Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Weak Pseudo – 2 – Absorbing Submodules And Related Concepts

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Article history: Received,19,May,2020, Accepted 23,June,2020, Published in April 2021

Doi: 10.30526/34.2.2615

Abstract

Let R be a commutative ring with identity and E be a unitary left R – module .We introduce and study the concept Weak Pseudo – 2 – Absorbing submodules as generalization of weakle – 2 – Absorbing submodules , where a proper submodule A of an R – module E is called Weak Pseudo – 2 – Absorbing if $0 \neq rsx \in A$ for $r, s \in R$, $x \in E$, implies that $rx \in A + soc(E)$ or $sx \in A + soc(E)$ or $rs \in [A + soc(E) :_R E]$. Many basic properties, characterizations and examples of Weak Pseudo – 2 – Absorbing submodule in some types of modules are introduced .

Key word : weakly – 2 – Absorbing submodules , essential submodule , socal of modules , multiplication modules , Z – regular modules , WP – 2 – Absorbing submodules .

1. Introduction

The concept of weakly – 2 – Absorbing submodule was first introduced by Darani and Soheilnia as generalization of weakly prime submodule , where a proper submodule A of an R – module E is called a weakly prime submodule of E if $0 \neq te \in A$, for $t \in R$, $e \in E$ implies that either $e \in A$ or $t \in [A :_R E]$, where $[A :_R E] = \{ s \in R : sE \subseteq A \}$ [1] , and a proper submodule A of an R- module E is called a weakly – 2 – Absorbing submodule of E , if $0 \neq ste \in A$, for $s, t \in R$, $e \in E$ implies that either $se \in A$ or $te \in A$ or $st \in [A :_R E]$ [2] .

Recently , several generalizations of weakly – 2 – Absorbing submodules have been introduced [3,4,5] . In our paper , we introduce a new generalization of weakly – 2 – Absorbing submodule which we callWeak Pseudo – 2 – Absorbing submodule , where a proper submodule A of an R – module E is said to be Weak Pseudo – 2 – Absorbing submodule if $0 \neq ste \in A$, for s , t $\in$ R , e $\in$ E , implies that either se $\in$ A + soc (E) or te $\in$ A + soc (E) or st $\in$ [A + soc (E) $:_R$ E] . Soc (E) is the intersection of all essential submodules of E [6] . A nonzero submodule N of an R – module E is called an essential if N $\cap$ K $\neq$ (0) for all nonzero submodules K of E [6]. Every weakly prime submodule of an R – module is weakly – 2 – Absorbing [7] . Recall that an R – module E is cyclic if E = < x > for x $\in$ E [8]. Recall that an R – module E is a semi simple if soc (E) = (0) [6]. It is well known that an R – module E is a semi simple if and only if soc ($\frac{E}{N}$) = ($\frac{soc(E)+N}{N}$) for each submodule N of E [6 , Ex.12(c)] . The set [N $:_E$ I] = [x $\in$ E : x I $\subseteq$ N } , where N is a submodule of E , and I is an ideal of R [N $:_E$ I] and is a submodule of E containing N. [N $:_E$ R] = N and [I $:_E$ R] = I [9]. Recall that an R – module E is a multiplication if every submodule A of E is of the form A = IE for some ideal I of R , Equivalently A = [A $:_R$ E] E [10]. Recall that an R – module E is a faithful if Ann (E) = { r $\in$ R : rE = (0)} [8].

2. Basic properties of WP – 2 – Absorbing submodules . In this part of the paper , we introduce the definition of WP – 2 – Absorbing submodules and, thus truth some of it's basic properties , examples and characterizations.

Definition .1.

A proper submodule A of an R – module E is said to be Weak Pseudo – 2 – Absorbing (for shorten WP – 2 – Absorbing) submodule of E , if $0 \neq ste \in A$, for s , t $\in$ R , e $\in$ E , implies that se $\in$ A + soc(E) or te $\in$ A + soc (E) or st $\in$ [A + soc (E) $:_R$ E] . And an ideal J of a ring R is said to be WP – 2 – Absorbing ideal of R, if J is a WP – 2 – Absorbing R – submodule of an R – module R.

Example and Remarks .2.

1. In the Z – module Z_{36} , the only essential submodules are $<\bar{2}>$, $<\bar{3}>$, $<\bar{6}>$ and Z_{36} itself thus Soc (Z_{36}) $=<\bar{6}>=$ { $\bar{0}, \bar{6}, \overline{12}, \overline{18}, \overline{24}, \overline{30}$ }

2 . It is clear that the submodules of the Z – module Z_{36} are $<\bar{4}>$, $<\bar{6}>$, $<\bar{9}>$, $<\overline{12}>$ and $<\overline{18}>$ are WP – 2 – Absorbing submodules .

3 . The submodules $<\overline{12}>$ and $<\overline{18}>$ of the Z – module Z_{36} are not weakly – 2 – Absorbing submodules , since $0 \neq 2 . 3 . \bar{2} \in <\overline{12}>$ for 2 , 3 $\in$ Z , $\bar{2} \in Z_{36}$ but $2 . \bar{2} = \bar{4} \notin <\overline{12}>$ and $3 . \bar{2} = \bar{6} \notin <\overline{12}>$ and $2. 3 = 6 \notin [<\overline{12}> :_Z Z_{36}] = 12Z$.

4 . The submodule $<\bar{2}>$, $<\bar{3}>$ of the Z- module Z_{36} are weakly – 2 – Absorbing submodules of Z_{36} because they are weakly prime submodules of Z - Z_{36} .

5 . It is clear that the submodules $<\bar{4}>$, $<\bar{6}>$ and $<\bar{9}>$ of the Z – module Z_{36} are weakly – 2 – Absorbing submodules .

6 . It is clear that every weakly – 2 – Absorbing submodule of an R – module E is a WP – 2 – Absorbing , but not conversely , the following example shows that : -- In the Z – module Z_{36} , the submodule $<\overline{18}>$ is a WP – 2- Absorbing by (2) , but $<\overline{18}>$ is not weakly – 2 – Absorbing submodule by (3) .

7 . It is clear that every weakly prime submodule of an R – module E is a WP – 2 – Absorbing but not conversely. The following example explains that in the Z- module Z_{36} , the submodule $<\bar{4}>$ is a WP – 2 – Absorbing by (2) . But $<\bar{4}>$ is not weakly prime submodule, since $0 \neq 2 . \bar{2} \in <\bar{4}>$ for $2 \in Z$, $\bar{2} \in Z_{36}$, but $\bar{2} \notin <\bar{4}>$ and $2 \notin [<\bar{4}> :_Z Z_{36}] = 4Z$.

8 . In general ,the submodule nZ of the Z – module Z is weakly – 2 – Absorbing if n = 0 , P, P^2 and pq by [7, Rems. And Exs. (1.2.2) (3)] . Hence the submodules nZ of the Z – module Z is a WP – 2 – Absorbing if n = 0 , P , P^2 and pq by (6) .

9 . The submodules 12Z and 18Z of the Z – module Z are not WP – 2 – Absorbing because soc (Z) = (0) [8]. That is , $0 \neq 2 . 3 . 2 \in 12Z$ for 2 , 3 , 2 $\in$ Z , but $2 . 2 \notin 12Z + soc(Z)$ and $3 .2 \notin 12Z + soc(Z)$ and $2 . 3 \notin [12Z + soc(Z) :_Z Z] = 12Z$.

Also , $0 \neq 2 . 3 . 3 \in 18Z$ for 2 , 3 $\in$ Z , but $2.3 \notin 18Z + soc(Z)$ and $3.3 \notin 18Z + soc(Z)$ and $2. 3 \notin [18Z + soc(Z) :_Z Z] = 18Z$.

10 . If A is a WP – 2 – Absorbing submodule of an R – module E , then $[A :_R E]$ need not to be WP – 2 – Absorbing ideal of R . For example the submodule $<\overline{18}>$ of the Z – module Z_{36} is a WP – 2 – Absorbing submodule by (2) , but $[<\overline{18}> :_Z Z_{36}] = 18Z$ is not WP – 2 – Absorbing ideal of Z by (9) .

11 . The intersection of two WP – 2 – Absorbing submodules of an R- module E need not to be WP – 2 – Absorbing submodule .For example the submodules 3Z , 4Z are WP – 2 – Absorbing submodule of the Z – module Z by (8) , but $3Z \cap 4Z = 12Z$ is not WP – 2 – Absorbing submodul by (9) . The following results are characterizations of WP – 2 – Absorbing submodules .

Proposition 3.

A proper submodule A of an R- module E is a WP – 2 – Absorbing submodule of E if and only if for any t , s $\in$ R with $ts \notin [A + soc(E) :_R E]$ we have $[A :_E ts] \subseteq [0 :_E ts] \cup [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$

Proof : ($\Rightarrow$)

Let $e \in [A :_E ts]$ with $ts \notin [A + soc(E) :_R E]$,then tse $\in$ A . If $0 \neq tse$, it follows that either $te \in A + soc(E)$ or $se \in A + soc(E)$, that is either $e \in [A + soc(E) :_E t]$ or $e \in [A + soc(E) :_E s]$. If tse = 0 then $e \in [0 :_E ts]$. Hence $e \in [0 :_E ts] \cup [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$. Therefore $[A :_E st] \subseteq [0 :_E ts] \cup [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$.

($\Leftarrow$) Let $0 \neq tse \in A$ for t, s $\in$ R , e $\in$ E with $ts \notin [A + soc(E) :_R E]$. It follows by hypothesis $e \in [A :_E ts]$ and $e \notin [0 :_E ts]$, implies that $e \in [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$. Hence either $te \in A + soc(E)$ or $se \in A + soc(E)$. Therefore A is aWP – 2 – Absorbing submodule of E .

Proposition 4.

A proper submodule A of an R – module E is a WP – 2 – Absorbing if $0 \neq tsK \subseteq A$ for t , $s \in R$ and K is a submodule of E , implies that either $tK \subseteq A + soc(E)$ or $sK \subseteq A + soc(E)$ or $ts \in [A + soc(E) :_R E]$.

Proof : (⇒)

Let $0 \neq tsK \subseteq A$, for $t, s \in R$, K is a submodule of E . Suppose that $ts \notin [A + soc(E) :_R E]$, $tK \not\subseteq A + soc(E)$ and $sK \not\subseteq A + soc(E)$.Then,there exists $e_1, e_2 \in K$ such that $te_1 \notin A + soc(E)$ and $se_2 \notin A + soc(E)$. Now $0 \neq tse_1 \in A$ and $ts \notin [A + soc(E) :_R E]$, then by proposition (3) we have $e_1 \in [A :_E st] \subseteq [0 :_E ts] \cup [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$. But $e_1 \notin [0 :_E ts]$ and $e_1 \notin [A + soc(E) :_E t]$ .It follows that $e_1 \in [A + soc(E) :_E s]$, that is, $se_1 \in A + soc(E)$. Again $0 \neq tse_2$ and $ts \notin [A + soc(E) :_R E]$ and $e_2 \notin [A + soc(E) :_E s]$, it follow tha $te_2 \in A + soc(E)$. Now, $0 \neq ts(e_1 + e_2) \in A$ and $ts \notin [A + soc(E) :_R E]$, then $(e_1 + e_2) \in [A :_E ts]$ and $(e_1 + e_2) \notin [0 :_E ts]$, it follows by proposition (3) either $(e_1 + e_2) \in [A + soc(E) :_E t]$ or $(e_1 + e_2) \in [A + soc(E) :_E s]$. That is either $t(e_1 + e_2) \in A + soc(E)$ or $s(e_1 + e_2) \in A + soc(E)$. If $t(e_1 + e_2) = te_1 + te_2 \in A + soc(E)$, and $te_2 \in A + soc(E)$, then $te_1 \in A + soc(E)$ which is a contradiction .

If $s(e_1 + e_2) = se_1 + se_2 \in A + soc(E)$ and $se_1 \in A + soc(E)$, then $se_2 \in A + soc(E)$ which is a contradiction.

Hence either $tK \subseteq A + soc(E)$ or $sK \subseteq A + soc(E)$ or $ts \in [A + soc(E) :_R E]$.

(⇐) Trivial , so we omitted it .

Proposition 5.

A proper submodule A of a cyclic R – module E is a WP – 2 – Absorbing if and only if for each $t, s \in R$ with $ts \notin [A + soc(E) :_R E]$ we have $[A :_R tse] \subseteq [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$.

Proof : (⇒)

Let $t, s \in R$,with $ts \notin [A + soc(E) :_R E]$ and let $r \in [A :_R tse]$, it follows that $ts(re) \in A$. If $0 \neq ts(re) \in A$ and A is a WP – 2 – Absorbing and $ts \notin [A + soc(E) :_R E]$, then either $tre \in A + soc(E)$ or $sre \in A + soc(E)$, that is either $r \in [A + soc(E) :_R te]$ or $r \in [A + soc(E) :_R se]$. If $tsre = 0$, implies that $r \in [0 : tse]$. Hence $r \in [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$. Therefore $[A :_R tse] \subseteq [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$.

(⇐) Since E is cyclic , then $E = \langle e_1 \rangle$ for some $e_1 \in E$. Let $0 \neq tse \in A$ for $t, s \in R$, $e \in E$ with $ts \notin [A + soc(E) :_R E]$.Since $e \in E$ then $e = re_1$ for some $r \in R$, that is $0 \neq ts(re_1) \in A$,it follows that $r \in [A :_R tse_1] \subseteq [0 :_R tse_1] \cup [A + soc(E) :_R te_1] \cup [A + soc(E) :_R se_1]$. But $r \notin [0 :_R tse_1]$ (since $0 \neq tsre_1$) , therefore , $r \in [A + soc(E) :_R te_1]$ or $r \in [A +$

soc (E) $:_R$ se_1], it follows that $tre_1 \in A + soc(E)$ or $sre_1 \in A + soc(E)$. That is $te \in A + soc(E)$ or $se \in A + soc(E)$. Therefore A is a WP – 2 – Absorbing submodule of E .

Proposition 6.

A proper submodule N of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq IJL \subseteq N$ for some ideals I , J of R and some submodule L of E implies that either $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$ or $IJ \subseteq [N + soc(E) :_R E]$.

Proof : (⇒)

Let $(0) \neq IJL \subseteq N$ for some ideals I,J of R and some submodule L of E with $IJ \not\subseteq [N + soc(E) :_R E]$. To prove that $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$ .Suppose that $IL \not\subseteq N + soc(E)$ and $JL \not\subseteq N + soc(E)$, that is there exist $a_1 \in I$ and $a_2 \in J$ such that $a_1L \not\subseteq N + soc(E)$ and $a_2L \not\subseteq N + soc(E)$ .Now , $(0) \neq a_1a_2L \subseteq N$, and N is aWP – 2 – Absorbing submodule of E , then by proposition (4) either $a_1 L \subseteq N + soc(E)$ or $a_2L \subseteq N + soc(E)$ or $a_1a_2 \in [N + soc(E) :_R E]$. Since $IJ \not\subseteq [N + soc(E) :_R E]$, there exists $b_1 \in I$ and $b_2 \in J$ such that $b_1b_2 \notin [N + soc(E) :_R E]$.But $(0) \neq b_1b_2L \subseteq N$ and N is a WP – 2 – Absorbing submodule of E , and $b_1b_2 \notin [N + soc(E) :_R E]$, then by proposition (4) either $b_1L \subseteq N + soc(E)$ or $b_2L \subseteq N + soc(E)$.

Now : -- **(1)** If $b_1L \subseteq N + soc(E)$ and $b_2L \not\subseteq N + soc(E)$. Since $(0) \neq a_1b_2L \subseteq N$ and $b_2L \not\subseteq N + soc(E)$ and $a_1L \not\subseteq N + soc(E)$, then by proposition (4) $a_1b_2 \in [N + soc(E) :_R E]$. Since $b_1L \subseteq N + soc(E)$ and $a_1L \not\subseteq N + soc(E)$, we get $(a_1 + b_1) L \not\subseteq N + soc(E)$ .For there more $(0) \neq (a_1 + b_1)b_2 L \subseteq N$ and N is a WP – 2 – Absorbing with $(a_1 + b_1) L \not\subseteq N + soc(E)$, $b_2L \not\subseteq N + soc(E)$, it follows that by proposition (4) $(a_1 + b_1) b_2 = a_1b_2 + b_1b_2 \in [N + soc(E) :_R E]$, but $a_1b_2 \in [N + soc(E) :_R E]$, then $b_1b_2 \in [N + soc(E) :_R E]$, this is a contradiction .

(2) If $b_2L \subseteq N + soc(E)$ and $b_1L \not\subseteq N + soc(E)$, so by similar steps of (1) we get a contradiction .

(3) If $b_1L \subseteq N + soc(E)$ and $b_2L \subseteq N + soc(E)$, since $b_2L \subseteq N + soc(E)$ and $a_2L \not\subseteq N + soc(E)$, we get $(a_2 + b_2)L \not\subseteq N + soc(E)$. But $(0) \neq a_1 (a_2 + b_2) L \subseteq N$ and N is a WP – 2 – Absorbing with $a_1L \not\subseteq N + soc(E)$ and $(a_2 + b_2) L \not\subseteq N + soc(E)$ then , we get $a_1 (a_2 + b_2) \in [N + soc(E) :_R E]$. Since $a_1a_2 \in [N + soc(E) :_R E]$ and $a_1a_2 + a_1b_2 \in [N + soc(E) :_R E]$, it follows that $a_1b_2 \in [N + soc(E) :_R E]$.

Now , $(0) \neq (a_1 + b_1) a_2 \in N$ and $a_2L \not\subseteq N + soc(E)$ and $(a_1 + b_1) L \not\subseteq N + soc(E)$, it follows by proposition (4) $(a_1 + b_1)a_2 = a_1a_2 + b_1a_2 \in [N + soc(E) :_R E]$ and since $a_1a_2 \in [N + soc(E) :_R E]$, we get $b_1a_2 \in [N + soc(E) :_R E]$. Since $(0) \neq (a_1 + b_1) (a_2 + b_2) L \subseteq N$ and $(a_1 + b_1) L \not\subseteq N + soc(E)$ and $(a_2 + b_2) L \not\subseteq N + soc(E)$ then by proposition (4) we have $(a_1 + b_1) (a_2 + b_2) = a_1a_2 + a_1b_2 + b_1a_2 + b_1b_2 \in [N + soc(E) :_R E]$. But $a_1a_2 , b_1a_2 , a_1b_2 \in [N + soc(E) :_R E]$, we get $b_1b_2 \in [N + soc(E) :_R E]$ which is a contradiction . Thus $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$.

($\Leftarrow$) Trivial, so we omittedit

The following corollaries are adirect consequence of proposition (6) .

Corollary 7.

A proper submodule A of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq IJx \subseteq A$ for some ideals I , J of R and $x \in E$, implies that either $Ix \subseteq A + soc(E)$ or $Jx \subseteq A + soc(E)$ or $IJ \subseteq [A + soc(E) :_R E]$.

Corollary 8.

A proper submodule A of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq sIL \subseteq A$ for some $s \in R$ and ideal I of R and some submodule L of E , implies that either $sL \subseteq A + soc(E)$ or $IL \subseteq A + soc(E)$ or $sI \subseteq [A + soc(E) :_R E]$.

Corollary 9.

A proper submodule A of an R – module E is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq sIx \subseteq A$ for some $s \in R$, ideal I of R and some $x \in E$, implies that either $sx \in A + soc(E)$ or $Ix \subseteq A + soc(E)$ or $sI \subseteq [A + soc(E) :_R E]$.

Proposition 10.

Let A be a WP – 2 – Absorbing submodule of an R – module E and B is a submodule of E with $B \subseteq A$ then $\frac{A}{B}$ is a WP – 2 – Absorbing submodule of an R – module $\frac{E}{B}$.

Proof : Let $0 \neq ts(x + B) = stx + B \in \frac{A}{B}$ for $s, t \in R$, $x + B \in \frac{E}{B}$, $x \in E$ .It follows that $tsx \in A$. If $tsx = 0$ then $ts(x + B) = 0$ which is a contradiction . thus $0 \neq tsx \in A$ implies that either $tx \in A + soc(E)$ or $sx \in A + soc(E)$ or $tsE \subseteq A + soc(E)$ .It follow that either $t(x + B) \in \frac{A + soc(E)}{B}$ or $s(x + B) \in \frac{A + soc(E)}{B}$ or $\frac{tsE}{B} \subseteq \frac{A + soc(E)}{B}$. That is either $t(x + B) \in \frac{A}{B} + \frac{A + soc(E)}{B} \subseteq \frac{A}{B} + soc(\frac{E}{B})$ or $s(x + B) \in \frac{A}{B} + \frac{A + soc(E)}{B} \subseteq \frac{A}{B} + soc(\frac{E}{B})$ or $ts\frac{E}{B} \subseteq \frac{A}{B} + \frac{A + soc(E)}{B} \subseteq \frac{A}{B} + soc(\frac{E}{B})$. Hence , $\frac{A}{B}$ is a WP – 2 – Absorbing submodule of an R – module $\frac{E}{B}$.

Proposition .11.

Let A , B be submodules of semi simple R – module E with $B \subseteq A$. If B and $\frac{A}{B}$ are WP – 2 – Absorbing submodules of E , $\frac{E}{B}$ respectively , then A is a WP – 2 – Absorbing submodule of E .

Proof :

Let $0 \neq tsx \notin A$ for $t, s \in R$, $x \in E$, then $0 \neq ts(x + B) = tsx + B \in \frac{A}{B}$. If $0 \neq tsx \in B$ and B is a WP – 2 – Absorbing , implies that either $tx \in B + soc(E) \subseteq A + soc(E)$ or $sx \in B + soc(E) \subseteq A + soc(E)$ or $tsE \subseteq B + soc(E) \subseteq A + soc(E)$ .Thus A is a WP – 2 –

Absorbing submodule of E . Assume that tsx $\notin$ B , it follows that $0 \neq ts(x + B) \in \frac{A}{B}$. But $\frac{A}{B}$ is a WP – 2 – Absorbing submodule of $\frac{E}{B}$ implies that either $t(x + B) \in \frac{A}{B} + soc(\frac{E}{B})$ or $s(x + B)\ \frac{A}{B} + soc(\frac{E}{B})$ or $\frac{tsE}{B} \subseteq \frac{A}{B} + soc(\frac{E}{B})$. Since E is a semi simple then $soc(\frac{E}{B}) = \frac{soc(E)+B}{B}$. It follows that either $t(x + B) \in \frac{A}{B} + \frac{B+soc(E)}{B}$ or $s(t + B) \subseteq \frac{A}{B} + \frac{B+soc(E)}{B}$ or

$\frac{tsE}{B} \subseteq \frac{A}{B} + \frac{B+soc(E)}{B}$. But B $\subseteq$ A , implies that B + soc (E) $\subseteq$ A + soc (E) , hence $\frac{A}{B} + \frac{B+soc(E)}{B} \subseteq \frac{A}{B} + \frac{A+soc(E)}{B}$. Since $\frac{A}{B} \subseteq \frac{A+soc(E)}{B}$ implies that $\frac{A}{B} + \frac{A+soc(E)}{B} = \frac{A+soc(E)}{B}$. that is either $t(x + B) \in \frac{A+soc(E)}{B}$ or $s(x + B) \in \frac{A+soc(E)}{B}$, $\frac{tsE}{B} \subseteq \frac{A+soc(E)}{B}$, it follows that either tx $\in$ A + soc (E) or sx $\in$ A + soc (E) or tsE $\subseteq$ A + soc (E) . Thus A is WP – 2 – Absorbing submodule of E .

Proposition 12.

Let A be a proper submodule of an R – module E with soc (E) $\subseteq$ A. Then A is a WP – 2 – Absorbing submodule of E so if and only if [A $:_E$ I] is a WP – 2 – Absorbing submodule of E for each ideal I of R .

Proof : (⇒)

Let (0) $\neq$ tsB $\subseteq$ [A $:_E$ I] for t , s $\in$ R , B is a submodule of E , then (0) $\neq$ tsIB $\subseteq$ A , implies that either tIB $\subseteq$ A + soc (E) or sIB $\subseteq$ A + soc (E) or tsE $\subseteq$ A + soc (E) . But soc (E) $\subseteq$ A , then A + soc (E) = A . that is either tIB $\subseteq$ A or sIB $\subseteq$ A or tsE $\subseteq$ A. Thus , either tB $\subseteq$ [A $:_E$ I] or sB $\subseteq$ [A $:_E$ I] or tsE $\subseteq$ A $\subseteq$ [A $:_E$ I] . It follows that either tB $\subseteq$ [A $:_E$ I] $\subseteq$ [A $:_E$ I] + soc (E) or sB $\subseteq$ [A $:_E$ I] $\subseteq$ [A $:_E$ I] + soc (E) or tsE $\subseteq$ [A $:_E$ I] $\subseteq$ [A $:_E$ I] + soc (E). Hence , [A $:_E$ I] is a WP – 2 – Absorbing submodule of E .

(⇐) Since [A $:_E$ I] is a WP – 2 – Absorbing subodule for every non zero ideal I of R . Put I = R , we get [A $:_E$ R] = A is a WP – 2 – Absorbing submodule of E .

We need to introduce the following definition.

Definition 13. Let A be a WP – 2 – Absorbing submodule of an R – module E and r, s $\in$ R , e $\in$ E , we say that (r , s , e) is WP – triple zero of A if rse = 0 , re $\notin$ A + soc (E) , se $\notin$ A + soc (E) and rs $\notin$ [A + soc (E) $:_R$ E] .

Proposition 14. If A is a WP – 2 – Absorbing submodule of E with (r , s , e) is a WP – trible zero of A for some r, s $\in$ R , e $\in$ E . Then rsA = (0) .

Proof : Suppose rsA $\neq$ (0) , then rsa $\neq$ 0 for some a $\in$ A. Since (r , s , e) is a WP – triple zero of A then rse = 0 , re $\notin$ A + soc (E) , se $\notin$ A + soc (E) and rs $\notin$ [A + soc (E) $:_R$ E]. Since 0 $\neq$ rsa $\in$ A and A is a WP – 2 – Absorbing submodule of E and rs $\notin$ [A + soc (E) $:_R$ E], then either ra$\in$ A + soc (E) or sa $\in$ A + soc (E).

Now, $0 \neq rs (e + a) = rse + rsa = rsa \in A$, and $rs \notin [A + soc (E) :_R E]$, then either $r(e + a) = re + ra \in A + soc (E)$ or $s(e + a) = se + sa \in A + soc (E)$. If $re + sa \in A + soc (E)$ and $ra \in A + soc (E)$ implies that $re \in A + soc (E)$ contradiction. If $se + sa \in A + soc (E)$ and $sa \in A + soc (E)$, implies that $se \in A + soc (E)$ contradiction . Hence , $rsA = (0)$.

Proposition 15. If A is a WP – 2 – Absorbing submodule of E with (r , s , e) is a WP – triple zero of A for some $r, s \in R$, $e \in E$, then $[A :_R E]re = [A :_R E]se = (0)$.

Proof : Suppose that $[A :_R E]se \neq (0)$ then $yse \neq o$ for some $y \in [A :_R E]$. Since (r , s , e) is a WP – triple zero of A, $rse = 0$ and $re \notin A + soc (E)$, $se \notin A + soc (E)$ and $rs \notin [A + soc (E) :_R E]$. We have $0 \neq yse \in A$ and A is a WP – 2 – Absorbing submodule of E , then either $ye \in A + soc (E)$ or $se \in A + soc (E)$ or $ys \in [A + soc (E) :_R E]$. Now , $0 \neq (r + y) se = rse + yse = yse \in A$ and A is a WP – 2 – Absorbing submodule , then either $(r + y)e = re + ye \in A + soc (E)$ or $se \in A + soc (E)$ or $(r + y)s \in [A + soc (E) :_R E]$.Since $ye \in A + soc (E)$ and if $re + ye \in A + soc (E)$, it follows that $re \in A + soc (E)$ a contradiction . If $(r + y)s = rs + ys \in [A + soc (E) :_R E]$ and $ys \in [A + soc (E) :_R E]$, then $rs \in [A + soc (E) :_R E]$ a contradiction . Thus $[A :_R E]se = (0)$. Similarly we can prove $[A :_R E]re = (0)$.

Proposition 16. If A is a WP – 2 – Absorbing submodule of E with (r , s , e) is a WP – triple zero of A for some $r, s \in R$, $e \in E$. Then $r[A :_R E]e = s[A :_R E]e = (0)$.

Proof : Suppose that $r[A :_R E]e \neq (0)$, then there exists $x \in [A :_R E]$ such that $rxe \neq 0$. But (r , s , e) is a WP – triple zero of A , $rse = 0$, $re \notin A + soc (E)$ or $se \notin A + soc (E)$ and $rs \notin [A + soc (E) :_R E]$.

For $0 \neq rxe \in A$ and A is a WP – 2 – Absorbing submodule of E, then either $re \in A + soc (E)$ or $xe \in A + soc (E)$ or $rx \in [A + soc (E) :_R E]$. Now , $0 \neq r(s + x)e = rse + rxe = rxe \in A$, and A is a WP – 2 – Absorbing submodule , then either $re \in A + soc (E)$ or $(s + x) e = se + xe \in A + soc (E)$ or $r(s + x) = rs + rx \in [A + soc (E) :_R E]$. That is $re \in A + soc (E)$ a contradiction. If $(s + x)e = se + xe \in A + soc (E)$, implies that $se \in A + soc (E)$ a contradiction. If $rs + rx \in [A + soc (E) :_R E]$, implies that $rs \in [A + soc (E) :_R E]$ a contradiction. Thus $r[A :_R E]e = (0)$. In similary way $s[A :_R E]e = (0)$.

As direct consequence of proposition (16) , we get the following corollary :

Corollary 17. If A is a WP – 2 – Absorbing submodule of an R – module E with (r , s , e) is a WP – triple zero of A for some $r, s \in R$, $e \in E$, then $r [A :_R E] A = s [A :_R E]A = (0)$.

Proposition 18. If A is a WP – 2 – Absorbing submodule of E with (r , s , e) is a WP – triple zero of A for some $r, s \in R$, $e \in E$, then $[A :_R E] sA = [A :_R E] rA = (0)$.

Proof : Suppose that $[A :_R E] sA \neq (0)$, then $xsa \neq (0)$ for some $x \in [A :_R E]$, $a \in A$. Since (r , s , e) is a WP – triple zero of A, then $rse = 0$, $re \notin A + soc (E)$, $se \notin A + soc (E)$ and $rs \notin [\notin A + soc (E) :_R E]$. For $0 \neq xsa \in A$, it follows that either $xa \in A + soc (E)$ or $sa \in A + soc (E)$ or $xs \in [A + soc (E) :_R E]$. We have $(r + x) s (a + e) = rsa + rse +$

xsa + xse = xsa $\in$ A (since rse = 0 , and rsa = 0 for proposition (14) and xsa = 0 from proposition (16)). That is 0 $\neq$ (r + x)(a + e) = ra + re + xa + xe $\in$ A , implies that re $\in$ A + soc (E) a contradiction or s(a + e) = sa + se $\in$ A + soc (E) implies that se $\in$ A + soc (E) a contradiction or (r + x)s = rs + xs $\in$[A + soc (E) $:_R$ E], implies that rs $\in$[A + soc (E) $:_R$ E] a contradiction .Thus [A $:_R$ E] sA = (0) .

In similar steps , we can show that [A $:_R$ E]rA = (0) .

Proposition 19. If A is a WP – 2 – Absorbing submodule of E with (r , s , e) is a WP – triple zero of A for some r, s $\in$ R , e $\in$ E , then [A $:_R$ E] [A $:_R$ E] e = (0) .

Proof : Suppose that [A $:_R$ E] [A $:_R$ E] e $\neq$ (0), then 0 $\neq$ xye $\in$ A for some x , y $\in$ [A $:_R$ E]. For (r , s , e) is a WP – triple zero of A, then rse = 0, re $\notin$ A + soc (E) , se $\notin$ A + soc (E) and rs $\notin$ [A + soc (E) $:_R$ E].

Now, 0 $\neq$ xye $\in$ A , implies that either xe $\in$ A + soc (E) or ye $\in$ A + soc (E) or xy $\in$ [A + soc (E) $:_R$ E].

Now , 0 $\neq$ (r + x)(s + y)e = rse + rye + xse + xye = xye $\in$ A (since rse = 0 , rye = 0 , xse = 0 by proposition (16)).It follows that either (r + x)e = re + xe $\in$ A + soc (E), implies that re $\in$ A + soc (E) a contradiction . or (s + y)e = se + ye $\in$ A + soc (E), implies that se $\in$ A + soc (E) a contradiction, or (r + x)(s + y) = rs + ry + xs + xy $\in$[A + soc (E) $:_R$ E], implies that rs $\in$[A + soc (E) $:_R$ E] a contradiction.

Hence [A $:_R$ E] [A $:_R$ E] e = (0) .

Proposition 20. If A is a WP – 2 – Absorbing submodule of E with (r , s , e) is a WP – triple zero of A for some r, s $\in$ R , e $\in$ E , then [A $:_R$ E] [A $:_R$ E]A = (0) .

Proof : By proposition (14) and proposition (19) .

Proposition 21 . Let A be a WP – 2 – Absorbing submodule of E and rsB $\subseteq$ A for some r, s $\in$ R, and some submodule B of E with (r , s , x) is not WP – triple zero of A for every x $\in$ B . If rs $\notin$ [A + soc (E) $:_R$ E], then rx $\in$ A + soc (E) or sx $\in$ A + soc (E) .

Proof : Suppose that (r , s , x) is not WP – triple zero of A for every x $\in$ B and suppose that rB $\not\subseteq$ A + soc (E) and sB $\not\subseteq$ A + soc (E), then $ry_1 \notin$ A + soc (E) or $sy_2 \notin$ A + soc (E) for some $y_1, y_2 \in$ B. If 0 $\neq$ $rsy_1 \in$A with rs $\notin$ [A + soc (E) $:_R$ E] and since $ry_1 \notin$ A + soc (E) then $sy_1 \in$ A + soc (E) (for A is a WP – 2 – Absorbing submodule) . If rsy_1 = 0 and $ry_1 \notin$ A + soc (E) , rs $\notin$ [A + soc (E) $:_R$ E] and (r , s , y_1) is not WP – triple zero of A , we get $sy_1 \in$ A + soc (E). By similar arguments since (r , s , y_2) is not WP – triple zero of A , we get $ry_2 \in$ A + soc (E). Now , rs ($y_1 + y_2$) $\in$ A and (r 1 s , $y_1 + y_2$) is not WP – triple zero of A and rs $\notin$ [A + soc (E) $:_R$ E], we get r ($y_1 + y_2$) $\in$ A + soc (E) or s($y_1 + y_2$) $\in$ A + soc (E) .

If r ($y_1 + y_2$) = $ry_1 + ry_2 \in$ A + soc (E) and $ry_2 \in$ A + soc (E), we get $ry_1 \in$ A + soc (E) is a contradiction .

If $s(y_1 + y_2) = sy_1 + sy_2 \in A + soc(E)$ and $sy_1 \in A + soc(E)$ then $sy_2 \in A + soc(E)$ is a contradiction .

Hence $rB \subseteq A + soc(E)$ or $sB \subseteq A + soc(E)$.

Proposition 22. Let A , B be WP – 2 – Absorbing submodule of E with B is not contained in A and either $soc(E) \subseteq A$ or $soc(E) \subseteq B$. Then $A \cap B$ is a WP – 2 – Absorbing submodule of E .

Proof : It is clear that $A \cap B$ is a proper submodule of B and B is a proper submodule of E , implies that $A \cap B$ is a proper submodule of E. Let $(0) \neq rsL \subseteq A \cap B$ for $r, s \in R$, L is a submodule of E, it follows that $(0) \neq rsL \subseteq A$ and $(0) \neq rsL \subseteq B$. But A, B are WP – 2 – Absorbing submodule of E, then either $rL \subseteq A + soc(E)$ or $sL \subseteq A + soc(E)$ or $rsE \subseteq A + soc(E)$ and $rL \subseteq B + soc(E)$ or $sL \subseteq B + soc(E)$ or $rsE \subseteq B + soc(E)$. Thus , either $rL \subseteq (A + soc(E)) \cap (B + soc(E))$ or $sL \subseteq (A + soc(E)) \cap (B + soc(E))$ or $rsE \subseteq (A + soc(E)) \cap (B + soc(E))$. If $soc(E) \subseteq B$ then $B + soc(E) = B$, it follows that either $rL \subseteq (A + soc(E)) \cap B$ or $sL \subseteq (A + soc(E)) \cap B$ or $rsE \subseteq (A + soc(E)) \cap B$. Again Since $soc(E) \subseteq B$, then by Modular Law $(A + soc(E)) \cap B = (A \cap B) + soc(E)$. Thus either $rL \subseteq (A \cap B) + soc(E)$ or $sL \subseteq (A \cap B) + soc(E)$ or $rsE \subseteq (A \cap B) + soc(E)$. Thus $A \cap B$ is a WP – 2 – Absorbing submodule of E .

Recall that for any submodules A , K a multiplication R – module E with $A = IE$, $B = JE$, for some ideals I, J of R, the product $AB = IJE = IB$. In particular $AE = IEE = IE = A$, and for any $x \in E$, $A = Ix$ [2] .

The following propositions are characterizations of WP – 2 – Absorbing submodules is class of multiplication modules.

Proposition 23. Let E be a multiplication R – module, and A be a proper submodule of E. Then A is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq L_1L_2L_3 \subseteq A$ for some submodules L_1, L_2, L_3 of E implies that either $L_1L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$.

Proof : (⇒) Let $(0) \neq L_1L_2L_3 \subseteq A$ for some submodules L_1, L_2, L_3 of E . But E is a multiplication , then $L_1 = I_1E$, $L_2 = I_2E$, $L_3 = I_3E$ for some ideals I_1, I_2, I_3 of R . That is $(0) \neq L_1L_2L_3 = I_1 I_2 I_3 E \subseteq A$. But A is a WP – 2 – Absorbing submodule of E, then by proposition (6) either $I_1 I_3E \subseteq A + soc(E)$ or $I_2 I_3E \subseteq A + soc(E)$ or $I_1I_2 \subseteq [A + soc(E) :_R E]$ (ie $I_1 I_2E \subseteq A + soc(E)$). It follows that either $L_1L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1 L_2 \subseteq A + soc(E)$.

(⇐) Let $(0) \neq I_1 I_2L \subseteq A$ for I_1, I_2 are ideals of R , L is submodule of E. Since E is a multiplication, then $L = I_3E$ for some ideal I_3 of R. That is $(0) \neq I_1 I_2 I_3 E \subseteq A$. Put $L_1 = I_1E$ and $L_2 = I_2E$, then $(0) \neq L_1L_2L \subseteq A$, it follows by hypothesis that either $L_1L \subseteq A + soc(E)$ or $L_2L \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$. That is either $I_1L \subseteq A + soc(E)$ or $I_2L \subseteq A + soc(E)$ or $I_1I_2E \subseteq A + soc(E)$, (ie $I_1I_2 \subseteq [A + soc(E) :_R E]$. Thus , by proposition (6) A is a WP – 2 – Absorbing submodule of E .

The following corollary is a direct consequence of proposition (23) .

Corollary 24. Let E be a multiplication R – module and A be a proper submodule of E. Then A is a WP – 2 – Absorbing submodule of E if and only if $(0) \neq L_1L_2\, e \subseteq A$ for some submodules L_1, L_2 of E and $e \in E$, implies that either $L_1e \subseteq A + soc(E)$ or $L_2e \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$.

It is well known that if E is a faithful multiplication R – module then soc (E) = soc (R)E [11,coro. (2.14) (i)].

Proposition 25. Let E be a faithful multiplication R – module and A be a proper submodule of E. Then A is a WP – 2 – Absorbing submodule of E if and only if $[A :_R E]$ is a WP – 2 – Absorbing ideal of R .

Proof : (⇒) Let $(0) \neq I_1I_2I_3 \subseteq [A :_R E]$ for I_1, I_2, I_3 are ideals of R , it follows that $(0) \neq I_1I_2I_3E \subseteq A$. But E is a multiplication then $(0) \neq I_1I_2I_3E = L_1L_2L_3 \subseteq A$ by taking $L_1 = I_1E$, $L_2 = I_2E$ and $L_3 = I_3E$. Now since A is a WP – 2 – Absorbing , then by proposition (23) either $L_1L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$. But E is a faithful multiplication then soc (E) = soc (R)E . Thus either $I_1I_3E \subseteq [A :_R E]E + soc(R)E$ or $I_2I_3E \subseteq [A :_R E]E + soc(R)E$ or $I_1I_2E \subseteq [A :_R E]E + soc(R)E$. That is either $I_1I_3 \subseteq [A :_R E] + soc(R)$ or $I_2I_3 \subseteq [A :_R E] + soc(R)$ or $I_1I_2 \subseteq [A :_R E] + soc(R) = [[A :_R E] + soc(R) :_E R]$. Therefore by proposition (6) $[A :_R E]$ is a WP – 2 – Absorbing ideal of R .

(⇐) Let $(0) \neq I_1I_2L \subseteq A$ for I_1, I_2 are ideals of R and L is submodule of E. Since E is a multiplication , then $L = I_3E$ for some ideal I_3 of R . That is $(0) \neq I_1I_2I_3E \subseteq A$, it follows that $(0) \neq I_1I_2I_3 \subseteq [A :_R E]$. But $[A :_R E]$ is a WP – 2 – Absorbing ideal of R , then by proposition (6) either $I_1I_3 \subseteq [A :_R E] + soc(R)$ or $I_2I_3 \subseteq [A :_R E] + soc(R)$ or $I_1I_2 \subseteq [A :_R E] + soc(R)$ .Thus either $I_1I_3E \subseteq [A :_R E]E + soc(R)E$ or $I_2I_3E \subseteq [A :_R E]E + soc(R)E$ or $I_1I_2E \subseteq [A :_R E]E + soc(R)E$. That is either $I_1L \subseteq A + soc(E)$ or $I_2L \subseteq A + soc(E)$ or $I_1I_2E \subseteq A + soc(E)$, (ie $I_1I_2 \subseteq [A + soc(E) :_R E]$. Hence by proposition (6) A is a WP – 2 – Absorbing submodule of E .

It is well known that cyclic R- module is multiplication [10]. We get the following corollary:

Corollary 26. Let E be faithful cyclic R – module and A be a proper submodule of E. Then A is a WP – 2 – Absorbing if and only if $[A :_R E]$ is a WP – 2 – Absorbing ideal of R .

Proposition .27. Let E be a faithful finitely generated multiplication R – module and I be a WP – 2 – Absorbing ideal of R . Then , IE is a WP – 2 – Absorbing submodule of E .

Proof : Let $(0) \neq rI_1K \subseteq IE$ for $r \in R$, I_1 be an ideal of R , K is a submodule of E. It follows that $0 \neq rI_1I_2E \subseteq IE$ f or some ideal I_2 of R. Since E is a finitely generated multiplication , then by[2 coro. of Theo. (9)] we have $0 \neq r I_1I_2 \subseteq I + ann(E) = I$. But I is a WP – 2 – Absorbing , then,by corollary (8) either $rI_2 \subseteq I + soc(R)$ or $I_1I_2 \subseteq I + soc(R)$ or $rI_1 \subseteq [I + soc(R) :_R R] = I + soc(R)$. That is either $rI_2E \subseteq IE + soc(R)E$ or $I_1I_2E \subseteq IE + soc(R)E$ or $rI_1E \subseteq IE + soc(R)E$. Thus either $rK \subseteq IE + soc(E)$

or $I_1K \subseteq IE + soc(E)$ or $r I_1 \subseteq [IE + soc(E) :_R E]$. Therefore ,by corollary (8) IE is a WP – 2 – Absorbing submodule of E .

It is well known that cyclic R – modules are finitely generated [8], we get the following corollary which is a direct consequence of proposition (27)

Corollary 28. Let E be a faithful cyclic R – module , and I be a WP – 2 – Absorbing ideal of R. Then , IE is a WP – 2 – Absorbing submodule of E .

3 . Conclusion . … A new generalization of weakly – 2 – Absorbing submodule was introduced , and many characterizations were given. The definition of WP – triple zero of WP – 2 – Absorbing submodules were introduced. A lot of basic properties of these concepts were established. Among the main new characterizations of WP – 2 – Absorbing submodules are the following :

• A proper submodule A of E is a WP- 2 – Absorbing if and only if for any $t, s \in R$ with $ts \notin [A + soc(E) :_R E]$; we have $[A :_E ts] \subseteq [0 :_E ts] \cup [A + soc(E) :_E t] \cup [A + soc(E) :_E s]$.

• A proper submodule A of E is a WP- 2 – Absorbing if and only if $0 \neq tsK \subseteq A$ for $t, s \in R$ and K is a submodule of E , implies that either $tK \subseteq A + soc(E)$ or $sK \subseteq A + soc(E)$ or $ts \in [A + soc(E) :_R E]$.

• A proper submodule A of a cyclic R – module E is a WP- 2 – Absorbing if and only if for each $t, s \in R$ with $ts \notin [A + soc(E) :_R E]$, we have $[A :_R tse] \subseteq [0 :_R tse] \cup [A + soc(E) :_R te] \cup [A + soc(E) :_R se]$.

• A proper submodule N of E is a WP- 2 – Absorbing if and only if $(0) \neq IJL \subseteq N$ for some ideal I , J of R and submodule L of E implies that either $IL \subseteq N + soc(E)$ or $JL \subseteq N + soc(E)$ or $IJ \subseteq [N + soc(E) :_R E]$.

• If A is a WP- 2 – Absorbing submodule of E with (r , s , e) is a WP – triple zero of A for some $t, s \in R$, $e \in E$. Then , $rsA = (0)$, $[A :_R E] re = (0)$, $r [A :_R E] e = (0)$, $r [A :_R E] A = (0)$, $[A :_R E] sA = (0)$ and $[A :_R E][A :_R E] A = (0)$.

• A proper submodule A of multiplication module E is a WP- 2 – Absorbing if and only if $(0) \neq L_1L_2L_3 \subseteq A$ for some submodules L_1, L_2, L_3 of E implies that either $L_1L_3 \subseteq A + soc(E)$ or $L_2L_3 \subseteq A + soc(E)$ or $L_1L_2 \subseteq A + soc(E)$.

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