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Some results on IS-algebras
بعض النتائج حول جبورIS

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Abstract

In this paper we study IS- algebra , subIS- algebra , IS-algebra homomorphism and congruence relations on IS-algebra , and prove some results about this .

المستخلص

في هذا البحث درسنا جبور IS و التشاكل الزمري والعلاقات على جبورIS وبرهنا العديد من النتائج المتعلقة بهذا الموضوع .

1.Introduction

The notion of BCK-algebras was proposed by Y.Imai and K.Iseki [1] in 1966 in the same year K.Iseki [2] introduced the notion of a BCI- algebras which is a generalization of a BCK-algebras for the general development of BCK / BCI- algebras , the ideal theory plays an important role in 1993 Y.B.Jun et al. [6] introduced a new class of algebras related to BCI- algebras and semigroups called a BCI- semigroup . from now on, we rename it as an IS- algebra for the convenience of study .

2. preliminary

we review some definitions and properties that will be useful in our results.

Definition **2.1** A ***Semigroup*** is an ordered pair $(X,\cdot)$,where X is a non empty set and " ." is an associative binary operation on X . [3]

Definition **2.2** A *BCI- algebra* is triple (X ,* ,0) where X is a non empty set "*" is binary operation on X , $0 \in X$ is an element such that the following axioms are satisfied for all $x, y, z \in X$:

1) $((x * y) * (x * z)) * (z * y) = 0$,
2) $(x * (x * y)) * y = 0$,
3) $x * x = 0$,
4) $0 * x = 0$
if $x * y = 0$ and $y * x = 0$ then $x = y$, $\forall x, y, z \in X$.[11]

Definition **2.3** An ***IS-algebra*** is a non empty set with two binary operation "*" and "." and constant 0 satisfying the axioms :

1. (X, ∗, 0) is a *BCI-algebra.*
2. (X, .) is a *semigroup,*
3. $x.(y * z) = (x.y) * (x.z)$ *and* $(x * y).z = (x.z) * (y.z)$, for all $x, y, z \in X$. [9]

Example **2.4** let X={0,a,b,c} define "*" operation and multiplication "." by the following tables:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	0	0	0

Then by routine calculations we can see that X is an ***IS-algebra***.

***Example* 2.5** let X={0,a,b,c} define "*" operation and multiplication "." by the following tables:

*	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

.	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	b	c
c	0	0	c	b

Then by routine calculations we can see that X is an ***IS-algebra***.

***Example* 2.6** let X={0,a,b,c} define "*" operation and multiplication "." by the following tables:

*	0	a	b	c
0	0	0	b	b
a	a	0	c	b
b	b	b	0	0
c	c	b	a	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	c	b

It is easy to prove that X is an ***IS-algebra***.

***Example* 2.7** let X={0,a,b,c} define "*" operation and multiplication "." by the following tables:

*	0	a	b	c
0	0	0	c	b
a	a	0	c	b
b	b	b	0	c
c	c	c	b	0

.	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	0	b	c

Then by routine calculations we can see that X is an ***IS-algebra***.

***Example* 2.8** let X={0,a,b,c} define "*" operation and multiplication "." by the following tables:

*	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	a	0
b	b	b	0	0	0
c	c	c	c	0	0
d	d	d	d	d	0

.	0	a	b	c	d
0	0	0	0	0	0
a	0	0	0	0	0
b	0	0	0	0	b
c	0	0	0	b	c
d	0	a	b	c	d

Then X is an ***IS-algebra***.

***Remark* 2.9** let X be an IS-algebra then we have
1) $0x = x0 = 0$
2) $x \leq y$ implies that $xz \leq yz \quad and \quad zx \leq zy \quad \forall x, y, z \in X$.

Definition* 2.10** let $(S, \cdot)$ be a semigroup P a non empty set proper subset of S is said to be a ***subsemigroup if $(P, \cdot)$ is semigroup .[3]

Definition* 2.11** A non empty subset S of X with binary operation "*" and " **.**" is called ***subIS-algebra of X if it satisfies the following condition :

1) $x*y\in S \quad \forall \; x,y\in S$.
2) $xy\in S \quad \forall \; x,y\in S$.

Definition* 2.12** Let X and Y be IS-algebra a mapping $f:X\to Y$ is called a ***IS-algebra homomorphism (briefly ***homomorphism***) if $f(x*y)=f(x)*f(y)$ and $f(xy)=f(x)f(y)$ for all $x,y\in X$.

Let $f:X\to Y$ IS-algebra homomorphism . then the set $\{x\in X: f(x)=0\}$ is called ***the kernel of f*** , and denote by *ker f* . moreover, the set $\{f(x)\in Y: x\in X\}$ is called ***the image of f*** and denote by ***Im f*** .

***Definition* 2.13** Let X,Y be a IS-algebra and $f:X\to Y$ IS-algebra homomorphism then :
1) f is a ***monomorphism*** iff one to one homomorphism .
2) f is an ***epimorphism*** iff onto homomorphism .
3) f is an ***isomorphism*** iff bijective homomorphism .

***Definition* 2.14** A BCI algebra is said to be commutative if $x*(x*y)=y*(y*x) \quad \forall \; x,y\in X$.

***Definition* 2.15** Let X be a IS-algebra and let ρ be binary relation on X then :

1) ρ is right (left) compatible if when every $(x,y)\in\rho$ then $(x*z,y*z)\in\rho \; [(z*x,z*y)\in\rho]$ and $(x\cdot z,y\cdot z)\in\rho \; [(z\cdot x,z\cdot y)\in\rho]$
$\forall \; x,y,z\in X$.
2) ρ is compatible if $(x,y)\in\rho$ and $(u,v)\in\rho$ imply
$(x*u,y*v)\in\rho$ *and* $(x\cdot u,y\cdot v)\in\rho \; \forall x,y,u,v\in X$.
3) A compatible equivalence relation is called a congruence relation .[10]

***Remark* 2.16**
$x\rho=\{y\in X:(x,y)\in\rho\}$ and $X/\rho=\{x\rho : x\in X\}$.

3. Main Results

In this section , we find some results about subIS-algebra , IS-algebra homomorphism and congruence relation on IS-algebra .

***Proposition* 3.1** Let *A* and *B* are subIS-algebra of X then $A\cap B$ is subIS-algebra of X .

Proof:
Let *A* and *B* be subIS-algebra of X ,
and let $x,y\in A\cap B$. Then
$x,y\in A$ *and* $x,y\in B$
so $x*y\in A$ *and* $x*y\in B$ [since A , B are subIS-algebra]
then $x*y\in A\cap B$
Now, *let* $x,y\in A\cap B$
then $x,y\in A$ *and* $x,y\in B$
so $xy\in A$ *and* $xy\in B$ [since A , B are subIS-algebra]
therefore $xy\in A\cap B$
Hence $A\cap B$ is a subIS-algebra .

***Proposition* 3.2** Let A and B are subIS-algebra of X then $A \bigcup B$ is a subIS-algebra *If* $A \subseteq B$ *or* $B \subseteq A$.

Proof:

Suppose that A and B are subIS-algebra of X , and $x, y \in A \bigcup B$

if $A \subseteq B$ then $A \bigcup B = B$

so $A \bigcup B$ is a subIS-algebra . [since B is a subIS-algebra]

if $B \subseteq A$ then $A \bigcup B = A$

so $A \bigcup B$ is a subIS-algebra . [since A is a subIS-algebra]

***Lemma* 3.3** Let $f : \mathrm{X} \to \mathrm{Y}$ be a IS-algebra homomorphism then $\ker f$ is a subIS-algebra .

Proof:

Let $x, y \in \ker f$. *Then* $f(x) = 0$ *and* $f(y) = 0$

$f(x * y) = f(x) * f(y) = 0 * 0 = 0$ [since f is a homomorphism]

so $x * y \in \ker f$, *also*

$f(xy) = f(x) f(y) = 0$,

Therefore $xy \in \ker f$

Hence $\ker f$ is a subIS-algebra .

***Lemma* 3.4** Let $f : \mathrm{X} \to \mathrm{Y}$ be a IS-algebra homomorphism then:

1) $f(0) = 0$

2) if $x \le y$ *then* $f(x) \le f(y)$

3) if $x \wedge y = x * (x * y)$ *then* $f(x \wedge y) = f(x) \wedge f(y)$.

Proof:

1) let $x \in \mathrm{X}$ $f(0) = f(x * x) = f(x) * f(x) = 0$.

2) let $x \le y \to x * y = 0$ *then* $f(x * y) = f(0) = 0$

$f(x * y) = 0 \to f(x) * f(y) = 0 \Rightarrow f(x) \le f(y)$.

3) let $x, y \in \mathrm{X}$ *and* $x \wedge y = x * (x * y) \Rightarrow f(x \wedge y) = f(x * (x * y))$

$= f(x) * (f(x) * f(y)) = f(x) \wedge f(y)$

***Proposition* 3.5** Let $f : \mathrm{X} \to \mathrm{Y}$ and $g : \mathrm{Y} \to \mathrm{Z}$ are IS-algebra homomorphism then $g \circ f : \mathrm{X} \to \mathrm{Z}$ is a IS-algebra homomorphism .

Proof:

Let $f : \mathrm{X} \to \mathrm{Y}$ and $g : \mathrm{Y} \to \mathrm{Z}$ are a IS-algebra homomorphism

Now,

$$
\begin{aligned}
(g \circ f)(xy) &= g(f(xy)) \\
&= g(f(x).f(y)) \\
&= g(f(x)).g(f(y)) \\
&= g \circ f(x).g \circ f(y)
\end{aligned}
$$

And

$$\begin{aligned}(g \circ f)(x * y) &= g(f(x * y)) \\ &= g(f(x) * f(y)) \\ &= g(f(x)) * g(f(y)) \\ &= g \circ f(x) * g \circ f(y)\end{aligned}$$

Hence $g \circ f$ is a IS-algebra homomorphism .

***Proposition* 3.6** Let $f : \mathrm{X} \to \mathrm{Y}$ be a IS-algebra homomorphism and $A \subseteq \mathrm{X}$ is subIS-algebra then $f(A)$ is subIS-algebra .

Proof:

Let a' , $b' \in f(A)$ $\exists$ $a, b \in A$ $s.t$ $f(a) = a'$, $(b) = b'$

$a' * b' = f(a) * f(b) = f(a * b) \in f(A)$ $\therefore a' * b' \in f(A)$

Now,

$a' \cdot b' = f(a) \cdot f(b) = f(a \cdot b) \in f(A)$ $\therefore a' \cdot b' \in f(A)$

Hence $f(A)$ is subIS-algebra .

***Proposition* 3.7** Let $f : \mathrm{X} \to \mathrm{X}'$ be a IS-algebra homomorphism if X is commutative then $f(\mathrm{X})$ is commutative .

Proof:

to prove $f(\mathrm{X})$ is IS-algebra

let y_1 , $y_2 \in f(x)$ $\exists$ $x_1, x_2 \in \mathrm{X}$ s.t $f(x_1) = y_1$, $f(x_2) = y_2$

so, $y_1 \cdot y_2 = f(x_1) \cdot f(x_2) = f(x_1 \cdot x_2) \in f(x)$

let $x, y, z \in f(x) \subseteq \mathrm{Y}$ *so* $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

hence ($f(x), \cdot$) is semigroup .

to prove $(f(\mathrm{X}), *, \cdot)$ is BCI algebra

let X is commutative then

$$\begin{aligned}f(x) * (f(x) * f(y)) &= f(x) * f(x * y) \\ &= f(x * (x * y)) \\ &= f(y * (y * x)) \\ &= f(y) * f(y * x) \\ &= f(y) * [f(y) * f(x)]\end{aligned}$$

Hence $f(\mathrm{X})$ is commutative .

***Proposition* 3.8** Let $f : \mathrm{X} \to \mathrm{X}'$ be a IS-algebra homomorphism if $f(\mathrm{X})$ is commutative then X is commutative .

Proof:

Let $x, y \in \mathrm{X}$ $\to$ $f(x)$, $f(y) \in f(\mathrm{X})$

Let $f(\mathrm{X})$ is commutative

$f(x)*[f(x)*f(y)] = f(y)*[f(y)*f(x)]$

$\Rightarrow f(x*(x*y)) = f(y*(y*x))$

$\Rightarrow x*(x*y) = y*(y*x)$

Hence X is commutative .

***Proposition* 3.9** Let $f : X \to Y$ epimorphism and let X is commutative with operation of semigroup then Y is commutative with operation of semigroup .

Proof:
Let x' , $y' \in Y$ $\quad \exists \quad x, y \in X \quad s.t \quad f(x) = x' \quad , \quad f(y) = y'$

$x' \cdot y' = f(x) \cdot f(y) = f(x \cdot y) = f(y \cdot x) = f(y) \cdot f(x) = y' \cdot x'$

Hence Y is commutative with operation of semigroup .

***Proposition* 3.10** Let X be a IS-algebra then an equivalence relation ρ on X is congruence if and only if is both left and right compatible .

Proof:
Let ρ is congruence relation and let $x, y \in X \quad s.t \quad (x, y) \in \rho$ then

$(z, z) \in \rho$ [since ρ is reflexive] and ρ is compatible then $(x*z, y*z) \in \rho$ and $(x \cdot z, y \cdot z) \in \rho$

Hence ρ is right compatible .

In a similar way , we can prove that ρ is left compatible .

Conversely
Let ρ is both left and right compatible and let $x, y, u, v \in X$ s.t $(x, y) \in \rho \quad and \quad (u, v) \in \rho$

Since ρ is right compatible then

$(x*u, y*u) \in \rho \quad and \quad (x \cdot u, y \cdot u) \in \rho$

Since ρ is left compatible then

$(y*u, y*v) \in \rho \quad and \quad (y \cdot u, y \cdot v) \in \rho$

Since ρ is transitive

So $(x*u, y*v) \in \rho \quad and \quad (x \cdot u, y \cdot v) \in \rho$

Hence ρ is congruence .

***Proposition* 3.11** Let ρ is congruence relation on IS-algebra X then X/ρ is IS-algebra under operations $x\rho * y\rho = (x*y)\rho$ and $(x\rho)\cdot(y\rho) = (x \cdot y)\rho$

Proof: Let ρ is congruence relation

It is clear the operation are well define then

$(X/\rho, *, 0)$ is BCI algebra and $(X/\rho, .)$ is semigroup

Let $x\rho, y\rho, z\rho \in X/\rho$ then

$$
\begin{aligned}
(x\rho \cdot y\rho) * z\rho &= (x \cdot y)\rho * z\rho \\
&= ((x \cdot y) * z)\rho \\
&= (x*z \cdot y*z)\rho \\
&= (x*z)\rho \cdot (y*z)\rho \\
&= (x\rho * z\rho) \cdot (y\rho * z\rho)
\end{aligned}
$$

Hence $(X/\rho, *, ., 0)$ is IS-algebra .

***Proposition* 3.12** Let ρ is congruence relation on IS-algebra X then the mapping $\Omega : X \to X/\rho$ define by $\Omega(x) = x\rho \quad \forall \; x \in X$ is IS-algebra homomorphism .

Proof: Let ρ is congruence relation and let $x, y \in X$ then
$\Omega(x * y) = (x * y)\rho = x\rho * y\rho = \Omega(x) * \Omega(y)$ and
$\Omega(x \cdot y) = (x \cdot y)\rho = x\rho \cdot y\rho = \Omega(x) \cdot \Omega(y)$
Hence Ω is IS-algebra homomorphism .

***Proposition* 3.13** Let X and Y be IS-algebra and $f : X \to Y$ homomorphism then Φ is congruence relation on X where $\Phi = \{(x, y) \in X \times X : f(x) = f(y)\}$.

Proof: Let $f : X \to Y$ homomorphism
To show that Φ is an equivalence relation
then $(x, x) \in \Phi$ [since $f(x) = f(x)$]
Let $(x, y) \in \Phi$ *s.t* $f(x) = f(y) \Rightarrow f(y) = f(x) \quad \therefore (y, x) \in \Phi$
Let $(x, y) \in \Phi$ and $(y, z) \in \Phi$
$\to f(x) = f(y)$ and $f(y) = f(z) \Rightarrow f(x) = f(z) \quad \therefore (x, z) \in \Phi$
$\therefore$ Φ is equivalence relation
Let $x, y, u, v \in X$ s.t $(x, y), (u, v) \in \Phi \Rightarrow f(x) = f(y)$, $f(u) = f(v)$
$\Rightarrow f(x * u) = f(x) * f(u) = f(y) * f(v) = f(y * v)$ and
$f(x \cdot u) = f(x) \cdot f(u) = f(y) \cdot f(v) = f(y \cdot v)$ then
$(x * u, y * v) \in \Phi$ *and* $(x \cdot u, y \cdot v) \in \Phi$
$\therefore$ Φ is congruence relation .

***Proposition* 3.14** Let $f : X \to Y$ be IS-algebra homomorphism and $\rho \subseteq \Phi$ a congruence relation of X then there exist a unique homomorphism $g : X/\rho \to Y$ where Φ as define above .

Proof:
Let g define by $g(x\rho) = f(x)$ then g is well define
$\because x\rho = y\rho \to (x, y) \in \rho \quad \because \rho \subseteq \Phi$ *and* $(x, y) \in \rho \to (x, y) \in \Phi$
$\Rightarrow f(x) = f(y) \qquad \therefore \; g(x\rho) = g(y\rho)$
Now ,

$$g(x\rho * y\rho) = g((x * y)\rho) = f(x * y) = f(x) * f(y) = g(x\rho) * g(x\rho)$$

And

$$g(x\rho \cdot y\rho) = g((x \cdot y)\rho) = f(x \cdot y) = f(x) \cdot f(y) = g(x\rho) \cdot g(x\rho)$$

Hence g is IS-algebra homomorphism and g is a unique .

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