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Approximation of Fixed Points of Nearly Non-expansive Mapping

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Abstract
The aim of this paper is to prove some weak and strong convergence results of a novel modified Picard-S hybrid iteration to converge to fixed point involving nearly non-expansive mapping in the setting of uniformly convex Banach space. Our results generalize and improve the known results of the existing literature.

Keywords: Picard-S hybrid iteration, fixed point, nearly non-expansive mapping, uniformly convex Banach space.

تقريب النقاط الثابتة لتطبيق اللامتمددة التقريبي

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الخلاصة:
هدف البحث هو أثبات بعض نتائج التقارب الضعيف والقوي لتكرار معدل جديد يدعى (Picard–S– hybrid) لتقريب الى نقطة ثابتة لتطبيق اللامتمددة التقريبي (nearly non–expansive) في فضاء بناخ المحدب بشكل موحد. نتائجنا تعمم وتحسن النتائج المعروفة للادبيات الموجودة.

1. Introduction

The first fixed point approximation method is Picard's iteration method proposed by Banach [1] in his renowned results Banach principle. Approximation theory provides very helpful tools to finding the fixed points of a problem. There are several problems in such various fields as finances, engineering, economics, informatics, chemistry, biology and physics which are not solved exactly by renowned methods. So, many authors worked in this direction and proposed different fixed point approximation methods for different kinds of mappings to find the solution of such problems, see [2, 3, 4].

On the other hand, Goebel and Kirk in [5] proposed the class of asymptotically non-expansive mappings as a generalization of the class of non-expansive mappings. The asymptotic fixed point theory has an essential role in non-linear functional analysis [6]. A branch of this theory attached to asymptotically non-expansive self and non-self mappings in uniformly convex Banach space, using modified Mann, Isikawa and three step iterations have been evolved by many authors see, [5 -15] in Banach space.

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In 2007, Agrawal et al. [16] proposed a novel iteration for approximation of fixed points of non-Lipschitzain nearly asymptotically non-expansive in a Banach space. It is defined as follows:

$$a_1 = a \in V$$
$$a_{n+1} = (1-\beta_n)T^n a_n + \beta_n T^n y_n$$
$$y_n = (1-\delta_n)a_n + \delta_n T^n a_n,$$

where $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in (0,1).

In 2014, Faik et al. [17] proposed a novel iteration named Picard-S hybrid iteration can be utilized to approximate fixed point of contraction mappings. As well, showed that novel iteration converges faster than CR iteration, Picard, Mann, Ishikawa, Noor, SP, S and some other iterations in the existing literature. It is defined as follows:

$$a_1 = a \in V$$
$$a_{n+1} = Tz_n$$
$$z_n = (1-\beta_n)Ta_n + \beta_n Ty_n$$
$$y_n = (1-\delta_n)a_n + \delta_n Ta_n,$$

where $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in (0,1) accomplishing condition $\sum_{n=1}^{\infty}\beta_n\delta_n(1-\delta_n) = \infty$.

It has been shown that a three-step iteration gives better numerical results than two-step and one-step iterations. Therefore, we conclude that three-step iteration plays an important role in resolving diverse problems which show in pure and applied sciences. These realities motivated us to study a class of three-step modified Picard-S hybrid iteration in uniformly convex Banach space. The proposed iteration is as follows:

$$a_1 = a \in V$$
$$a_{n+1} = T^n z_n$$
$$z_n = (1-\beta_n)T^n a_n + \beta_n T^n y_n$$
$$y_n = (1-\delta_n)a_n + \delta_n T^n a_n \qquad \ldots (1)$$

where $\{\beta_n\}$ and $\{\delta_n\}$ are real sequences in (0,1) accomplishing condition $\sum_{n=1}^{\infty}\beta_n\delta_n(1-\delta_n) = \infty$.

In this paper, some weak and strong convergence theorems for a novel proposed iteration (1) of nearly non-expansive mapping in the setting of uniformly convex Banach space are proved. As well, a numerical example to elucidate our work is provided. The results presented in this paper extend and improved some previous well known results from the literature e.g. [16 -19].

2. Preliminaries

We now recall several preliminary notions, definitions and lemmas which are beneficial in proving our main results.

Let V be a non-empty convex subset of a real Banach space M and T be a self-mapping of V with one fixed point set $F(T)$. A mapping $T: V \to V$ is named Lipschitzain if for all $n \in N$, there is a constant $l_n > 0$ such as

$$\|T^n a - T^n b\| \le l_n \|a - b\|, \forall\, a, b \in V, n \in N.$$

A Lipschitzain mapping T is named uniformly L-Lipschitzain if $l_n = L$, $\forall\, n \in N$ and Asymptotically non-expansive mapping [20] if $l_n \ge 1$, $\forall\, n \in N$ with $\lim_{n\to\infty} l_n = 1$.

Definition 2.1 [21]: Assume that V is a non-empty convex subset of a real Banach space M. A mapping $T: V \to V$ is named to be non-expansive mapping if

$$\|T^n a - T^n b\| \leq \|a - b\|, \forall\, a, b \in V, n \in N.$$

In 2005, Sahu [22] established the class of nearly Lipschitzain which is an important generalization of the class of Lipschitzain mapping.

Definition 2.2 [23]: Let V be a non-empty convex subset of a real Banach space M. A mapping $T: V \to V$ is named to be nearly uniformly L-Lipschitzain with respect to $\{\omega_n\}$[$\{\omega_n\}$ be a sequence in [0,1] with $\lim_{n\to\infty} \omega_n = 0$], $\exists L \geq 0$ such as

$$\|T^n a - T^n b\| \leq l_n(\|a - b\| + \omega_n), \forall\, a, b \in V, n \geq 1.$$

Notice that, the infimum of the constant l_n is denoted by $\eta(T^n) \leq L, n \in N$ and is called nearly Lipschitz constant.

A nearly uniformly L-Lipschitzain mapping T with $[\{\omega_n\}, \eta(T^n)]$ is said to be**:**

- Nearly non-expansive if $\eta(T^n) = 1, \forall n \geq 1$.
- Nearly uniformly L-Lipschitzain if $\eta(T^n) \leq L, n \in N$**.**

It easy to remark that every non-expansive is asymptotically non-expansive mapping with sequence $l_n = 1$ and very asymptotically non-expansive mapping is uniformly L-Lipschitzain with $L = sup_{n\in V} l_n$. Every asymptotically non-expansive mapping is nearly non-expansive mapping if V is a bounded domain of an asymptotically non-expansive T.

Example 2.3: Let $M = R, V = \left[0, \frac{1}{2}\right]$ and $T: V \to V$ be a mapping defined by

$$Ta = \begin{cases} \frac{1}{4}\ if\ a\ \in [0, \frac{1}{4}] \\ 0\ if\ a\ \in \left(\frac{1}{4}, \frac{1}{2}\right]. \end{cases}$$

Clearly that T is discontinuous and non-Lipschitzain mapping. However, it is a nearly non-expansive mapping.

Definition 2.4 [24]: The function $\vartheta_M: [0, 2] \to [0,1]$ is named modulus of convexity of M if $\vartheta_M = \inf\{1 - \frac{\|a+b\|}{2}, \|a\| = 1, \|b\| = 1, \|a - b\| = \mathrm{q}\}$.

A Banach space M is named uniformly convex [25] if $\vartheta_M(0) = 0$ and $\vartheta_M(\mathrm{q}) > 0, \forall\, \mathrm{q} \in [0, 2]$.

Definition 2.5 [26]: A Banach space M is named satisfy Opial property if for each sequence $\{v_n\}$ in M, converging weakly to $x \in M$, we have $\lim_{n\to\infty} sup\|v_n - x\| \leq \lim_{n\to\infty} sup\|v_n - y\|, \forall y \in M$ such as $x \neq y$.

Definition 2.6 [27]: A mapping $T: V \to V$ is named to be demiclosed at zero, if for any sequence $\{s_n\}$ in V, the condition $\{s_n\}$ converges weakly to $s \in V$ and $\{Ts_n\}$ converges strongly to 0 imply that $Ts = 0$.

Definition 2.7 [28]: A mapping $T: V \to V$ is named condition (Z) if there is a non-decreasing function $\gamma: [0, \infty) \to [0, \infty)$ with $\gamma(0) = 0$ and $\gamma(e) > 0, e > 0$ such as $d(a, Ta) \geq \gamma(d(a, F(T))), \forall a \in V$.

Lemma 2.8 [20]: Let $\{\rho_n\},\{\tau_n\}$ and $\{\beta_n\}$ be sequences of non-negative real numbers accomplishing the inequality

$$\rho_{n+1} \leq (1+\beta_n)\rho_n + \tau_n, \forall\, n \in N.$$

If $\sum_{n=0}^{\infty} \beta_n < \infty$ and $\sum_{n=0}^{\infty} \tau_n < \infty$, thus:

- $\lim_{n\to\infty} \rho_n$ exists.
- If $\{\rho_n\}$ has a subsequence that converges strongly to zero, hence $\lim_{n\to\infty} \rho_n = 0$.

Lemma 2.9 [29]: Let M be a real uniformly convex Banach space (UCBS) and $0 \leq r \leq k_n \leq e \leq 1, \forall n \in N$. Let $\{r_n\},\{e_n\}$ be sequences in M such as $\lim_{n\to\infty} \sup\|r_n\| \leq \varphi$, $\lim_{n\to\infty} \sup\|e_n\| \leq \varphi$ and $\lim_{n\to\infty} \|(1-k_n)r_n + k_n e_n\| = \varphi$ hold for some $\varphi \geq 0$. Therefore, $\lim_{n\to\infty} \|r_n - e_n\| = 0$.

3. Main Results

In this section, we prove several strongly and weakly convergence theorems of a modified Picard-S hybrid iteration for nearly non-expansive mapping in real uniformly convex Banach space.

Theorem 3.1: Let V be a non-empty convex subset of a real UCBS. Let $T: V \to V$ be a uniformly L-Lipschitzain, nearly non-expansive mapping. Let $\{a_n\}$ be the modified Picard-S hybrid iteration defined by (1). Therefore, the following:

- $\lim_{n\to\infty} \|a_n - h\|$ exists $\forall h \in F(T)$.
- $\lim_{n\to\infty} \|a_n - T^n a_n\| = 0$.
- $\{a_n\}$ converges strongly to h.

Proof: firstly, we obtain $F(T) \neq \emptyset$ and $h \in F(T)$, we have

$$\begin{aligned}\|a_{n+1} - h\| &= \|T^n z_n - h\| \\ &\leq \|z_n - h\| + \omega_n\end{aligned}$$

$$\begin{aligned}\|z_n - h\| &= \|(1-\beta_n)T^n a_n + \beta_n T^n y_n - h\| \\ &\leq (1-\beta_n)\|a_n - h\| + \beta_n\|y_n - h\| + (1-\beta_n)\omega_n + \beta_n\omega_n \\ &\leq (1-\beta_n)\|a_n - h\| + \beta_n\|y_n - h\| + \omega_n\end{aligned}$$

$$\begin{aligned}\|y_n - h\| &= \|(1-\delta_n)a_n + \delta_n T^n a_n - h\| \\ &\leq (1-\delta_n)\|a_n - h\| + \delta_n\|a_n - h\| + \delta_n\omega_n \\ &\leq \|a_n - h\| + \delta_n\omega_n.\end{aligned}$$

So, we get

$$\begin{aligned}\|a_{n+1} - h\| &\leq (1-\beta_n)\|a_n - h\| + \beta_n[\|a_n - h\| + \delta_n\omega_n] + 2\omega_n \\ &\leq \|a_n - h\| + (1-\beta_n\delta_n)\omega_n.\end{aligned}$$

From Lemma 2.8, $\lim_{n\to\infty} \|a_n - h\|$ exists, $\forall h \in F(T)$.

Now, to prove $\lim_{n\to\infty} \|a_n - T^n a_n\| = 0$. From firstly, $\lim_{n\to\infty} \|a_n - h\|$ exists and $\{a_n\}$ is bounded, presume that $\lim_{n\to\infty} \|a_n - h\| = \alpha$.

We have,

$$\begin{aligned}\lim_{n\to\infty} sup\|y_n - h\| &\leq \lim_{n\to\infty} sup\|a_n - h\| \\ &= \alpha.\end{aligned}$$

And

$$\begin{aligned}\lim_{n\to\infty} sup\|Ta_n - h\| &\leq \lim_{n\to\infty} sup\|a_n - h\| \\ &= \alpha.\end{aligned}$$

So,

$$
\begin{aligned}
\|a_{n+1}-h\| &= \|T^n z_n - h\| \\
&\le (1-\beta_n)\|a_n-h\| + \beta_n\|y_n-h\| + \omega_n \\
&\le (1-\beta_n)\|a_n-h\| + \beta_n\|y_n-h\|.
\end{aligned}
$$

It follows that,

$$
\begin{aligned}
\|a_{n+1}-h\| - \|a_n-h\| &\le \frac{\|a_{n+1}-h\| - \|a_n-h\|}{\beta_n} \\
&\le \|a_n-h\| + \|y_n-h\|
\end{aligned}
$$

we get,

$$\|a_{n+1}-h\| \le \|y_n-h\|,$$

therefore,

$$\alpha \le \lim_{n\to\infty} inf \|y_n-h\|,$$

hence,

$$\lim_{n\to\infty}\|y_n-h\| = \alpha$$

$$\lim_{n\to\infty}((1-\delta_n)\|a_n-h\| + \delta_n\|T^n a_n - h\|) = \alpha.$$

By Lemma 2.9, we obtain that

$$\lim_{n\to\infty}\|a_n - T^n a_n\| = 0.$$

Lastly, T is uniformly L-Lipschitzain mapping we have,

$$
\begin{aligned}
\|a_n - Ta_n\| &\le \|a_n - a_{n+1}\| + \|a_{n+1} - T^{n+1}a_{n+1}\| + \|T^{n+1}a_{n+1} - T^{n+1}a_n\| \\
&\quad + \|T^{n+1}a_n - Ta_n\| \\
&\le \|a_n - a_{n+1}\| + \|a_{n+1} - T^{n+1}a_{n+1}\| + L\|a_{n+1}-a_n\| + L\|a_n - T^n a_n\|.
\end{aligned}
$$

Since, T is uniformly continuous and $\lim_{n\to\infty}\|a_n - T^n a_n\| = 0$. So,

$$\lim_{n\to\infty}\|a_n - Ta_n\| = 0.$$

By the compactness there is a subsequence $\{a_{nj}\}$ of $\{a_n\}$ such as $\lim_{n\to\infty} a_{nj} = h^*$. Since T is continuous so, $h^* \in F(T)$. We conclude that $\lim_{n\to\infty} a_n = h^* \in F(T)$.

Theorem 3.2: Presume that all the conditions of Theorem 3.1 are accomplished. Therefore, the sequence $\{a_n\}$ converges strongly to a fixed point of T iff $\lim_{n\to\infty}\inf d(a_n, F(T)) = 0$ where $d(a_n, F(T) = \inf\{d(a,h); h \in F(T)\}$.

Proof: Necessity is obvious. Conversely, presume that $\lim_{n\to\infty}\inf d(a_n, F(T)) = 0$. From Theorem 3.1 that $\lim_{n\to\infty}\|a_n - h\|$ exists $\forall h \in F(T)$. Therefore, by the hypothesis $\lim_{n\to\infty} d(a_n, F(T)) = 0$. Now, we show that $\{a_n\}$ is Cauchy sequence in M. We have,

$$\|a_{n+1}-h\| \le \|a_n-h\| + (1-\beta_n\delta_n)\omega_n$$

We set $Q_n = (1-\beta_n\delta_n)\omega_n$. $\forall\, n,j \in N, j > n \ge 1$. Then, we obtain

$$
\begin{aligned}
\|a_n-h\| &\le \left\|a_{j-1}-h\right\| + Q_{j-1} \\
&\le \left\|a_{j-1}-h\right\| + Q_{j-1} + Q_{j-2} \\
&\quad . \\
&\quad . \\
&\le \|a_n-h\| + \textstyle\sum_{e=0}^{j-1} Q_e \\
&\le \|a_n-h\| + \textstyle\sum_{e=0}^{\infty} Q_e.
\end{aligned}
$$

Since, $\lim_{n\to\infty} d\big(a_n, F(T)\big) = 0$ and $\sum_{e=0}^{\infty} Q_e < \infty$. There is a positive integer n_0 such as

$$d\big(a_n, F(T)\big) < \frac{\epsilon}{4}, \quad \sum_{e=n_0}^{\infty} Q_e < \frac{\epsilon}{4}.$$

Hence, there is $h^* \in F(T)$, such as

$$\left\|a_{n_0} - h^*\right\| < \frac{\epsilon}{4}, \quad \sum_{e=n_0}^{\infty} Q_e < \frac{\epsilon}{4}.$$

Then, $\forall j, n \geq n_0$. We get,

$$\begin{aligned}\left\|a_j - a_n\right\| &\leq \left\|a_j - h^*\right\| + \|a_n - h^*\| \\ &\leq \left\|a_{n_0} - h^*\right\| + \textstyle\sum_{e=n_0}^{\infty} Q_e + \left\|a_{n_0} - h^*\right\| + \sum_{e=n_0}^{\infty} Q_e \\ &= 2\big(\left\|a_{n_0} - h^*\right\| + \textstyle\sum_{e=n_0}^{\infty} Q_e\big) \\ &< 2\left(\frac{\epsilon}{4} + \frac{\epsilon}{4}\right) \\ &= \epsilon.\end{aligned}$$

So, $\{a_n\}$ is Cauchy sequence in M.

Presume that $\lim_{n\to\infty} a_n = h^*$, we will show that $h^* \in F(T)$. Since, V is closed subset of Banach space, therefore $h^* \in V$. Also, $F(T)$ is closed of V and $\lim_{n\to\infty} d(a_n, F(T)) = 0$, we obtain $h^* \in F(T)$. Hence, $\{a_n\}$ converges strongly to a fixed point of T.

Theorem 3.3: Presume that all the conditions of Theorem 3.1 are accomplished. Therefore, $\lim_{n\to\infty} \inf d\big(a_n, F(T)\big) = \lim_{n\to\infty} \sup d\big(a_n, F(T)\big) = 0$ if $\{a_n\}$ converges to a unique point of T.

Proof: Let $h \in F(T)$. Since, $\{a_n\}$ converges to h, $\lim_{n\to\infty} d(a_n, h) = 0$. So, $\forall \varepsilon > 0, \exists n \in N$ such as $d(a_n, h) < \varepsilon$, $\forall n \geq n_1$. Taking the infimum over $h \in F(T)$, we obtain $d\big(a_n, F(T)\big) < \varepsilon$, $\forall n \geq n_1$. This intends $\lim_{n\to\infty} d(a_n, F(T)) = 0$. Therefore, we obtain
$\lim_{n\to\infty} \inf d\big(a_n, F(T)\big) = \lim_{n\to\infty} \sup d\big(a_n, F(T)\big)$.
This completes the proof.

Theorem 3.4: Presume that all the conditions of Theorem 3.1 are accomplished. Therefore, let T be a mapping accomplishing condition (Z). Therefore, the sequence $\{a_n\}$ converges strongly to a fixed point of T.

Proof: We showed in Theorem 3.1 that $\lim_{n\to\infty} \|a_n - h\|$ exists $\forall h \in F(T)$ and $\lim_{n\to\infty} \|a_n - T^n a_n\| = 0$.
From condition (Z)

$$\lim_{n\to\infty} \gamma(d(a_n, F(T)) \leq \lim_{n\to\infty} \gamma d(a_n, Ta_n)) = 0.$$

Since, function $\gamma: [0, \infty) \to [0, \infty)$ is a non − decreasing function accomplishing $\gamma(0) = 0$ and $\gamma(\beta) > 0, \forall \beta \in (0, \infty)$, we obtain $\lim_{n\to\infty} (a_n, F(T)) = 0$.
Therefore, $\{a_n\}$ converges strongly to a fixed point of T.

Theorem 3.5: Let V be a non-empty convex subset of a real UCBS which is satisfying Opials condition. Let $T: V \to V$ be a uniformly L-Lipschitzain, nearly non-expansive mapping. Let $\{a_n\}$ be the modified Picard-S hybrid iteration defined by (1). If the mapping $I - T$ [where I is the identity mapping] is demiclosed, therefore, $\{a_n\}$ converges weakly to a fixed point of T.

Proof: From Theorem 3.1 that $\lim_{n\to\infty}\|a_n - h\|$ exists and bounded. Since M is uniformly convex, then every bounded subset of M is weakly compact. Hence, there is a subsequence $\{a_{nj}\} \subset \{a_n\}$ such as $\{a_{nj}\}$ converges weakly to $h^* \in V[\, h \in F]$. From Theorem 3.1, we get

$$\lim_{n\to\infty}\|a_{nj} - h\| = 0.$$

Since the mapping $I - T$ is demiclosed at zero, then $Th^* = h^*$ that means $h^* \in F$. Lastly, let as show that $\{a_n\}$ converges weakly to $h^* \in F$.

Presume on contrary $\exists\, \{a_{nz}\} \subset \{a_n\}$ such as $\{a_{nz}\}$ converges weakly to $p^* \in V$ and $p^* \neq h^*$. Therefore by the similar method as given above, we can also show that $p^* \in F$. From Theorem 3.1 that $\lim_{n\to\infty}\|a_n - h^*\|$ and $\lim_{n\to\infty}\|a_n - p^*\|$ exist. By Opials condition:

$$\begin{aligned}\lim_{n\to\infty}\|a_n - h^*\| &= \lim_{n\to\infty}\|a_{nj} - h^*\| \\ &< \lim_{n\to\infty}\|a_{nj} - p^*\| \\ &= \lim_{n\to\infty}\|a_n - p^*\| \\ &= \lim_{n\to\infty}\|a_{nz} - p^*\| \\ &< \lim_{n\to\infty}\|a_{nz} - h^*\| \\ &= \lim_{n\to\infty}\|a_n - h^*\|.\end{aligned}$$

This is a contradiction, so $p^* = h^*$. Hence, $\{a_n\}$ converges weakly to a fixed point of T.

Example 3.5: Let $T: [0.1] \to [0,1]$ defined by $Ta = \frac{a}{2}, \forall a \in [0,1]$ be any mapping. Choose $\beta_n = \delta_n = 0.25$ for each $n \in N$ with initial value $a_n = 0.9$. T is a nearly non-expansive mapping. The modified Picard-S hybrid iteration converges to the fixed point 0. It is obvious of Table (1) and Figure (1).

Table 1: Numerical results for 10 steps.

n	***Modified Picard-S hybrid iteration***
0	0.9
1	0.2180
2	0.0528
3	0.0128
4	0.0031
5	0.0007
6	0.0002
7	**0.0000**
8	0.0000
9	0.0000
10	0.0000

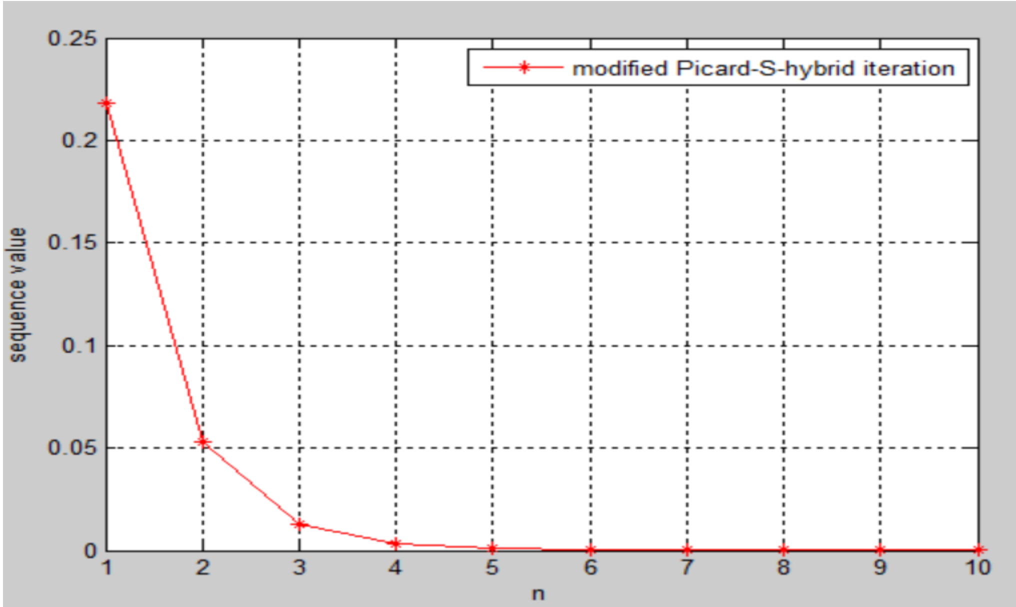

Figure 1: Convergence behaviors for 10 steps.

4. Conclusions

In this paper, a novel iteration named modified Picard-S hybrid iteration is introduced. As well, several strongly and weakly convergence theorems of nearly non-expansive mapping in uniformly convex Banach space are proved. Our results generalized and unified the results declared by many others.

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