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Extended exponential function method in rational form for exact solution of coupled Burgers equation

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Abstract.

The aim of this paper is to present an extension of exponential function method in rational form to find an exact solution of coupled Burgers equation. This extended exponential function method in rational form allows us to find extra travelling wave solutions of coupled Burgers equation instead of exponential function method in rational form.

Key Words: extended exp-function method, rational form, coupled Burgers equation, exact solution, travelling wave.

1-Introduction

In mathematics many nonlinear partial differential equations widely described important phenomena in many branches of sciences such as physical, chemical, economical and biological. Most of these equations have been studied to find the exact or approximate solutions by using different methods. He and Wu as in reference [1] are proposed a straightforward and concise method called exp-function method. The exp-function method used recently [2] to find the exact solution to the system of nonlinear partial differential equations, by Kazeminia, *et al.* used the exp-function method to find the exact solution of Benjamin–Bona–Mahony–Burgers (BBMB) equations. Thus, He and Abdou studied systematically new periodic solutions for nonlinear evolution equations using exp-function method [3]. Zhang in [4] is used the exp-function method for solving Maccari’s system. Jawad, *et al.* are used complex tanh method to find the soliton solution of the coupled burger’s equation [5]. Demiray in [6] is used the exponential rational function approach to present a travelling wave solution to the KdV-Burgers equation, but some researchers like [7] and [8] think that the exponential function method in rational form is a spatial case from [1].

The purpose of this work is to improve the exp-function method in rational form by extended it and illustrate an application to show the advantage of this extended of the exp-function method in rational form, we investigate the existence of the traveling wave solutions of the homogeneous form of nonlinear coupled burgers equation [5] of the form,

$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}+2u\frac{\partial u}{\partial x}+\alpha\frac{\partial}{\partial x}(uv)=0$$
$$\frac{\partial v}{\partial t}-\frac{\partial^2 v}{\partial x^2}+2v\frac{\partial v}{\partial x}+\beta\frac{\partial}{\partial x}(uv)=0 \qquad (1.1)$$

where, α and β are parameters.

The paper is organized as follows: - Section one concerns review on exp-function method and the rational form of this method. Section two is related to explain exp-function method in rational form and illustrate an example to explain the procedure of the method. Section three devoted to present extended exp-function method in rational form and resolves the example in section two. Section four presents the advantage of the extended present in section three.

2- Exponential function method in rational form

To apply the exponential function method in rational form to the equations (1.1) we make use of the traveling wave transformation $\zeta = kx+qt,\quad u=u(\zeta),\quad v=v(\zeta)$.
where k and q are constants to be determined later. Then, equations (1.1) reduce to ordinary differential equations.

$$qu' - k^2u'' + 2kuu' + k\alpha(uv)' = 0$$
$$qv' - k^2v'' + 2kvv' + k\beta(uv)' = 0 \qquad (2.1)$$

The exponential function method in rational form is based on the assumption that travelling wave solutions can be expressed in the following rational form,

$$u=\sum_{i=0}^{m}\frac{a_i}{(1+exp(\zeta))^i} \qquad (2.2a)$$

$$v=\sum_{i=0}^{n}\frac{b_i}{(1+exp(\zeta))^i} \qquad (2.2b)$$

where m and n are positive integers which are unknown to be further determined, a_i and b_i are unknown constants. In order to determine the values of m and n, we balance the linear term u'' with the nonlinear term uu' in the first equation of (2.1), and the linear term v'' with the nonlinear term vv' in second equation of (2.1), by normal calculation, we have

$$u''=\frac{K1}{\left(1+e^{\zeta}\right)^{m+2}} \qquad (2.3)$$

$$uu'=\frac{K2}{\left(1+e^{\zeta}\right)^{2m+1}} \qquad (2.4)$$

$$v''=\frac{K3}{\left(1+e^{\zeta}\right)^{n+2}} \qquad (2.5)$$

$$vv'=\frac{K4}{\left(1+e^{\zeta}\right)^{2n+1}} \qquad (2.6)$$

where $K1, K2, K3$ and $K4$ are determined coefficients only for simplicity. Balancing highest order of Exp-function in equations (2.3) and (2.4) we have $m=1$. Similarly balancing equations (2.5) and (2.6), we obtain $n=1$. Equations (2.2a) and (2.2b) become,

$$u=a_0+\frac{a_1}{1+exp(\zeta)} \qquad (2.7a)$$

$$v=b_0+\frac{b_1}{1+exp(\zeta)} \qquad (2.7b)$$

Substituting equations (2.7a) and (2.7b) into equations (2.1), by the help of software Mathematica 6.0, we have,

$$\frac{1}{A}\left[C_1\exp(\zeta)+C_2\exp(2\zeta)\right]=0$$

and

$$\frac{1}{A}\left[D_1\exp(\zeta)+D_2\exp(2\zeta)\right]=0$$

where $A=\left(1+\exp(\zeta)\right)^3$

$$C_1=-(2ka_1^2+(-k^2+((b_0+2b_1)\alpha+2a_0)k+q)a_1+k\alpha b_1 a_0)$$

$$C_2=-((k^2+(2a_0+\alpha b_0)k+q)a_1+k\alpha b_1 a_0)$$

$$D_1=-(2kb_1^2+(-k^2+((2a_1+a_0)\beta+2b_0)k+q)b_1+k\beta a_1 b_0)$$

$$D_2=-((k^2+(2b_0+\alpha a_0)k+q)b_1+k\beta a_1 b_0)$$

Equating the coefficients of all powers of $\exp(n\zeta)$ to be zero, we obtain

$$\left[C_2=0,\ C_1=0,\ D_1=0,\ D_2=0\right] \tag{2.8}$$

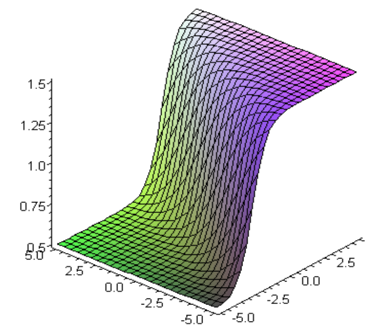
Solving the system, equations (2.8), simultaneously, we get the following solution

$$b_1=\frac{a_1 b_0}{a_0},\ \alpha=\frac{-a_1+k}{b_1},\ \beta=\frac{-b_1+k}{a_1}$$
$$q=\frac{1}{a_1^2}(-a_1^2 b_0 k+a_0 a_1 b_1 k-2a_0 a_1 k^2-a_1^2 k^2+a_1 b_0 k^2-a_0 b_1 k^2) \tag{2.9}$$

Inserting equation (2.9) into equations (2.7a) and (2.7b) yields the following exact solution,

$$u=a_0+\frac{a_1}{1+e^{kx+\frac{-2a_0 a_1 k-a_1^2 k^2}{a_1^2}t}},\quad v=b_0+\frac{\frac{a_1 b_0}{a_0}}{1+e^{kx+\frac{-2a_0 a_1 k-a_1^2 k^2}{a_1^2}t}} \tag{2.10}$$

The graph of the solution (2.10) is given in figure 1.



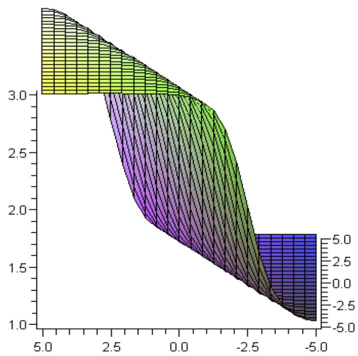


Fig.1 The solution of Eq. (2.10)

with $a_0=0.5, a_1=1, b_0=1, k=1$

3- Extended exponential function method in rational form

Suppose the solution $u=u(\zeta),\ v=v(\zeta)$ of equations (2.2a) and (2.2b) can be expressed as an a finite series in an extended form

$$u = \sum_{i=-M}^{M} \frac{a_i}{(1+exp(\zeta))^i} \qquad (3.1a)$$

$$v = \sum_{i=-N}^{N} \frac{b_i}{(1+exp(\zeta))^i} \qquad (3.1b)$$

where M and N are positive integers which are unknown to be further determined, a_i and b_i are unknown constants. In order to determine the values of M and N, we balance the highest and the lowest order of the linear term $u^{//}$ with the highest and the lowest order of the nonlinear term $uu^{/}$ in the first equation of (2.1), and the highest and the lowest order of the linear term $v^{//}$ with the highest and the lowest order of the nonlinear term $vv^{/}$ in second equation of (2.1), by normal calculation, we have $M=1$, $N=1$ and equations (3.1a) and (3.1b) become

$$u = \frac{a_{-1}}{(1+exp(\zeta))^{-1}} + a_0 + \frac{a_1}{1+exp(\zeta)} \qquad (3.2a)$$

$$v = \frac{b_{-1}}{(1+exp(\zeta))^{-1}} + b_0 + \frac{b_1}{1+exp(\zeta)} \qquad (3.2b)$$

Substituting equations (3.2a) and (3.2b) into equations (2.1), by the help of software Mathematica 6.0, we have,

$$\frac{1}{A}\left[C_1 \exp(\zeta) + C_2 \exp(2\zeta) + C_3 \exp(3\zeta) + C_4 \exp(4\zeta) + C_5 \exp(5\zeta)\right] = 0$$

and

$$\frac{1}{A}\left[D_1 \exp(\zeta) + D_2 \exp(2\zeta) + D_3 \exp(3\zeta) + D_4 \exp(4\zeta) + D_5 \exp(5\zeta)\right] = 0$$

where $A = (1+\exp(\zeta))^3$

$$C_1 = 3(-\frac{a_{-1}}{3} + \frac{a_1}{3} + (\frac{2a_{-1}^2}{3} + (\frac{b_0}{3} + \frac{2b_{-1}}{3})\alpha + \frac{2a_0}{3})a_{-1} + ((-\frac{b_1}{3} + \frac{b_{-1}}{3})a_0 - \frac{a_1}{3}(2b_1 + b_0))\alpha - \frac{2a_1}{3}(a_1 + a_0))k + \frac{q}{3}(-a_1 + a_{-1}))$$

$$C_2 = 3((-a_{-1} - \frac{a_1}{3})k^2 + (\frac{8a_{-1}^2}{3} + ((b_0 + \frac{8b_{-1}}{3})\alpha + 2a_0)a_{-1} + ((b_{-1} - \frac{b_1}{3})a_0 - \frac{a_1 b_0}{3})\alpha - \frac{2a_1 a_0}{3})k + q(a_{-1} - \frac{a_1}{3}))$$

$$C_3 = 3(-k^2 a_{-1} + (4a_{-1}^2 + ((4b_{-1} + b_0)\alpha + 2a_0)a_{-1} + \alpha a_0 b_{-1})k + qa_{-1})$$

$$C_4 = 3(-\frac{k^2 a_{-1}}{3} + (\frac{8a_{-1}^2}{3} + ((\frac{b_0}{3} + \frac{8b_{-1}}{3})\alpha + \frac{2a_0}{3})a_{-1} + \frac{\alpha a_0 b_{-1}}{3})k + \frac{qa_{-1}}{3})$$

$$C_5 = 2ka_{-1}(a_{-1} + \alpha b_{-1})$$

$$D_1 = 3((-\frac{b_{-1}}{3}+\frac{b_1}{3})+(\frac{2b_{-1}^2}{3}+((\frac{a_0}{3}+\frac{2a_{-1}}{3})\beta+\frac{2b_0}{3})b_{-1}+((-\frac{a_1}{3}+\frac{a_{-1}}{3})b_0-\frac{2}{3}(a_1+\frac{a_0}{2})b_1)\beta$$
$$-\frac{2b_1}{3}(b_1+b_0))k+\frac{q}{3}(b_{-1}-b_1))$$

$$D_2 = 3((-b_{-1}-\frac{b_1}{3})k^2+(\frac{8b_{-1}^2}{3}+((\frac{8a_{-1}}{3}+a_0)\beta+2b_0)b_{-1}+((a_{-1}-\frac{a_1}{3})b_0$$
$$-\frac{a_0b_1}{3})\beta-\frac{2b_0b_1}{3})k+q(b_{-1}-\frac{b_1}{3}))$$

$$D_3 = 3(-k^2b_{-1}+(4b_{-1}^2+((4a_{-1}+a_0)\beta+2b_0)b_{-1}+\beta a_{-1}b_0)k+qb_{-1})$$

$$D_4 = 3(-\frac{k^2b_{-1}}{3}+(\frac{8b_{-1}^2}{3}+((\frac{8a_{-1}}{3}+\frac{a_0}{3})\beta+\frac{2b_0}{3})b_{-1}+\frac{a_{-1}b_0\beta}{3})k+\frac{qb_{-1}}{3})$$

$$D_5 = 2kb_{-1}(b_{-1}+\beta a_{-1})$$

Equating the coefficients of all powers of $\exp(n\zeta)$ to be zero, we obtain

$$[C_1=0,\ C_2=0, C_3=0,\ C_4=0, C_5=0, D_1=0,\ D_2=0, D_3=0,\ D_4=0, D_5=0] \qquad (3.3)$$

Solving the system, equations (3.3), simultaneously, we get the following solutions

$$a_1=a_1,\ \alpha=\alpha,\ k=k,\ a_{-1}=0,\ b_0=\frac{(k^2+1-2ka_1)a_0}{2\alpha ka_1},\ b_1=\frac{k^2+1-2ka_1}{2k\alpha},$$
$$b_{-1}=0,\ \beta=\frac{k^2\alpha-k^2+2ka_1+\alpha-1}{2k\alpha a_1},\ q=-\frac{k^2a_1+a_0k^2+a_0}{a_1}, a_0=a_0 \qquad (3.4a)$$

$$a_0=a_0,\ a_1=0,\ a_{-1}=-b_{-1},\ b_0=b_0,\ b_1=0,$$
$$b_{-1}=b_{-1},\ \alpha=1,\ \beta=1,\ k=1,\ q=1-b_0-a_0 \qquad (3.4b)$$

$$a_0=a_0,\ a_1=-\frac{\alpha(-2\alpha a_0^2+a_0^2+\alpha^2a_0^2+4\alpha^2)}{a_0(1+\alpha)(\alpha-1)^2},\ k=-\frac{a_0(\alpha-1)}{2\alpha},$$
$$a_{-1}=\frac{\alpha(-2\alpha a_0^2+a_0^2+\alpha^2a_0^2+4\alpha^2)^2}{4a_0(\alpha a_0-2\alpha-a_0)(\alpha a_0+2\alpha-a_0)(\alpha-1)^2(1+\alpha)^2},\ \alpha=\alpha,$$
$$b_{-1}=-\frac{(-2\alpha a_0^2+a_0^2+\alpha^2a_0^2+4\alpha^2)^2}{4a_0(\alpha a_0-2\alpha-a_0)(\alpha a_0+2\alpha-a_0)(\alpha-1)^2(1+\alpha)^2}, \qquad (3.4c)$$
$$q=\frac{a_0^2(\alpha-1)^2}{4\alpha^2},\ \beta=\frac{1}{\alpha}, b_0=\frac{-a_0}{\alpha}, b_1=\frac{-2\alpha a_0^2+a_0^2+\alpha^2a_0^2+4\alpha^2}{4(1+\alpha)(\alpha-1)^2\alpha a_0}$$

$$a_0=a_0,\ a_1=0,\ a_{-1}=-\alpha b_{-1},\ b_0=\frac{-a_0}{\alpha}, b_1=0,$$
$$b_{-1}=b_{-1},\ \alpha=\alpha,\ \beta=\frac{1}{\alpha},\ k=-1,\ q=1 \qquad (3.4d)$$

we can show the equation (3.4a) reduces exact solution equivalent to the exact solution (2.10). Equations (3.4b)-(3.4d) reduce another exact solution. Inserting equations (3.4b)-(3.4d) into equations(3.2a) and (3.2b) respectively yields the following exact solutions

$$u_1 = \frac{-b_{-1}}{\left(1+e^{x+(1-b_0-a_0)t}\right)^{-1}} + a_0 \quad \text{and} \quad v_1 = \frac{b_{-1}}{\left(1+e^{x+(1-b_0-a_0)t}\right)^{-1}} + b_0 \qquad (3.5)$$

Thus,

$$u_2 = \frac{\alpha(-2\alpha a_0^2 + a_0^2 + \alpha^2 a_0^2 + 4\alpha^2)^2}{4a_0(\alpha a_0 - 2\alpha - a_0)(\alpha a_0 + 2\alpha - a_0)(\alpha-1)^2(1+\alpha)^2\left(1+e^{-\frac{a_0(\alpha-1)}{2\alpha}x+\frac{a_0^2(\alpha-1)^2}{4\alpha^2}t}\right)^{-1}}$$

$$+ a_0 - \frac{\alpha(-2\alpha a_0^2 + a_0^2 + \alpha^2 a_0^2 + 4\alpha^2)}{a_0(1+\alpha)(\alpha-1)^2\left(1+e^{-\frac{a_0(\alpha-1)}{2\alpha}x+\frac{a_0^2(\alpha-1)^2}{4\alpha^2}t}\right)}$$

$$v_2 = -\frac{(-2\alpha a_0^2 + a_0^2 + \alpha^2 a_0^2 + 4\alpha^2)^2}{4a_0(\alpha a_0 - 2\alpha - a_0)(\alpha a_0 + 2\alpha - a_0)(\alpha-1)^2(1+\alpha)^2\left(1+e^{-\frac{a_0(\alpha-1)}{2\alpha}x+\frac{a_0^2(\alpha-1)^2}{4\alpha^2}t}\right)^{-1}} \qquad (3.6)$$

$$-\frac{a_0}{\alpha} + \frac{-2\alpha a_0^2 + a_0^2 + \alpha^2 a_0^2 + 4\alpha^2}{4(1+\alpha)(\alpha-1)^2\alpha a_0\left(1+e^{-\frac{a_0(\alpha-1)}{2\alpha}x+\frac{a_0^2(\alpha-1)^2}{4\alpha^2}t}\right)}$$

$$u_3 = \frac{-\alpha b_{-1}}{\left(1+e^{-x+t}\right)^{-1}} + a_0 \quad \text{and} \quad v_3 = \frac{b_{-1}}{\left(1+e^{-x+t}\right)^{-1}} - \frac{a_0}{\alpha} \qquad (3.7)$$

we can see that equations (3.5)-(3.7) are equivalent solutions, the graph of the solution (3.7) is given in figure 2, and this solution may be new.

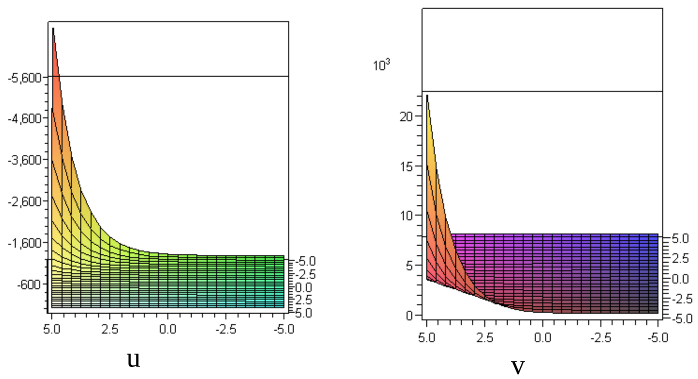


Fig.2 The solution of Eq. (3.7)
with $a_0 = 1, \;\; b_{-1} = 1, \;\; \alpha = 0.25$

4. Conclusion:

In this work we have assumed a symmetric form $\sum_{i=-N}^{N}\frac{a_i}{(1+\exp(\zeta))^i}$ to the exponential function method in rational form instead of the old assumption $\sum_{i=0}^{N}\frac{a_i}{(1+\exp(\zeta))^i}$ for the solution. This new form was successfully used to obtain travelling wave solutions of the coupled Burgers equation. Moreover this extension finds extra solutions for the nonlinear partial differential equations. The solution procedure is better than the previous method in section two, and some of these solutions may be new.

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