



Generalize Partial Metric Spaces

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Abstract

The purpose of this research is to introduce a concept of general partial metric spaces as a generalization of partial metric space, give some results and properties and find relations between the general partial metric space, partial metric spaces and D-metric spaces.

Keywords: partial metric space, D-metric space, general partial metric space.

1 Introduction and preliminaries

Metric spaces are very important in mathematics introduced and studied by the French mathematicians. Many researchers tried to generalized the metric space to different types for example, b-metric space defined by Stefan Czerwik [1], G-metric space defined by Mustafa and Sims [2], 2-metric space defined by Gahler [3], D*-metric space [4], partial metric space defined by Mathews [5] and D-metric space defined by Dhage [11]

First, give some definitions and properties of the partial metric spaces that can be found in [5-10]

A non-empty set Y is said to be partial metric space if there exists a function $p: Y^2 \rightarrow [0, \infty)$ satisfies the following condition:

$$p1 \quad \alpha = \beta \Leftrightarrow p(\alpha, \alpha) = p(\alpha, \beta) = p(\beta, \beta)$$

$$p2 \quad p(\alpha, \alpha) \leq p(\alpha, \beta)$$

$$p3 \quad p(\alpha, \beta) = p(\beta, \alpha)$$

$$p4 \quad p(\alpha, \beta) \leq p(\alpha, \mu) + p(\mu, \beta) - p(\mu, \mu)$$

$\forall \alpha, \beta$ and $\mu \in Y$, where p is a partial metric on Y .

A basic example of a partial metric space is (R^+, p) , where $p(\alpha, \beta) = \max\{\alpha, \beta\} \quad \forall \alpha, \beta \in R^+$



Also, the partial metric space p on Y generates a T_0 topology τ_p on Y , which has as a base of family open balls $\{B_p(\alpha, \epsilon) : \alpha \in Y, \epsilon > 0\}$, where $B_p(\alpha, \epsilon) = \{\beta \in Y : p(\alpha, \beta) < p(\alpha, \alpha) + \epsilon\} \forall \alpha \in Y$ and $\epsilon > 0$. (see [9])

The sequence $\{\alpha_n\}$ in a partial metric space (Y, p) converge sequence if

$$\lim_{n \rightarrow \infty} p(\alpha, \alpha_n) = p(\alpha, \alpha)$$

The sequence $\{\alpha_n\}$ in a partial metric space (Y, p) is said to be Cauchy sequences if

$$\lim_{n, m \rightarrow \infty} p(\alpha_n, \alpha_m) \text{ exists (finite).}$$

The partial metric space (Y, p) is said to be complete if every cauchy sequence $\{\alpha_n\}$ convergent to a point α in Y .

A mapping $F: (Y, p) \rightarrow (Y', p')$ is said to be continuous at $\alpha_0 \in Y$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$F(B_p(\alpha_0, \delta)) \subseteq B_{p'}(F\alpha_0, \epsilon). \text{ (See [10])}$$

If p is a partial metric space, then the function $p^s: Y^2 \rightarrow R^+$ defined by

$$p^s(\alpha, \beta) = 2p(\alpha, \beta) - p(\alpha, \alpha) - p(\beta, \beta)$$

is a metric on Y (See [6])

Not that, the sequence $\{\alpha_n\}$ is a Cauchy sequence in a partial metric space (Y, p) if and only if $\{\alpha_n\}$ is a Cauchy sequence in the metric space (Y, p^s) . (See [9], [10])

A partial metric space (Y, p) is said complete if and only if the metric space (Y, p^s) is complete. Furthermore, $\lim_{n \rightarrow \infty} p^s(\alpha_n, \alpha) = 0$ if and only if

$$p(\alpha, \alpha) = \lim_{n \rightarrow \infty} p(\alpha_n, \alpha) = \lim_{n, m \rightarrow \infty} p(\alpha_n, \alpha_m). \text{ (see [9], [10])}$$

Second, we recall definition of D-metric space that can be found in [11], [12], [13]

A non-empty set Y is said to be D-metric space if there exists a function $D: Y^3 \rightarrow [0, \infty)$ satisfies the following conditions:

$$D1. D(\alpha, \beta, \gamma) = 0 \Leftrightarrow \alpha = \beta = \gamma$$

$$D2. D(\alpha, \beta, \gamma) = D(\beta, \alpha, \gamma) = D(\gamma, \alpha, \beta) = D(\beta, \gamma, \alpha) = \dots \text{ (S}\beta\text{mmetr}\beta\text{)}$$

$$D3. D(\alpha, \beta, \gamma) \leq D(\mu, \beta, \gamma) + D(\alpha, \mu, \gamma) + D(\alpha, \beta, \mu)$$

$\forall \alpha, \beta, \gamma$ and $\mu \in Y$, where D is D-metric on Y .

Y if for given $\epsilon > 0$, A sequence $\{\alpha_n\}$ in D-metric space (Y, D) is said converge to $\alpha \in Y$ there exists a positive integer m_0 such that $D(\alpha_n, \alpha_m, \alpha) < \epsilon \forall m \geq m_0, n \geq m_0$. [13]

A sequence $\{\alpha_n\}$ in D-metric space (Y, D) is said Cauchy if for given $\epsilon > 0$; there exists a positive integer m_0 such that $D(\alpha_n, \alpha_m, \alpha_l) < \epsilon \forall m \geq m_0, n \geq m_0, l \geq m_0$. [13]

A sequence $\{\alpha_n\}$ in D-metric space (Y, D) is said to be complete if every Cauchy sequence in Y converges to a point α in Y .

A sequence $\{\alpha_n\}$ in D-metric space (Y, D) converges strongly to an element α in Y if (i) $D(\alpha_n, \alpha_m, \alpha) \rightarrow 0$ as $n, m \rightarrow \infty$. (ii) $\{D(\beta, \beta, \alpha_n)\}$ converges to $D(\beta, \beta, \alpha) \forall \beta \in Y$. [13]

A sequence $\{\alpha_n\}$ in D-metric space (Y, D) is said to be very strongly converges to an element α in Y if

(i) $D(\alpha_n, \alpha_m, \alpha) \rightarrow 0$ as $n, m \rightarrow \infty$. (ii) $\{D(\beta, z, \alpha_n)\}$ converges to $D(\beta, z, \alpha) \forall \beta, z \in Y$. [13]

2. Generalize Partial Metric Spaces

Definition 1

A non-empty set Y is said to be a general partial metric space if there exists a function $D_p: Y^3 \rightarrow [0, \infty)$ satisfy the following condition:

- Dp1. $D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \beta, \gamma) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma) \Leftrightarrow \alpha = \beta = \gamma$
- Dp2. $D_p(\alpha, \alpha, \alpha) \leq D_p(\alpha, \beta, \gamma)$
- Dp3. $D_p(\alpha, \beta, \gamma) = D_p(\alpha, \gamma, \beta) = D_p(\beta, \alpha, \gamma) = D_p(\beta, \gamma, \alpha) = \dots$ (Symmetry)
- Dp4. $D_p(\alpha, \beta, \gamma) \leq D_p(\mu, \beta, \gamma) + D_p(\alpha, \mu, \gamma) + D_p(\alpha, \beta, \mu) - D_p(\mu, \mu, \mu)$

$\forall \alpha, \beta, \gamma$ and $\mu \in Y$, where D_p is general partial metric space on Y .

So, given an example of a general partial metric space we obtain

Example 2

Let $Y = [0, \infty)$ and define a function D_p on Y^3 by $D_p(\alpha, \beta, \gamma) = \max\{\alpha, \beta, \gamma\}$

Then D_p is a general partial metric space on Y .

Solution:

1) since $\max\{\alpha, \beta, \gamma\} = \max\{\alpha, \alpha, \alpha\} = \max\{\beta, \beta, \beta\} = \max\{\gamma, \gamma, \gamma\}$ if and only if $\alpha = \beta = \gamma$ then $D_p(\alpha, \beta, \gamma) = D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma)$ if and only if $\alpha = \beta = \gamma$

2) $D_p(\alpha, \alpha, \alpha) = \max\{\alpha, \alpha, \alpha\} = \alpha \leq \max\{\alpha, \beta, \gamma\} = D_p(\alpha, \beta, \gamma)$.

3) Trivial

4) Since $\max\{\alpha, \beta, \gamma\} \leq \max\{\mu, \beta, \gamma\} + \max\{\alpha, \mu, \gamma\} + \max\{\alpha, \beta, \mu\} - \max\{\mu, \mu, \mu\}$

Then $D_p(\alpha, \beta, \gamma) + D_p(\mu, \mu, \mu) \leq D_p(\mu, \beta, \gamma) + D_p(\alpha, \mu, \gamma) + D_p(\alpha, \beta, \mu)$ □

Definition 3

Let (Y, D_p) be a general partial metric space, then

(1) A sequence $\{\alpha_n\}$ in (Y, D_p) converges to a point $\alpha \in Y$ if

$$\lim_{n, m \rightarrow \infty} D_p(\alpha_n, \alpha_m, \alpha) = D_p(\alpha, \alpha, \alpha)$$

(2) A sequence $\{\alpha_n\}$ in (Y, D_p) is Cauchy sequence if

$\lim_{n, m, l \rightarrow \infty} D_p(\alpha_n, \alpha_m, \alpha_l)$ exists (finite)

(3) A general partial metric space (Y, D_p) is said to be complete if every Cauchy sequence is converge to a point α in Y .

(4) A sequence $\{\alpha_n\}$ in (Y, D_p) is a strongly converge to α if

- i. $D_p(\alpha_n, \alpha_m, \alpha) \rightarrow D_p(\alpha, \alpha, \alpha)$ as $n, m \rightarrow \infty$
- ii. $D_p(\beta, \beta, \alpha_n) \rightarrow D_p(\beta, \beta, \alpha)$ as $n \rightarrow \infty \forall \beta \in Y$

(5) A sequence $\{\alpha_n\}$ in (Y, D_p) is a very strongly converge to α if

- i. $D_p(\alpha_n, \alpha_m, \alpha) \rightarrow D_p(\alpha, \alpha, \alpha)$ as $n, m \rightarrow \infty$
- ii. $D_p(\beta, \gamma, \alpha_n) \rightarrow D_p(\beta, \gamma, \alpha)$ as $n \rightarrow \infty \forall \beta, \gamma \in Y$

(6) For $\alpha \in Y$ and $\epsilon > 0$, the open ball of the general partial metric space with center α and radius ϵ is $B_{D_p}(\alpha, \epsilon) = \{\beta \in Y : D_p(\alpha, \beta, \beta) < D_p(\alpha, \alpha, \alpha) + \epsilon\}$.

(7) A mapping $F: (Y, D_p) \rightarrow (Y', D_{p'})$ is said to be continuous at α if for each open ball

$B_{D_{p'}}(F(\alpha), \epsilon')$ in $(Y', D_{p'})$ there exists a ball $B_{D_p}(\alpha, \epsilon)$ in (Y, D_p) such that

$$F(B_{D_p}(\alpha, \epsilon)) \subseteq B_{D_{p'}}(F(\alpha), \epsilon').$$

Not that, if $\{\alpha_n\}$ is a converge sequence in (Y, D_p) then the converge point is not unique.

Example4

Let $Y = [0, \infty)$ with $D_p(\alpha, \beta, \gamma) = \max\{\alpha, \beta, \gamma\}$ then (Y, D_p) is a general partial metric observe that if the sequence $\{1 + \frac{1}{n^2}\}, \alpha \geq 1$ then $D_p(\alpha_n, \alpha_m, \alpha) = \lim_{n,m \rightarrow \infty} \max\{1 + \frac{1}{n^2}, 1 + \frac{1}{m^2}, \alpha\} = \alpha = D_p(\alpha, \alpha, \alpha)$.

Hence, every $\alpha \in [1, \infty)$ is a convergent point for the sequence $\{1 + \frac{1}{n^2}\}$.

Thus, the converge point is not unique.

Theorem 5

Every converge sequence in (Y, D_p) is a Cauchy sequence.

Proof:

Let (Y, D_p) be a general partial metric space and $\{\alpha_n\}$ is a converge sequence to α and $\epsilon > 0$.

Since $\{\alpha_n\}$ is converge to α then there exists $k \in N$ such that

$$|D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha, \alpha, \alpha)| < \epsilon \forall n, m > k$$

So that

$$\begin{aligned} D_p(\alpha_n, \alpha_m, \alpha_l) &\leq D_p(\alpha, \alpha_m, \alpha_l) + D_p(\alpha_n, \alpha, \alpha_l) + D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha, \alpha, \alpha) \\ &\leq 2D_p(\alpha, \alpha, \alpha) + \epsilon \quad \forall n, m, l > K \end{aligned}$$

Hence, $\lim_{n,m,l \rightarrow \infty} D_p(\alpha_n, \alpha_m, \alpha_l)$ exists, thus $\{\alpha_n\}$ is a Cauchy sequence. □

Remark 6

It is clear from the definition that every strong converge sequence is a converge but the opposite is not true, as we see in example (4) that a sequence $\{\alpha_n\} = \{1 + \frac{1}{n^2}\}$ is a converge to 2 but not strongly a converge to 2, to see this

Take $1 < \beta < 2, \lim_{n \rightarrow \infty} D_p(\beta, \beta, \alpha_n) = \beta \neq 2 = D_p(\beta, \beta, 2)$

Thus, $\alpha_n = \{1 + \frac{1}{n^2}\}$ is not strongly converging to 2.

Theorem 7

If $\{\alpha_n\}$ is a strongly converge, then the converge point is unique.

Proof

Let $\{\alpha_n\}$ be a strongly converge to w, z

$\{D_p(\beta, \beta, \alpha_n)\}$ is real sequence converge to $D_p(\beta, \beta, w), D_p(\beta, \beta, z)$ since the converge point is unique

$$\therefore D_p(\beta, \beta, w) = D_p(\beta, \beta, z) \quad \forall \beta$$

Take $\beta = z$ then $D_p(z, z, w) = D_p(z, z, z) \dots 1$

Take $\beta = w$ then $D_p(w, w, w) = D_p(w, w, z) \dots 2$

by defying $D_p(z, z, z) \leq D_p(\beta, \beta, z) \quad \forall \beta$

If $\beta = w$ then $D_p(z, z, z) \leq D_p(w, w, z) = D_p(w, w, w)$

$$\therefore D_p(z, z, z) \leq D_p(w, w, w) \dots 3$$

Also, $D_p(w, w, w) \leq D_p(\beta, \beta, w) \quad \forall \beta$

If $\beta = z$ then $D_p(w, w, w) \leq D_p(z, z, w) = D_p(z, z, z)$

$$\therefore D_p(w, w, w) \leq D_p(z, z, z) \dots 4$$

$b\beta$ 3, 4; we have $D_p(w, w, w) = D_p(z, z, z)$

$$\therefore D_p(w, w, z) = D_p(w, w, w) = D_p(z, z, z), \text{ thus } w = z. \quad \square$$

Example 8

Let $Y = [0, \infty)$ and $D_p(\alpha, \beta, \gamma) = \max\{\alpha, \beta, \gamma\}$ then (Y, D_p) is general partial metric space as we see in example 2, if $\alpha_n = \frac{1}{2^n}$, then the sequence $\{\alpha_n\}$ is a very strongly converge to 1.

Indeed $D_p(\alpha_n, \alpha_m, \alpha_l) = \max\left\{\frac{1}{2^n}, \frac{1}{2^m}, 1\right\} \rightarrow 1 = D_p(1, 1, 1)$ as $n, m \rightarrow \infty$

Also $D_p(\beta, \gamma, \alpha_n) = \max\{\beta, \gamma, \frac{1}{2^n}\} \rightarrow \max\{\beta, \gamma, 1\}$ as $n \rightarrow \infty \quad \forall \beta, \gamma \in Y$

Remark 9

It is clear from definition that every very strong converge sequence is a strongly converge, so that If $\{\alpha_n\}$ is very strongly converge then the converge point is unique.

Theorem 10

Let (Y, D_p) be general partial metric space, if $D_p(\alpha, \beta, \gamma) = 0$ then $\alpha = \beta = \gamma$

Proof

We have $D_p(\alpha, \alpha, \alpha) \leq D_p(\alpha, \beta, \gamma) = 0$, $D_p(\beta, \beta, \beta) \leq D_p(\alpha, \beta, \gamma) = 0$ and

$$D_p(\gamma, \gamma, \gamma) \leq D_p(\alpha, \beta, \gamma) = 0$$

$$\Rightarrow D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma) = D_p(\alpha, \beta, \gamma) = 0$$

Therefore $\alpha = \beta = \gamma$. □

Remark 11

If $\alpha = \beta = \gamma$, then $D_p(\alpha, \beta, \gamma)$ may not be zero.

3. Relations between D-metric, partial metric and general partial metric spaces

Theorem 12

Let (Y, D_p) be a general partial metric space, then the functions $D^g: Y^3 \rightarrow [0, \infty)$ given by $D^g(\alpha, \beta, \gamma) = D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha) + D_p(\beta, \beta, \gamma) + D_p(\gamma, \gamma, \alpha) + D_p(\gamma, \gamma, \beta) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\beta, \beta, \beta) - 2D_p(\gamma, \gamma, \gamma)$. (1.1)

Is a D-metric space on Y

Proof

1) Since $D_p(\alpha, \alpha, \beta) - D_p(\alpha, \alpha, \alpha) \geq 0$, $D_p(\alpha, \alpha, \gamma) - D_p(\alpha, \alpha, \alpha) \geq 0$,

$$D_p(\beta, \beta, \alpha) - D_p(\beta, \beta, \beta) \geq 0,$$

$$D_p(\beta, \beta, \gamma) - D_p(\beta, \beta, \beta) \geq 0,$$

$$D_p(\gamma, \gamma, \alpha) - D_p(\gamma, \gamma, \gamma) \geq 0 \text{ and } D_p(\gamma, \gamma, \beta) - D_p(\gamma, \gamma, \gamma) \geq 0 \text{ so } D^g(\alpha, \beta, \gamma) \geq 0.$$

2) If $D^g(\alpha, \beta, \gamma) = 0$

$$\text{then } D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha) + D_p(\beta, \beta, \gamma) + D_p(\gamma, \gamma, \alpha) + D_p(\gamma, \gamma, \beta) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\beta, \beta, \beta) - 2D_p(\gamma, \gamma, \gamma) = 0$$

$$\text{Take } D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) - 2D_p(\alpha, \alpha, \alpha) = 0$$

$$\Rightarrow 2D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) \dots 1$$

$$D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha) - 2D_p(\alpha, \alpha, \alpha) = 0$$

$$\Rightarrow 2D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha) \dots 2$$

$$\text{From 1\&2, we get } D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) = D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha)$$

$$\text{then } D_p(\alpha, \alpha, \beta) = D_p(\beta, \beta, \alpha) \dots 3$$

$$\begin{aligned} \text{Since } D_p(\alpha, \alpha, \alpha) &= D_p(\alpha, \alpha, \beta) + D_p(\alpha, \beta, \beta) = D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \beta) \\ &= 2D_p(\alpha, \alpha, \beta) \end{aligned}$$

$$D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \alpha, \beta) \dots 4$$

$$\text{Now, take } D_p(\beta, \beta, \gamma) + D_p(\beta, \beta, \alpha) - 2D_p(\beta, \beta, \beta) = 0$$

$$\Rightarrow 2D_p(\beta, \beta, \beta) = D_p(\beta, \beta, \gamma) + D_p(\beta, \beta, \alpha) \dots 5$$

$$D_p(\alpha, \alpha, \beta) + D_p(\beta, \beta, \gamma) - 2D_p(\beta, \beta, \beta) = 0$$

$$\Rightarrow 2D_p(\beta, \beta, \beta) = D_p(\alpha, \alpha, \beta) + D_p(\beta, \beta, \gamma) \dots 6$$

From 5&6, we get $D_p(\beta, \beta, \alpha) + D_p(\beta, \beta, \gamma) = D_p(\alpha, \alpha, \beta) + D_p(\beta, \beta, \gamma)$

$$\text{then } D_p(\beta, \beta, \alpha) = D_p(\alpha, \alpha, \beta) \dots 7$$

$$\begin{aligned} \text{Since } 2D_p(\beta, \beta, \beta) &= D_p(\beta, \beta, \alpha) + D_p(\alpha, \alpha, \beta) = D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \beta) \\ &= 2D_p(\alpha, \alpha, \beta) \end{aligned}$$

$$\Rightarrow D_p(\beta, \beta, \beta) = D_p(\alpha, \alpha, \beta) \dots 8$$

From 4&8, we get $D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \alpha, \beta) = D_p(\beta, \beta, \beta)$

so by definition $\alpha = \beta \dots 9$

$$\text{and take } D_p(\gamma, \gamma, \alpha) + D_p(\gamma, \gamma, \beta) - 2D_p(\gamma, \gamma, \gamma) = 0$$

$$\Rightarrow 2D_p(\gamma, \gamma, \gamma) = D_p(\gamma, \gamma, \alpha) + D_p(\gamma, \gamma, \beta) \text{ if } \beta = \alpha$$

Then $2D_p(\gamma, \gamma, \gamma) = D_p(\gamma, \gamma, \alpha) + D_p(\gamma, \gamma, \alpha) = 2D_p(\gamma, \gamma, \alpha)$

$$\Rightarrow D_p(\gamma, \gamma, \gamma) = D_p(\gamma, \gamma, \alpha) \dots 10$$

$$\text{and } D_p(\alpha, \alpha, \beta) + D_p(\alpha, \gamma, \gamma) - 2D_p(\alpha, \alpha, \alpha) = 0$$

$$\Rightarrow 2D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \alpha, \beta) + D_p(\alpha, \gamma, \gamma) \text{ if } \beta = \alpha$$

Then, $2D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \alpha, \alpha) + D_p(\alpha, \gamma, \gamma)$

$$\Rightarrow D_p(\alpha, \alpha, \alpha) = D_p(\alpha, \gamma, \gamma) \dots 11$$

From 10&11, we get $D_p(\gamma, \gamma, \gamma) = D_p(\gamma, \gamma, \alpha) = D_p(\alpha, \alpha, \alpha)$

so by definition $\alpha = \gamma \dots 12$

Then by 9&12 we get $\alpha = \beta = \gamma$.

3) Trivial

4) by definition since $D_p(\mu, \mu, \beta) + D_p(\mu, \mu, \gamma) - 2D_p(\mu, \mu, \mu) \geq 0$

$$D_p(\beta, \beta, \mu) + D_p(\beta, \beta, \mu) - 2D_p(\beta, \beta, \beta) \geq 0$$

$$D_p(\gamma, \gamma, \mu) + D_p(\gamma, \gamma, \mu) - 2D_p(\gamma, \gamma, \gamma) \geq 0$$

$$D_p(\alpha, \alpha, \mu) + D_p(\alpha, \alpha, \mu) - 2D_p(\alpha, \alpha, \alpha) \geq 0,$$

$$D_p(\mu, \mu, \alpha) + D_p(\mu, \mu, \gamma) - 2D_p(\mu, \mu, \mu) \geq 0$$

$$D_p(\mu, \mu, \alpha) + D_p(\mu, \mu, \beta) - 2D_p(\mu, \mu, \mu) \geq 0$$

When combined, it is more than and equal to zero and when adding these values

$$\begin{aligned}
 & D_p(\alpha, \alpha, \beta), D_p(\alpha, \alpha, \gamma), D_p(\beta, \beta, \alpha), D_p(\beta, \beta, \gamma), D_p(\gamma, \gamma, \alpha), D_p(\gamma, \gamma, \beta), -2D_p(\alpha, \alpha, \alpha), \\
 & -2D_p(\beta, \beta, \beta), -2D_p(\gamma, \gamma, \gamma) \text{ to two parties, we get} \\
 & D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha) + D_p(\beta, \beta, \gamma) + D_p(\gamma, \gamma, \alpha) \\
 & \quad + D_p(\gamma, \gamma, \beta) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\beta, \beta, \beta) - 2D_p(\gamma, \gamma, \gamma) \\
 & \leq D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha) + D_p(\beta, \beta, \gamma) \\
 & \quad + D_p(\gamma, \gamma, \alpha) \\
 & \quad + D_p(\gamma, \gamma, \beta) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\beta, \beta, \beta) - 2D_p(\gamma, \gamma, \gamma) + D_p(\mu, \mu, \beta) + D_p(\mu, \mu, \gamma) \\
 & \quad + D_p(\beta, \beta, \mu) + D_p(\beta, \beta, \mu) + D_p(\gamma, \gamma, \mu) + D_p(\gamma, \gamma, \mu) + D_p(\alpha, \alpha, \mu) + D_p(\alpha, \alpha, \mu) \\
 & \quad + D_p(\mu, \mu, \alpha) + D_p(\mu, \mu, \gamma) + D_p(\mu, \mu, \alpha) + D_p(\mu, \mu, \beta) \\
 & \quad - 2D_p(\mu, \mu, \mu) - 2D_p(\beta, \beta, \beta) - 2D_p(\gamma, \gamma, \gamma) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\mu, \mu, \mu) - 2D_p(\mu, \mu, \mu) \\
 & = [D_p(\mu, \mu, \beta) + D_p(\mu, \mu, \gamma) + D_p(\beta, \beta, \mu) + D_p(\beta, \beta, \gamma) \\
 & \quad + D_p(\gamma, \gamma, \mu) \\
 & \quad + D_p(\gamma, \gamma, \beta) - 2D_p(\mu, \mu, \mu) - 2D_p(\beta, \beta, \beta) - 2D_p(\gamma, \gamma, \gamma)] + [D_p(\alpha, \alpha, \mu) \\
 & \quad + D_p(\alpha, \alpha, \gamma) + D_p(\mu, \mu, \alpha) + D_p(\mu, \mu, \gamma) + D_p(\gamma, \gamma, \alpha) \\
 & \quad + D_p(\gamma, \gamma, \mu) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\mu, \mu, \mu) - 2D_p(\gamma, \gamma, \gamma)] \\
 & \quad + [D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \mu) + D_p(\beta, \beta, \alpha) + D_p(\beta, \beta, \mu) \\
 & \quad + D_p(\mu, \mu, \alpha) + D_p(\mu, \mu, \beta) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\beta, \beta, \beta) - 2D_p(\mu, \mu, \mu)] \\
 & \Rightarrow D^g(\alpha, \beta, \gamma) \leq D^g(\mu, \beta, \gamma) + D^g(\alpha, \mu, \gamma) + D^g(\alpha, \beta, \mu) \quad \square
 \end{aligned}$$

Corollary 13

Let (Y, D_p) be a general partial metric space, the function $D^g: Y^3 \rightarrow [0, \infty)$ given by

$$\begin{aligned}
 D^g(\alpha, \beta, \gamma) = & D_p(\alpha, \beta, \gamma) + D_p(\alpha, \alpha, \beta) + D_p(\alpha, \alpha, \gamma) + D_p(\beta, \beta, \alpha) + D_p(\beta, \beta, \gamma) \\
 & + D_p(\gamma, \gamma, \alpha) + D_p(\gamma, \gamma, \beta) - 2D_p(\alpha, \alpha, \alpha) - 2D_p(\beta, \beta, \beta) - 3D_p(\gamma, \gamma, \gamma)
 \end{aligned}$$

is D – metric space. (1.2)

Proof: the prove is similar to theorem 12

Remark 14

It is clear from definition that every D -metric space (Y, D) is a general partial metric space (Y, D_p) , but the converse is not true. as we saw in example 2, by definition of D -metric space if $\alpha = \beta = \gamma$ then $D(\alpha, \beta, \gamma) = 0$
 Suppose $\alpha = 5 = \beta = \gamma$ then $D_p(5, 5, 5) = ma\alpha\{5, 5, 5\} = 5 \neq 0$

Lemma 15

Let (Y, D_p) be a general partial metric space if $\{D_p(\alpha_m, \alpha_m, \alpha_m)\} \rightarrow \alpha$ as $m \rightarrow \infty$ and $\{D^g(\alpha_n, \alpha_m, \alpha_l)\}$ is a Cauchy sequence as $n, m, l \rightarrow \infty$, then $D_p(\alpha_n, \alpha_m, \alpha_l) \rightarrow \alpha$ as $n, m, l \rightarrow \infty$ where D^g define in corollary 13.

Proof

Since $D_p(\alpha_n, \alpha_m, \alpha_l) \rightarrow \alpha$ as $m \rightarrow \infty$, then from every $\epsilon > 0$ there exists $n_0 \in N$ such that

$$|D_p(\alpha_m, \alpha_m, \alpha_m) - \alpha| < \frac{\epsilon}{2} \quad \forall m > n_0, \text{ and } D^g(\alpha_n, \alpha_m, \alpha_l) < \frac{\epsilon}{2} \quad \forall n, m, l > n_0$$

$$\frac{\epsilon}{2} > D^g(\alpha_n, \alpha_m, \alpha_l)$$

$$\begin{aligned} = & D_p(\alpha_n, \alpha_m, \alpha_l) + D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_l) + D_p(\alpha_m, \alpha_m, \alpha_n) \\ & + D_p(\alpha_m, \alpha_m, \alpha_l) + D_p(\alpha_l, \alpha_l, \alpha_n) \\ & + D_p(\alpha_l, \alpha_l, \alpha_m) - 2D_p(\alpha_n, \alpha_n, \alpha_n) - 2D_p(\alpha_m, \alpha_m, \alpha_m) - 3D_p(\alpha_l, \alpha_l, \alpha_l). \end{aligned}$$

$$\Rightarrow D_p(\alpha_n, \alpha_m, \alpha_l) - D_p(\alpha_m, \alpha_m, \alpha_m) < \frac{\epsilon}{2}$$

$$\begin{aligned} \text{So that } |D_p(\alpha_n, \alpha_m, \alpha_l) - \alpha| = & |D_p(\alpha_n, \alpha_m, \alpha_l) - D_p(\alpha_m, \alpha_m, \alpha_m) + D_p(\alpha_m, \alpha_m, \alpha_m) - \alpha| \leq \\ & |D_p(\alpha_n, \alpha_m, \alpha_l) - D_p(\alpha_m, \alpha_m, \alpha_m)| + |D_p(\alpha_m, \alpha_m, \alpha_m) - \alpha| < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon \end{aligned}$$

Hence $D_p(\alpha_n, \alpha_m, \alpha_l) \rightarrow \alpha$ as $n, m, l \rightarrow \infty$ □

Theorem 16

Let (Y, D_p) be a general partial metric space, then

i) A sequence $\{\alpha_n\}$ is a Cauchy sequence in a general partial metric space (Y, D_p) if and only if $\{\alpha_n\}$ is a Cauchy sequence in (Y, D^g) .

ii) A general partial metric space (Y, D_p) is complete if (Y, D^g) is complete.

Where D^g define in corollary 13

Proof i

First, we must prove that each Cauchy sequence in (Y, D_p) is Cauchy in (Y, D^g) .

Then, there exists $\alpha \in R$ such that, $\forall \epsilon > 0$ there is $n_0 \in N$ with

$$|D_p(\alpha_n, \alpha_m, \alpha_l) - \alpha| < \frac{\epsilon}{14} \quad \forall n, m, l \geq n_0. \text{ Hence,}$$

$$\begin{aligned} |D^g(\alpha_n, \alpha_m, \alpha_l)| = & |D_p(\alpha_n, \alpha_m, \alpha_l) + D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_l) + D_p(\alpha_m, \alpha_m, \alpha_n) + \\ & D_p(\alpha_m, \alpha_m, \alpha_l) + D_p(\alpha_l, \alpha_l, \alpha_n) + D_p(\alpha_l, \alpha_l, \alpha_m) - 2D_p(\alpha_n, \alpha_n, \alpha_n) - 2D_p(\alpha_m, \alpha_m, \alpha_m) - \\ & 3D_p(\alpha_l, \alpha_l, \alpha_l)| \leq |D_p(\alpha_n, \alpha_m, \alpha_l) - \alpha| + |D_p(\alpha_n, \alpha_n, \alpha_m) - \alpha| + |D_p(\alpha_n, \alpha_n, \alpha_l) - \alpha| + \\ & |D_p(\alpha_m, \alpha_m, \alpha_n) - \alpha| + |D_p(\alpha_m, \alpha_m, \alpha_l) - \alpha| + |D_p(\alpha_l, \alpha_l, \alpha_n) - \alpha| + |D_p(\alpha_l, \alpha_l, \alpha_m) - \\ & \alpha| - 2|D_p(\alpha_n, \alpha_n, \alpha_n) - \alpha| - 2|D_p(\alpha_m, \alpha_m, \alpha_m) - \alpha| - 3|D_p(\alpha_l, \alpha_l, \alpha_l) - \alpha| \frac{\epsilon}{14} < \epsilon \quad \forall n, m, l \\ & \geq n_0. \end{aligned}$$

Hence, $\{\alpha_n\}$ is a Cauchy sequence in (Y, D^g) .

Conversely, now we must prove $\{\alpha_n\}$ is Cauchy sequence in (Y, D_p)

Since $\{\alpha_n\}$ is a Cauchy sequence in (Y, D^g) so $\forall \epsilon > 0, \exists n_0 \in N$ such that

$$D^g(\alpha_n, \alpha_m, \alpha_l) < \frac{\epsilon}{2} \quad \forall n, m, l > n_0$$

$$\frac{\epsilon}{2} > D^g(\alpha_n, \alpha_m, \alpha_l)$$

$$\begin{aligned} = & D_p(\alpha_n, \alpha_m, \alpha_l) + D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_l) + D_p(\alpha_m, \alpha_m, \alpha_n) \\ & + D_p(\alpha_m, \alpha_m, \alpha_l) + D_p(\alpha_l, \alpha_l, \alpha_n) \\ & + D_p(\alpha_l, \alpha_l, \alpha_m) - 2D_p(\alpha_n, \alpha_n, \alpha_n) - 2D_p(\alpha_m, \alpha_m, \alpha_m) - 3D_p(\alpha_l, \alpha_l, \alpha_l) \end{aligned}$$

$$\Rightarrow D_p(\alpha_n, \alpha_m, \alpha_l) - D_p(\alpha_n, \alpha_n, \alpha_n) \leq D^g(\alpha_n, \alpha_m, \alpha_l) < \frac{\epsilon}{2}$$

By compensation $D_p(\alpha_n, \alpha_n, \alpha_n)$ to two parties, we have

$$D_p(\alpha_n, \alpha_m, \alpha_l) \leq D^g(\alpha_n, \alpha_m, \alpha_l) + D_p(\alpha_n, \alpha_n, \alpha_n) < \frac{\epsilon}{2} + D_p(\alpha_n, \alpha_n, \alpha_n)$$

And since $D_p(\alpha_m, \alpha_m, \alpha_m) \leq D_p(\alpha_n, \alpha_m, \alpha_l)$ so

$$D_p(\alpha_m, \alpha_m, \alpha_m) \leq D_p(\alpha_n, \alpha_m, \alpha_l) \leq D^g(\alpha_n, \alpha_m, \alpha_l) + D_p(\alpha_n, \alpha_n, \alpha_n) < \frac{\epsilon}{2} + D_p(\alpha_n, \alpha_n, \alpha_n)$$

$$\Rightarrow D_p(\alpha_m, \alpha_m, \alpha_m) \leq \frac{\epsilon}{2} + D_p(\alpha_n, \alpha_n, \alpha_n) \quad \forall n, m > n_0$$

Let $\alpha_n = D_p(\alpha_n, \alpha_n, \alpha_n) \in R$ such that $|\alpha_m - \alpha_n| < \frac{\epsilon}{2}$

$\therefore \{\alpha_m\}$ is a Cauchy sequence, $\therefore \{\alpha_n\} \rightarrow \alpha, \therefore D_p(\alpha_m, \alpha_m, \alpha_m) \rightarrow \alpha \in R$

Then by lemma 15, $D_p(\alpha_m, \alpha_m, \alpha_m)$ is Cauchy sequence in (Y, D_p) . □

ii :

If $\{\alpha_n\}$ is a Cauchy sequence in (Y, D_p) , then it is a Cauchy sequence in (Y, D^g) and since D-metric (Y, D^g) is complete then there exists $\alpha \in Y$ such that

$\lim_{n,m \rightarrow \infty} D^g(\alpha_n, \alpha_m, \alpha) = 0$, hence

$$\begin{aligned} \lim_{n,m \rightarrow \infty} [D_p(\alpha_n, \alpha_m, \alpha) + D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha) + D_p(\alpha_m, \alpha_m, \alpha_n) + \\ D_p(\alpha_m, \alpha_m, \alpha) + D_p(\alpha, \alpha, \alpha_n) + \\ D_p(\alpha, \alpha, \alpha_m) - 2D_p(\alpha_n, \alpha_n, \alpha_n) - 2D_p(\alpha_m, \alpha_m, \alpha_m) - 3D_p(\alpha, \alpha, \alpha)] = 0 \end{aligned}$$

There for $\lim_{n,m \rightarrow \infty} [D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha, \alpha, \alpha)] = 0$

$\Rightarrow \lim_{n,m \rightarrow \infty} D_p(\alpha_n, \alpha_m, \alpha) = D_p(\alpha, \alpha, \alpha)$ hence (Y, D_p) is converge

Thus (Y, D_p) is complete.

Corollary 17

we can get from the proof of theorem 16, (ii), $\lim_{n,m \rightarrow \infty} [D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha, \alpha, \alpha)] = \lim_{n,m \rightarrow \infty} [D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha_m, \alpha_m, \alpha_m)] = \lim_{n,m \rightarrow \infty} [D_p(\alpha_n, \alpha_m, \alpha) - D_p(\alpha_n, \alpha_n, \alpha_n)] = 0$, such that $\lim_{n,m \rightarrow \infty} D_p(\alpha_n, \alpha_m, \alpha) = \lim_{m \rightarrow \infty} D_p(\alpha_m, \alpha_m, \alpha_m) = \lim_{n \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_n) = D_p(\alpha, \alpha, \alpha)$.

Propositions 18

If $\{\alpha_n\}$ is Cauchy sequence in (Y, D^g) , then $\lim_{n,m \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_m) = \lim_{n \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_n)$.

Proof

Since $\{\alpha_n\}$ is Cauchy sequence in (Y, D^g) then $\lim_{n,m,l \rightarrow \infty} D^g(\alpha_n, \alpha_m, \alpha_l) = 0$

And $D^g(\alpha_n, \alpha_m, \alpha_l) = D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_l) + D_p(\alpha_m, \alpha_m, \alpha_n) + D_p(\alpha_m, \alpha_m, \alpha_l) + D_p(\alpha_l, \alpha_l, \alpha_n) + D_p(\alpha_l, \alpha_l, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) - 2D_p(\alpha_m, \alpha_m, \alpha_m) - 2D_p(\alpha_l, \alpha_l, \alpha_l)$.

And $D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) \leq D^g(\alpha_n, \alpha_m, \alpha_l)$ then

$$\lim_{n,m \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) \rightarrow 0$$

Similarly $\lim_{n,m \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_m, \alpha_m, \alpha_m) \rightarrow 0$

Since $D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n) = D_p(\alpha_m, \alpha_m, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)$ then $\lim_{n,m \rightarrow \infty} [D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] = \lim_{n,m \rightarrow \infty} [D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_m)] + \lim_{n,m \rightarrow \infty} [D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] \rightarrow 0$

So that $\lim_{n,m \rightarrow \infty} [D_p(\alpha_m, \alpha_m, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] \rightarrow 0$

Let $\alpha_n = D_p(\alpha_n, \alpha_n, \alpha_n)$

$$\therefore |\alpha_m - \alpha_n| \rightarrow 0 \text{ as } n, m \rightarrow \infty$$

Hence $\{\alpha_n\}$ is a Cauchy sequence in R, therefor $\{D_p(\alpha_n, \alpha_n, \alpha_n)\}$ converge to α .

Also, $\lim_{n \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_m) =$

$$\lim_{n,m \rightarrow \infty} [D_p(\alpha_n, \alpha_n, \alpha_m) + D_p(\alpha_n, \alpha_n, \alpha_n) - D_p(\alpha_n, \alpha_n, \alpha_n)]$$

Then $[D_p(\alpha_n, \alpha_n, \alpha_m) - D_p(\alpha_n, \alpha_n, \alpha_n)] \rightarrow 0$ so $\lim_{n,m \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_m) = \alpha$

Thus $\lim_{n,m \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_m) = \lim_{n \rightarrow \infty} D_p(\alpha_n, \alpha_n, \alpha_n)$. □

Theorem 19

If (Y, p) be partial metric space then

$$D_p(\alpha, \beta, \gamma) = p(\alpha, \beta) + p(\alpha, \gamma) + p(\beta, \gamma) - p(\alpha, \alpha) - P(\beta, \beta) - p(\gamma, \gamma) \quad (1.3)$$

is general partial metric space.

1) Since $p(\alpha, \beta) - p(\alpha, \alpha) \geq 0, p(\alpha, \gamma) - p(\gamma, \gamma) \geq 0, p(\beta, \gamma) - p(\beta, \beta) \geq 0$

then $D_p(\alpha, \beta, \gamma) \geq 0$

2) let $D_p(\alpha, \beta, \gamma) = D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma)$

since $D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) = D_p(\gamma, \gamma, \gamma) = 0$

$$\Rightarrow D_p(\alpha, \beta, \gamma) = 0$$

$$\Rightarrow p(\alpha, \beta) + p(\beta, \gamma) + p(\alpha, \gamma) - p(\alpha, \alpha) - p(\beta, \beta) - p(\gamma, \gamma) = 0$$

$$\Rightarrow p(\alpha, \beta) - p(\alpha, \alpha) = 0 \Rightarrow p(\alpha, \beta) = p(\alpha, \alpha) \dots 1,$$

$$\Rightarrow p(\alpha, \gamma) - p(\gamma, \gamma) = 0 \Rightarrow p(\alpha, \gamma) = p(\gamma, \gamma) \dots 2,$$

and $p(\beta, \gamma) - p(\beta, \beta) = 0 \Rightarrow p(\beta, \gamma) = p(\beta, \beta) \dots 3$

From 1 $p(\alpha, \alpha) = p(\alpha, \beta)$ and by definition $p(\alpha, \alpha) = p(\alpha, \beta) \leq p(\alpha, \gamma) + p(\gamma, \beta) - p(\gamma, \gamma)$

Since $p(\alpha, \gamma) = p(\gamma, \gamma)$ & $p(\beta, \gamma) = p(\beta, \beta)$ we get $p(\alpha, \alpha) = p(\beta, \beta)$

From 2 $p(\beta, \beta) = p(\beta, \gamma)$ and by definition $p(\beta, \beta) = p(\beta, \gamma) \leq p(\beta, \alpha) + p(\alpha, \gamma) - p(\alpha, \alpha)$

by 1&2 we get $p(\beta, \beta) = p(\gamma, \gamma)$

From 3 $p(\gamma, \gamma) = p(\alpha, \gamma)$ and by definition $p(\gamma, \gamma) = p(\alpha, \gamma) \leq p(\alpha, \beta) + p(\beta, \gamma) - p(\beta, \beta)$

by 1&3 we get $p(\gamma, \gamma) = p(\alpha, \alpha)$

Hence $p(\alpha, \alpha) = p(\beta, \beta) = p(\gamma, \gamma)$

Thus $p(\alpha, \alpha) = p(\alpha, \beta) = p(\beta, \beta)$ by definition $\alpha = \beta$, and $p(\beta, \beta) = p(\beta, \gamma) = p(\gamma, \gamma)$ by definition $\beta = \gamma$ then we get $\alpha = \beta = \gamma$.

3) Trivial

4) since $p(\mu, \gamma) - p(\gamma, \gamma) \geq 0, p(\mu, \beta) - p(\mu, \mu) \geq 0, p(\alpha, \mu) - p(\alpha, \alpha) \geq 0,$
 $p(\mu, \gamma) - p(\mu, \mu) \geq 0, p(\beta, \mu) - p(\beta, \beta) \geq 0,$
 $p(\alpha, \mu) - p(\mu, \mu) \geq 0$

When combined, it is more than and equal to zero and when added these values

$p(\alpha, \beta), p(\beta, \gamma), p(\alpha, \gamma), -p(\alpha, \alpha), -p(\beta, \beta), -p(\gamma, \gamma)$ to both said, we get

$$p(\alpha, \beta) + p(\beta, \gamma) + p(\alpha, \gamma) - p(\alpha, \alpha) - p(\beta, \beta) - p(\gamma, \gamma) \leq p(\alpha, \beta) + p(\beta, \gamma) + p(\alpha, \gamma) - p(\alpha, \alpha) - p(\beta, \beta) - p(\gamma, \gamma) + p(\mu, \gamma) - p(\mu, \mu) + p(\alpha, \mu) - p(\alpha, \alpha) + p(\beta, \mu) - p(\beta, \beta) + p(\mu, \beta) - p(\mu, \mu) + p(\mu, \gamma) - p(\mu, \mu) + p(\alpha, \mu) - p(\mu, \mu).$$

$\Rightarrow D_p(\alpha, \beta, \gamma)$

$$\leq [p(\mu, \beta) + p(\mu, \gamma) + p(\gamma, \beta) - p(\mu, \mu) - p(\beta, \beta) - p(\gamma, \gamma)] + [p(\alpha, \mu) + p(\alpha, \gamma) + p(\mu, \gamma) - p(\alpha, \alpha) - p(\mu, \mu) - p(\gamma, \gamma)] + [p(\alpha, \beta) + p(\alpha, \mu) + p(\beta, \mu) - p(\alpha, \alpha) - p(\beta, \beta) - p(\mu, \mu)]$$

$$D_p(\alpha, \beta, \gamma) \leq D_p(\mu, \beta, \gamma) + D_p(\alpha, \mu, \gamma) + D_p(\alpha, \beta, \mu) - D_p(\mu, \mu, \mu) \quad \square$$

Proposition 20

Let (Y, D_p) be a general partial metric space and

$$D_p(\alpha, \beta, \beta) \leq D_p(\alpha, \gamma, \gamma) + D_p(\gamma, \beta, \beta) - D_p(\gamma, \gamma, \gamma) \tag{1.4}$$

holds then the function $p: Y^2 \rightarrow [0, \infty)$ which is defined by $p(\alpha, \beta) = D_p(\alpha, \beta, \beta)$ is a partial metric on Y .

Proof

1) since $p(\alpha, \beta) = p(\alpha, \alpha) = p(\beta, \beta) \Leftrightarrow D_p(\alpha, \beta, \beta) = D_p(\alpha, \alpha, \alpha) = D_p(\beta, \beta, \beta) \Leftrightarrow \alpha = \beta$.

2) since $D_p(\alpha, \alpha, \alpha) \leq D_p(\alpha, \beta, \beta)$ then $p(\alpha, \alpha) \leq p(\alpha, \beta) \quad \forall \alpha, \beta \in \alpha$

3) Trivial

4) $p(\alpha, \beta) = D_p(\alpha, \beta, \beta) \leq D_p(\alpha, \gamma, \gamma) + D_p(\gamma, \beta, \beta) - D_p(\gamma, \gamma, \gamma)$ from (1.3)

$$= p(\alpha, \gamma) + p(\gamma, \beta) - p(\gamma, \gamma). \quad \square$$

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