means that $(u, u) \in F$.

**Dunning.4.** The set $D = \{u \in K: (v, u) \in F, \forall\, v \in K\}$ is compact .
It clear $K$ is a weakly compact subset of reflexive space $X$ as it is bounded and convex. We notice that the set $D$ can berecount written in the following form:

$$D = \cap_{v \in k} H(v)\,,$$

which implies that $D$ is ditto a weakly compact set as it is an intersection of weakly closed subset of $K$ ,therefore, by
applying Theorem **2.8** ,we obtain that there exists $u^o \in D \subset k$ such that $K \times \{u^o\} \subset F$.Which implies

$$\langle Tu^o, \eta(u^o, v)\rangle + B(u^o, v) + J^o(u^o, \eta(u^o, v)) \geq 0\,, \qquad \forall v \in K.$$

So,$u^o$ is a solution of (**VHEP**) and ,from Remark **3.1,** it is a solution of the system of nonlinear hemiequilibrium problems(**SNHEP**).
We will show next that if we change able the conditions on the nonlinear functional $B_i$ we are still able to prove the existence of at least one solution for our system. Let us consider that alternatively of $(R_1)$ we have the following set of conditions:
$(R_4)$ For every $i \in \{1, \dots n\}$, the functional $B_i: X_1 \times \dots \times X_i \times \dots \times X_n \to \mathbb{R}$ holds
(**i**) $B_i(u_1, \dots, u_i, \dots, u_n, 0) = 0\,, \quad \forall z_i \in X_i\,;$
(**ii**) For every $i \in \{1, \dots, n\}$ and any couple $(u_1, \dots, u_i, \dots, u_n), (v_1, \dots, v_i, \dots, v_n) \in X_1 \times \dots \times X_i \times \dots \times X_n$ we have
$B_i(u_1, \dots, u_i, \dots, u_n, \eta_i(u_i, v_i)) + B_i(\eta_i(u_i, v_i), u_1, \dots, u_i, \dots, u_n) \geq 0,$
(iii) For every $(u_1, \dots, u_n) \in X_1 \times \dots \times X_n$ the mapping $v_i \to B_i(u_1, \dots, u_n, \eta_i(u_i, v_i))$ is weakly lower semi continuous;
(**iv**) For every $v_i \in X_i$ the mapping $(u_1, \dots, u_n) \to B_i(u_1, \dots, u_n, \eta_i(u_i, v_i))$ is concave.
**Theorem 3.2.** Suppose that the nonempty ,bounded, closed and convex set $K_i \subset X_i$ for each $i \in \{1, \dots, n\}$ .If the conditions $(R_2)$_ $(R_4)$ hold true. Then the system of nonlinear hemiequilibrium problems (**SNHEP**) admits at least one solution.
For proof Theorem **3.2.** we must prove the following Lemma;
**Lemma 3.3.**Let condition $(R_4)$ be satisfied:
(1)$B(u, v) + B(v, u) \geq 0\,, \quad \forall\, u, v \in X;$
(2) For every $v \in X$ the mapping $u \to -B(v, u)$ is weakly upper semi continuous;
(3) For every $u \in X$ the mapping $v \to -B(v, u)$ is convex.
**Proof:**
(1) From $(R_4)$_(**ii**) ,we have

$$B(u, v) + B(v, u) = \sum_{i=1}^{n}[B_i(u_1, \dots, u_n, \eta_i(u_i, v_i)) + B_i(\eta_i(u_i, v_i), u_1, \dots, u_n)] \geq o\,.$$

(2) Assume that $\{u^m\} \subset X$ is a sequence which coverages weakly to some point $u \in X$ .From $(R_4)$_(**iii**) and the fact $z^m \to z$ ,one can get

$$\limsup_{m \to \infty} [-B(v, u^m)] = -\liminf_{m \to \infty} B(v, u^m)$$
$$= -\liminf_{m \to \infty} \sum_{i=1}^{n} B_i(\eta_i(u_i, v_i), u_i^m)$$

$$\leq -\sum_{i=1}^{n} \liminf_{m\to\infty} B_i(\eta_i(u_i,v_i),u_i^m)$$
$$\leq -\sum_{i=1}^{n} B_i\,(B_i(u_i,v_i),u_i)$$
$$= -B(v,u)$$

(3) **Let** $u, v^1, v^2 \in X$ .Using $(R_4)$_(**iv**) such that , $v^t = tv^1 + (1-t)v^2, \forall t \in (0,1)$ , then

$B(v^t,u) = \sum_{i=1}^{n}[B_i(\eta_i(u_i, tv_i^1 + (1-t)v_i^2), u_i)]$ $\geq \sum_{i=1}^{n}[tB_i(\eta_i(u_i,v_i^1),u_1,\dots,u_n) + (1-t)B_i(\eta_i(u_i,v_i^2),u_1,\dots,u_n)]$ $= tB(v^1,u) + (1-t)B(v^2,u).$

This means that the mapping $v \to B(v^t,u)$ is concave, then $v \to -B(v,u)$ must be convex.

**Proof of Theorem 3.2.**Assume that the set $F \subset K \times K$ ,the following defined by

$$F = \{(v,u) \in K \times K: -B(v,u) + \langle Tu, \eta(u,v)\rangle + J^o(\bar{u}, \bar{\eta}(u,v)) \geq 0\}.$$

In the same way as the proof of Theorem 3.1 to deduce that the assumption is bespoken in Theorem 2.8 are satisfies. So, there exists $u^o \in K$ such that $K \times \{u^o\} \subset F$ .Then

$$-B(v,u^o) + \langle Tu^o, \eta(u^o,v)\rangle + J^o(\bar{u}^o, \bar{\eta}(u^o,v)) \geq 0 \qquad \forall v \in K. \qquad (3.1)$$

From Lemma **3.3,** we get

$$B(u^o,v) + B(v,u^o) \geq 0, \qquad \forall v \in k. \qquad (3.2)$$

Adding relation $(3.1)$ and $(3.2)$ we conclude that $u^o$**is** a so**l**utions of the problem (**VHEP**), which applying Remark $(3.1)$.Then $u^o$ is a solution of problem (**SNHEP**) **.**

Now, let us assume the state when at least one of subset $K_i$ unbounded and either hypotheses $R_1$, $R_2$ and $R_3$ or $R_2$,
$R_3$ and $R_4$satisfies. Let $B_s(0;P)$be the closed ball of the space **S** centered in the origin point and of radius **P**,defined
as follows:

$$B_s(0,P) = \{v \in S: \|v\|_S \leq P\}\ .$$

Let us consider $P > o$ such that the set $K_{i,P} = K_i \cap B_{X_i}(0,P)$ is a nonempty for every $i \in \{1,\dots,n\}$ .Therefor, the set $K_{i,P}$ is a nonempty, bounded, closed and convex and from Theorem **3.1** or Theorem **3.2** the following problem:

($SP$) Find $(u_1,\dots,u_n) \in K_{1,p} \times \dots \times K_{n,P}$ such that for all $(v_1,\dots,v_n) \in K_{1,P},\dots,K_{n,P}$

$$\begin{cases} B_1(u_1,\dots,u_n,\eta_1(u_1,v_1)) + \langle T_1(u_1,\dots,u_n),\eta_1(u_1,v_1)\rangle + J_{,1}^o\,(\bar{u}_1,\dots,\bar{u}_n,\bar{\eta}_1(u_1,v_1)) \geq 0 \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ B_n(u_1,\dots,u_n,\eta_n(u_n,v_n)) + \langle T_n(u_1,\dots,u_n),\eta_n(u_n,v_n)\rangle + J_{,n}^o(\bar{u}_1,\dots,\bar{u}_n,\bar{\eta}_n(u_n,v_n)) \geq 0 \end{cases}$$

admits at least one solution.

Now, we have below existence result includes the state of at least one unbounded subset.

**Theorem 3.4.**Assume that the nonempty, bounded, closed and convex set $K_i \subset X_i$,for every $i \in \{1,\dots,n\}$ and
supposed there exists at least one index $i \in \{1,\dots n\}$such that $K_{i_o}$is not bounded and either$R_1$ ,$R_2$ and $R_3$ or $R_2$ , $R_3$ and $R_4$ fulfilled ,then the system of nonlinear hemiequilibrium problem (**SNHEP**) admits at least one solution if and only if the following assumption satisfies

$(\mathbf{R_5})$ $\exists\, \mathbf{P} > \mathbf{0}$ such that $\mathbf{K_{i,P}}$ is a nonempty for each $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$ and admits at least one solve $(\mathbf{u_1^o}, \dots, \mathbf{u_n^o})$ of problem
$(\boldsymbol{SP})$ holds:

$$\mathbf{u_i^o} \in \mathbf{int}\, \mathbf{B_{X_i}}(\mathbf{0}, \mathbf{P}), \qquad \forall \mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$$

**Proof.** The necessity is obvious.
In order to proof the a dequation for every $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$ assume that fix $\mathbf{v_i} \in \boldsymbol{K_i}$ and defined the scalar

$$\boldsymbol{\alpha}_\mathbf{i} = \begin{cases} \frac{1}{2} & \text{if } u_i^o = v_i \\ \min\left\{\frac{1}{2}, \frac{P - \|u_i^o\|_{X_i}}{\|v_i - u_i^o\|_{X_i}}\right\} & \text{otherwise} \end{cases}$$

Assumption $(\mathbf{R_5})$ secures that $\boldsymbol{\alpha}_\mathbf{i} \in (\mathbf{0}, \mathbf{1})$, so $\mathbf{w}_{\alpha_i} = \mathbf{u_i^o} + \boldsymbol{\alpha}_\mathbf{i}(\mathbf{v_i} - \mathbf{u_i^o})$ is an element of $\mathbf{K_{i,P}}$ due to the convexity of the set $\mathbf{K_i}$ .
**State 1.**$(\boldsymbol{R_1})$ _$(\boldsymbol{R_3})$holds.
Applying verity $(\mathbf{u_1^o}, \dots, \mathbf{u_n^o})$ is a solution of $(\mathbf{SP})$ for every $\mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$ we get

$$\langle \mathbf{T_i}(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}), \boldsymbol{\eta}_\mathbf{i}(\mathbf{u_i^o}, \mathbf{w}_{\alpha_i})\rangle + \mathbf{B_i}\left(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}, \boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \mathbf{w}_{\alpha_i})\right) + \mathbf{J_{,i}^o}\left(\overline{\boldsymbol{u}}_1^o, \dots, \overline{\boldsymbol{u}}_n^o, \overline{\boldsymbol{\eta}}_i(\boldsymbol{u_i^o}; \mathbf{w}_{\alpha_i}\right) \geq \mathbf{0} \qquad (\mathbf{3.3})$$

From $(\mathbf{3.3})$,we get

$$\mathbf{0} \leq \boldsymbol{\alpha}_\mathbf{i}[\langle \mathbf{T_i}(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}, \boldsymbol{\eta_i}(\mathbf{u_i^o}, \mathbf{v_i})\rangle + \mathbf{B_i}\left(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}, \boldsymbol{\eta}_\mathbf{i}(\boldsymbol{u_i^o}, \mathbf{v_i})\right) + \mathbf{J_{,i}^o}(\overline{\mathbf{u}}_1^o, \dots, \overline{\mathbf{u}}_n^o, \overline{\boldsymbol{\eta}}_\mathbf{i}(\mathbf{u_i^o}, \mathbf{v_i})]$$
$$+(\mathbf{1} - \boldsymbol{\alpha_i})[\langle \boldsymbol{T_i}(\boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}), \boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{u_i^o})\rangle + \boldsymbol{B_i}\left(\boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}, \boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{u_i^o})\right) + \boldsymbol{J_{,i}^o}(\overline{\boldsymbol{u}}_i^o, \dots, \overline{\boldsymbol{u}}_i^o, \overline{\boldsymbol{\eta}}_i(\boldsymbol{u_i^o}, \boldsymbol{u_i^o}))]$$
$$= \boldsymbol{\alpha}_\mathbf{i}\left[\langle \mathbf{T_i}(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}), \boldsymbol{\eta}_\mathbf{i}(\mathbf{u_i^o}, \mathbf{v_i})\rangle + \mathbf{B_i}\left(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}, \boldsymbol{\eta}_\mathbf{i}(\mathbf{u_i^o}, \mathbf{v_i})\right) + \mathbf{J_{,i}^o}(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}, \overline{\boldsymbol{\eta}}_\mathbf{i}(\mathbf{u_i^o}, \mathbf{v_i})\right]$$

Portioning by $\boldsymbol{\alpha_i}$ the relation above, since $\mathbf{v_i} \in \boldsymbol{K_i}$ fixed we deduce that $(\mathbf{u_1^o}, \dots, \mathbf{u_n^o})$ is a solution of problem $(\mathbf{SNHEP})$.
**State 2.**$(\boldsymbol{R_2})$_$(\boldsymbol{R_4})$ satisfies.
From Theorem $\mathbf{3.2}$ secures that $(\mathbf{3.1})$

$$-\mathbf{B}(\boldsymbol{\eta}(\mathbf{u^o}, \mathbf{w}), \mathbf{u^o}) + \langle \mathbf{Tu^o}, \boldsymbol{\eta}(\mathbf{u^o}, \mathbf{w})\rangle + \mathbf{J^o}\left(\overline{\mathbf{u}}^o, \overline{\boldsymbol{\eta}}(\mathbf{u^o}, \mathbf{w})\right) \geq \mathbf{0}, \qquad \forall\, \mathbf{w} \in \mathbf{K_P} = \mathbf{K_{1,P}} \times \dots \times \mathbf{K_{n,P}}\,.$$

Let us consider $\mathbf{w_i} = \boldsymbol{w}_{\alpha_i}$ and $\mathbf{w_j} = \mathbf{u_j^o}$ for $\mathbf{j} \neq \mathbf{i}$ in the haut relation we get

$$0 \leq \boldsymbol{\alpha_i}\left[-\mathbf{B}(\boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o},), \boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}) + \langle \boldsymbol{T}(\boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}), \boldsymbol{\eta}(\boldsymbol{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o})\rangle + \boldsymbol{J_{,i}^o}(\overline{\boldsymbol{u}}_1^o, \dots, \boldsymbol{u_n^o}, \overline{\boldsymbol{\eta}}_\mathbf{i}(\boldsymbol{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o}))\right]$$

$$= \sum_{\mathbf{j=1}}^{\mathbf{n}}[-\mathbf{B_j}\left(\boldsymbol{\eta}_\mathbf{j}(\boldsymbol{u_j^o}, \boldsymbol{w_j} - \boldsymbol{u_j^o}), \boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}\right) + \langle \boldsymbol{T_j}(\boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}), \boldsymbol{\eta_j}(\boldsymbol{u_j^o}, \boldsymbol{w_j} - \boldsymbol{u_j^o})\rangle$$
$$+ \boldsymbol{J_{,j}^o}\left(\overline{\mathbf{u}}_1^o, \dots, \overline{\mathbf{u}}_n^o; \overline{\boldsymbol{\eta}}(\boldsymbol{u_j^o}, \boldsymbol{w_j} - \boldsymbol{u_j^o})\right)$$
$$= -\mathbf{B_i}\left(\boldsymbol{\eta_i}(\mathbf{u_i^o}, \boldsymbol{w}_{\alpha_i} - \boldsymbol{u_i^o}), \boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}\right) + \langle \boldsymbol{T_i}(\boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}), \boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{w}_{\alpha_i} - \boldsymbol{u_i^o})\rangle + \mathbf{J_{,i}^o}(\overline{\mathbf{u}}_1^o, \dots, \overline{\mathbf{u}}_n^o; \mathbf{w}_{\alpha_i} - \mathbf{u_i^o})$$
$$\leq \boldsymbol{\alpha_i}[-\mathbf{B_i}(\boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o}), \mathbf{u_i^o}, \dots, \boldsymbol{u_n^o}) + \langle \boldsymbol{T_i}(\boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}), \boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o})\rangle + \mathbf{J_{,i}^o}(\overline{\mathbf{u}}_1^o, \dots, \overline{\mathbf{u}}_n^o; \overline{\boldsymbol{\eta}}_\mathbf{i}(\mathbf{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o}))]$$

Partitioning by $\boldsymbol{\alpha}_\mathbf{i}$ we get that

$$-\mathbf{B_i}(\boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o}), \mathbf{u_i^o}) + \langle \boldsymbol{T_i}(\boldsymbol{u_1^o}, \dots, \boldsymbol{u_n^o}), \boldsymbol{\eta_i}(\boldsymbol{u_i^o}, \boldsymbol{v_i} - \boldsymbol{u_i^o})\rangle + \mathbf{J_{,i}^o}(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}; \mathbf{v_i} - \mathbf{u_i^o}) \geq \mathbf{0}$$

Addition the relation above and $(\mathbf{R_4})$_$(\mathbf{ii})$ we obtain that

$\langle \mathbf{T_i}(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}), \boldsymbol{\eta}_i(\mathbf{u_i^o}, \mathbf{v_i} - \mathbf{u_i^o}) \rangle\ \mathbf{B_i}\big(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}, \boldsymbol{\eta}_i(u_i^o, v_i - u_i^o)\big) + \mathbf{J_{,i}^o}(\mathbf{u_1^o}, \dots, \mathbf{u_n^o}; \mathbf{v_i} - \mathbf{u_i^o}) \geq \mathbf{0} \quad \forall \mathbf{i} \in \{\mathbf{1}, \dots, \mathbf{n}\}$

This implies that $(\mathbf{u_1^o}, \dots, \mathbf{u_n^o})$ is a solution of problem $(\mathbf{SNHEP})$**,**since $\mathbf{v_i} \in \mathbf{K_i}$was arbitrary fixed .

---

**REFERENCES:**

**[1]** M.Alimohammady and A.E.Hashoosh,Existence theorems for $\boldsymbol{\alpha}(\mathbf{u}, \mathbf{v})$ –monotone of nonstandard hemivariaational inequality, Advances in Math.10(2) (2015) 3205-3212.

**[2]** B.E. Breckner, Cs. Varga, A multiplicity result for gradient-type systems with non-differentiable term, Acta.Math.
Hungarica 118 (2008),85-104.

**[3]** B.E. Breckner, A. Horvath, Cs. Varga, A multiplicity result for a special of gradient-type systems with non-differentiable term, Nonlinear Analysis T.M.A.70 (2009) ,606-620.

**[4] N.** Costea, C. Varga, Systems of nonlinear hemivariational inequalities and applications, Topological Methods in
Nonlinear Analysis.1(2003) 39-67.

**[5]** S. Carl, V.k. Le, D. Motreanu, Evolutionary variational-hemivariational inequalities; existence and comparison results, J. Math.Apple.345 (2008),545-558.

[6] S. Carl and D. Motreanu, comparison for quasilinear parabolic inclusions with clarkes gradient, Adv.Nonlinear Stud.9(2009),69-80.

[7] S. Cal and D. Motreanu, General Comparison principle for quasilinear elliptic inclusions, Nonlinear Analysis
T.M.A.70 (2009),1105-1112.

**[8]** F.H. Clarke, Optimization and Nonsmooth Analysis, John Wiley (1983).

**[9**] N. Costea and A. Matei, Weak solutions for nonlinear antiplane problems leading to hemivariational inequalities
Nonlinear Analysis T.M.A.72 (2010),3669-3680.

**[10]** N.Costea and V.Radulescu, Existence results for hemivariational inequalities involving relaxed $\boldsymbol{\eta} - \boldsymbol{\alpha}$ monotone mappings,Commum.Appl.Anal.13(2009),293-304.

**[11]** N. Costea, Existence and uniqueness results for a class of quasi-hemivariational inequalities, J. Math.Anal. Appl,373 (1) (2011),305-311.

**[12**] A. Eva Molnar and Orsolya Vas, An existence result for nonlinear hemivaraiational-like inequality systems, Stud.Univ. Babes-Bolyai Math.58(2013), No.3,381-392

**[13]** G Fichera, Problemi electrostatici con vincoli unilaterali:il problema de Signorini con ambigue condizioni al
Contorno, Mem.Acad. Naz.Lincei,7(1964),91-140**.**

**[14]** D. Goeleven, D. Motreanu, Y. Dumont, and M. Rochdi, Variational and Hemivariational Inequalities, Theory, MethodsBoston/London, (2003).

**[15] A.E.Ha**shoosh ,M.Alimohammady and M.K.Kalleji.Existence Results for Some Equilibrium Problems involving α-Monotone Bifunction,International Journal of Mathemtics and Mathemtical Sciences,(2016) 1-5.

**[16]** A.E. Hashoosh and M. Alimohammady, and G.A. Almusawi, Existence Results for Nonlinear Quasi-hemivariational Inequality Systems, Journal of Thi-Qar University, Vol.11 No.4 (2016).

**[17]** A.E. Hashoosh and M. Alimohammady, Existence and uniqueness results for a nonstandard variation-hemivariational inequalities with application, Int.J. Industrial Mathematics (2016), accepted.

**[18**] P. Hartman, G. Stampacchia, On some nonlinear elliptic differential functional equations, Acta Math.,115(1966),271-310**.**

**[19] Kr**istaly, An existence result for gradient-type systems with a nondifferentiable term on unbounded strips, J.Math.Anal.Appl.229 (2004),186-204.

**[20] A. Kristaly**, V. Radulescu and Cs. Varge, Variational Principles in Mathematical physics, Geometry, and Economics: Qualitative Analysis of Nonlinear Equations and Unilateral problems, Encyclopedia of Mathematics(No.136), Cambridge University Press, Cambridge, (2010).

**[21] T**.C. Lin, Convex sets, fixed points, variational and minimax inequalities, Ball.Austral.Math.Soc.34(1986),107-117.

**[22]** J.L. Lions, Stampacchia, G., Variational inequalities, Comm.Pure Appl.Math.,20(1967),493-519**.**

**[23]** S. Migorski, a class of hemivariational inequalities for electroelastic contact problems with slip dependent friction, Discrete and Continuous Dynamical Systems Series S 1(1) (2008), 117-126.

**[24]** D. Moteanu and P.D. Panagiotopoulos, Minimax Theorems and Qualitative Properties of the Solutions of Hemi-
Variational Inequalitaties and Application, Kluwer Academic Publishers, Nonconvex Optimization and its **A**pplications, **vol.**29, Boston/Dordrecht/London, (1999).

**[25]** D. Motreanu and V. Radulescu, Variational and Non-variational Methods in Nonlinear Analysis and Boundary Value problems, Kluwer A cademic Publishers, Boston/Dordrecht/London, (2003).

**[26]** Z. Naniewicz and P.D. Panagiotopoulos, Mathematical Theory of Hemivariational Inequalities and Applications,
Marcel, Dekker, New York, (1995).

**[27]** P.D. Panagiotopoulos, Nonconvex energy functions Hemivariational inequalities and substationarity principle,
Acta Mechanica 42(1983),160-183.

**[28]** P.D. Panagiotopoulos, Inequality Problems in Mechanics and Applications.Convex and Nonconvex Energy Functions, Birkhauser, Basel, (1985).

**[29]** P.D. Panagiotopoulous, Hemivariational Inequalities: Applications to Mechanics and Engineering, Springer-Verlag, New York/Boston/Berlin, (1993).