Al-Mustansiriyah Journal of Science
ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 32, Issue 4, 2021 DOI: http://doi.org/10.23851/mjs.v32i4.1019

Research Article Open Access

Z-transform Solution for Nonlinear Difference Equations

Eman A. Hussain, Asmaa S. Jasim*

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, IRAQ.

*Correspondent contact: asmaa93salih@gmail.com

Article Info

Received
12/06/2021

Accepted
27/08/2021

Published
20/11/2021

ABSTRACT

The aim of this paper is to study Z-transform to solve non-linear difference equations, after converting them to linear difference equations by one of the conversion methods. This is because the z-transform cannot be directly applied to the nonlinear difference equations.

KEYWORDS: difference equations, nonlinear difference equations, Z-transform.

الخلاصة

الهدف في هذا البحث هو دراسة تحويل زد لحل معادلات الفروق الغير خطيه ،وذلك بعد تحويلها الى معادلات الفروق الخطية بأحدى طرق التحويل. وذلك لأنه لا يمكن تطبيق تحويل زد مباشرتا على معادلات الفروق غير الخطية .

INTRODUCTION

Transformation is a very powerful mathematical tool so using it in mathematical treatment of problem is arising in many applications. The idea of Z-transform back to 1730 when De Moivre introduced the concept of "generating functions" to probability theory [1]. In 1947 a transform of sampled signal or sequence defined by W. Hurewicz as a tractable way to solve linear difference equations. The transformation named "Z-transform" by Ragazzini and Lotfi Zadeh in the sampled-data control group at Columbia University in 1952. Z-transform is transformation for discrete data equivalent to the Laplace transform of continuous data and it is a generalization of discrete Fourier transform [2].

Z-transform is used in many areas of applied mathematics as digital signal processing, control theory, economics and some other fields [3].

Difference equation are models of the world around us [4, 5]. From clocks to computers to chromosomes, processing discrete objects in discrete steps is a common theme, and are the discrete equivalent of differential equations and arise whenever an independent variable can have only discrete values [6]. The Difference equations used in situations of real life, in various sciences population models, genetics, psychology, economics, sociology, stochastic time series, combinatorial analysis, queuing problems, number theory, geometry, radiation quanta and electrical networks.

Recently there has been a great interest in studying nonlinear difference equations [7] and one of the reasons is a necessity for some techniques, which can be used in investigating equations arising in mathematical models describing real-life situations in population biology, economy, probability theory, genetics, psychology, sociology, and so forth. Some nonlinear difference equations, especially the boundedness, global attract, oscillatory and some other properties of second order nonlinear difference equations have been investigated by many authors see [8, 9, 10]. We need to study nonlinear equations of difference because almost all biological processes are nonlinear. In this paper, we use the Z-transformation to solve nonlinear difference equations. This paper consists of several sections, the first section includes the introduction, the second section contains Z-transfer and the third section includes the conversion of nonlinear difference equations into linear difference equations.

Z-transform

Definition:

Let $\{u_k\}$ be a sequence of numbers such that $u_k = 0\ for\ k < 0$, The Z-transform of this sequence is the series $Z(u_k)=\sum_{k=0}^{\infty}\frac{u_k}{z^k}$, Where Z is the transform variable.

A Table of Properties of Z-transform.

N	Sequence	Z-transform
1	1	$Z(1)=\sum_{k=0}^{\infty}\frac{1}{z^k}=1+z^{-1}+z^{-2}+z^{-3}+\cdots=\frac{1}{1-z^{-1}}$, $for\ \lvert z\rvert>1,$
2	u_{k+1}	$Z(u_{k+1})=\sum_{k=0}^{\infty}u_{k+1}z^{-k}$ $=z\sum_{k=0}^{\infty}u_{k+1}z^{-k-1}$ $=z\sum_{k=0}^{\infty}u_{k+1}z^{-(k+1)}$ $=z\sum_{m=1}^{\infty}u_m z^{-m}, m=k+1$ $=z[-u_0+\sum_{m=0}^{\infty}u_m z^{-m}]$ $=z[-u_0+U(z)],$ $=zU(z)-zu_0$
3	a^k	$Z(a^k)=\sum_{k=0}^{\infty}\frac{a^k}{z^k}=1+az^{-1}+a^2z^{-2}+a^3z^{-3}+\cdots$ $=\frac{1}{1-az^{-1}},\quad for\ \lvert z\rvert>a,$ $=\frac{z}{z-a}.$

Converting nonlinear difference equations into linear difference equations.

There are several ways to convert:

a. The first form of nonlinear difference equation homogeneous in u_k [11] can be expressed in the following form:

$$f\left(\frac{u_{k+1}}{u_k},k\right)=0,\qquad (a.1)$$

If the nonlinear function f is a polynomial function of u_{k+1}/u_k, then equation (a.1) can be written as

$$\prod_{i=1}^{n}[w_k-A_i(k)]=0,\qquad (a.2)$$

Where $w_k=u_{k+1}/u_k$, $A_i(k)$ is a known function of k, and n is the order of the polynomial function of w_k. the solution to each of the linear equations

$$w_k-A_i(k)=0,\qquad (a.3)$$

$$u_{k+1}-A_i(k)u_k=0\qquad (a.4)$$

Provide a solution to equation (a.1).

b. The second form of nonlinear difference equations

Consider the special class of nonlinear difference equations [12]

$$(u_{k+n})^{\gamma_1}(u_{k+n-1})^{\gamma_2}\dots(u_k)^{\gamma_{n+1}}=f(k),\quad (b.1)$$

Where the γ_i are constants and $f(k)$ is a given function. This nth-order, nonlinear equation can be transformed into an nth-order linear equation.

We can solve equation (b.1) by taking the logarithm of equation:

$$\gamma_1\log u_{k+n}+\gamma_2\log u_{k+n-1}+\cdots+\gamma_{n+1}\log u_k=\log f(k)\qquad (b.2)$$

And define

$$x_k=\log u_k\ \ and\ \ g(k)=\log f(k).\qquad (b.3)$$

Thus , x_k satisfies the following linear , inhomogeneous nth-order equation with constant coefficients:

$$\gamma_1 x_{k+n}+\gamma_2x_{k+n-1}+\cdots+\gamma_{n+1}x_k=g(k).\qquad (b.4)$$

c. The third form of nonlinear difference equations is of the Riccati type [13].

$$u_{k+1}u_k+p_ku_{k+1}+q_ku_k+r_k=0.\qquad (c.1)$$

Let $u_k=\frac{y_{k+1}}{y_k}-p_k$. Direct substitution of this expression into Equation (c.1) yields the linear equation

$$y_{k+2}+[q_k-p_{k+1}]y_{k+1}+[r_k-p_kq_k]y_k=0,\qquad (c.2)$$

Through this substitution, we obtain the equation of the linear differences that can be solved using the Z-transform.

Then solutions of Equation (c.1) are obtained from the relationship between u and y.

d. The four form of nonlinear difference equations

$$u_{k+1}=f(u_k),\qquad (d.1)$$

This technique is based on Lie's transformation group.

To convert equation (d.1) into a linear equation, we follow the following:

Let's begin by assuming that a solution $\xi(u)$ of the functional equation

$$D\xi\big(f(u)\big)=\xi(u)\frac{df}{du}(u)\qquad (d.2)$$

Is known for some constant D. Then we define a new dependent variable y by

$$\frac{dy}{du}=\frac{1}{\xi(u)}\qquad (d.3)$$

For u belonging to an open interval I in which $\xi(u)$ is different from 0. Using the chain rule , we have

$$\frac{d}{du}y(f(u)) = \frac{dy}{du}(f(u))\frac{df}{du}(u)$$
$$= \frac{1}{\xi(f(u))}\frac{df}{du}(u) \qquad (by\ Eq.(d.3))$$
$$= \frac{D}{\xi(u)} \qquad (by\ Eq.(d.2))$$
$$= D\,\frac{dy}{du}\,(u). \qquad (by\ Eq.(d.3))$$

Now we integrate to obtain

$$y(f(u)) = Dy(u) + C,$$

or

$$y(u_{k+1}) = Dy(u_k) + C,$$

Which is a linear equation of first order with constant coefficients. After the conversion, we can apply Z-transform, as shown in the following examples.

APPLICATION

Example 1. Consider the nonlinear difference equation

$$u_{k+1}^2 - 4u_{k+1}u_k - 5u_k^2 = 0. \qquad (E.1)$$

Now, we will convert the nonlinear difference equation to linear difference equation

We are going to take $w_k = \frac{u_{k+1}}{u_k}$ $(E.2)$ and plug it into the equation (E.1)

$w_k^2 - 4w_k - 5 = 0$

$(w_k - 5)(w_k + 1) = 0$

$w_k = 5 \qquad (E.3)$

Or

$w_k = -1 \qquad (E.4)$

We will substitute the equation (E.3) and (E.4) into equation (E.2)

$$5 = \frac{u_{k+1}}{u_k}$$

Or

$$-1 = \frac{u_{k+1}}{u_k}$$

$$u_{k+1} - 5u_k = 0 \qquad (E.5)$$

Or

$$u_{k+1} + u_k = 0 \qquad (E.6)$$

The equations (E.5) and (E.6)are linear difference equations [14]

We will take Z-transform of equations $(E.5)$

$Z\{u_{k+1}\} - 5Z\{u_k\} = 0$

Where $u_0 = 1$

$zZ\{u_k\} - zu_0 - 5Z\{u_k\}$
$= 0$

$(z-5)Z\{u_k\} = z$

$$Z\{u_k\} = \frac{z}{z-5}$$

$$u_k = Z^{-1}\left\{\frac{z}{z-5}\right\}$$

$u_k = 5^k.$

We will take Z-transform of equations $(E.6)$

$Z\{u_{k+1}\} + Z\{u_k\} = 0$

Where $u_0 = 1$

$zZ\{u_k\} - zu_0 + Z\{u_k\}$
$= 0$

$(z+1)Z\{u_k\} = z$

$$Z\{u_k\} = \frac{z}{z+1}$$

$$u_k = Z^{-1}\left\{\frac{z}{z+1}\right\}$$

$u_k = (-1)^k.$

Example 2. Consider the nonlinear difference equation

$$u_{k+1}^2 + u_k^2 = 0 \qquad (e.1)$$

Now, we will convert the nonlinear difference equation to linear difference equation

If we set $x_k = log u_k$, then x_k satisfies the $eq\ (e.1)$

$[\,2x_{k+1} + 2x_k = 0\ \ ] \div 2$

$x_{k+1} + x_k = 0 \qquad (e.2)$

Where $x_0 = 1$

Now, can be taking Z-transform of $eq\ (e.2)$

$Z\{x_{k+1}\} + Z\{x_k\} = 0$

$zZ\{x_k\} - zx_0 + Z\{x_k\} = 0$

$(z+1)Z\{x_k\} = z$

$$Z\{x_k\} = \frac{z}{z+1},$$

$$x_k = Z^{-1}\{\frac{z}{z+1}\}$$

$x_k = (-1)^k.$

Since $x_k = log u_k$

$u_k = e^{(-1)^k}.$

Example 3. Consider the nonlinear difference equation

$$u(k+1)u(k) + 2u(k+1) + 4u(k) + 9 = 0. \quad (E.1)$$

Let

$$u(k) = \frac{y(k+1)}{y(k)} - 2 \qquad (E.2)$$

We will substitute the (E.2) into the equation (E.1)

$$y(k+2) + 2y(k+1) + y(k) = 0. \qquad (E.3)$$

Where $y_0 = 1$

Now, can be taking Z-transform of $eq\ (E.3)$

$Z\{y_{k+2}\} + 2Z\{y_{k+1}\} + Z\{y_k\} = 0,$

$z^2Z\{y_k\} - z^2y_0 - zy(1) + 2zZ\{y_k\} - 2zy_0$
$+ Z\{y_k\} = 0,$

$$\{z^2+2z+1\}Z\{y_k\} = z^2-2z+2z$$
$$\frac{Z\{y_k\}}{z} = \frac{z}{z^2+2z+1} = \frac{z}{(z+1)(z+1)}$$
$$= \frac{A}{z+1} + \frac{B}{z+1}$$
$$y_k = Z^{-1}\left\{\frac{A}{z+1}\right\} + Z^{-1}\left\{\frac{B}{z+1}\right\}$$
$$y_k = A(-1)^k + kB(-1)^k$$

The general solution of the Riccati equation is

$$u(k) = \frac{A(-1)^{k+1} + B(k+1)(-1)^{k+1}}{A(-1)^k + Bk(-1)^k} - 2$$
$$= \frac{-1-C(k+1)}{1+Ck} - 2$$
$$= \frac{-3-C(3k+1)}{1+Ck}$$

Where C is arbitrary.

Example 4. Consider the nonlinear difference equation

$$u_{k+1}^2 - 3u_{k+1}u_k - 4u_k^2 = 0. \qquad (E.1)$$

Now, we will convert the nonlinear difference equation to linear difference equation

We are going to take $w_k = \frac{u_{k+1}}{u_k}$ $(E.2)$ and plug it into the equation (E.1)

$$w_k^2 - 3w_k - 4 = 0$$
$$(w_k-4)(w_k+1) = 0$$
$$w_k = 4 \qquad (E.3)$$

Or

$$w_k = -1 \qquad (E.4)$$

We will substitute the equation (E.3) and (E.4) into equation (E.2)

$$4 = \frac{u_{k+1}}{u_k}$$

Or

$$-1 = \frac{u_{k+1}}{u_k}$$
$$u_{k+1} - 4u_k = 0 \qquad (E.5)$$

Or

$$u_{k+1} + u_k = 0 \qquad (E.6)$$

The equations (E.5) and (E.6)are linear difference equations. We will take Z-transform of equations $(E.5)$

$$Z\{u_{k+1}\} - 4Z\{u_k\} = 0$$

Where $u_0 = 1$

$$zZ\{u_k\} - zu_0 - 4Z\{u_k\} = 0$$
$$(z-4)Z\{u_k\} = z$$
$$Z\{u_k\} = \frac{z}{z-4}$$
$$u_k = Z^{-1}\left\{\frac{z}{z-4}\right\}$$
$$u_k = 4^k.$$

We will take Z-transform of equations $(E.6)$

$$Z\{u_{k+1}\} + Z\{u_k\} = 0$$

Where $u_0 = 1$

$$zZ\{u_k\} - zu_0 + Z\{u_k\} = 0$$
$$(z+1)Z\{u_k\} = z$$
$$Z\{u_k\} = \frac{z}{z+1}$$
$$u_k = Z^{-1}\left\{\frac{z}{z+1}\right\}$$
$$u_k = (-1)^k.$$

Example 5. Consider the nonlinear difference equation

$$u(k+1) = au(k)\big(1-u(k)\big). \qquad (E.1)$$

Where a is a constant and $f(u) = au(1-u)$,

From equation

$$D\xi\big(f(u)\big) = \xi(u)\frac{df}{du}(u)$$

We have

$$D\xi\big(au(1-u)\big) = \xi(u)a(1-2u).$$

The form of this last equation suggests that we try a linear expression for ξ −say, $\xi(u) = cu + d$. We obtain

$$-Dcau^2 + Dcau + Dd$$
$$= -2acu^2 + (ac-2ad)u + ad.$$

Equation coefficients leads to

$D = a = 2$ and $c = -2d$

Let $c = -1$ and $d = \frac{1}{2}$; from equation $\frac{dy}{du} = \frac{1}{\xi(u)}$

$$\frac{dy}{du} = \frac{1}{-u+\frac{1}{2}},$$

Therefore, we take

$$y = -\log\left(\frac{1}{2} - u\right)$$

Or

$$u = \frac{1}{2} - e^{-y}.$$

Now we substitute the last expression into

$$u(k+1) = 2u(k)\big(1-u(k)\big)$$

To obtain

$$\frac{1}{2} - e^{-y(k+1)} = 2\left(\frac{1}{2} - e^{-y(k+1)}\right)\left(e^{-y(k)} + \frac{1}{2}\right)$$

Or

$$e^{-y(k+1)} = 2e^{-2y(k)}$$

Or

ISSN: 1814-635X (print), ISSN:2521-3520 (online) Volume 32, Issue 4, 2021 DOI: http://doi.org/10.23851/mjs.v32i4.1019

$y(k+1) = 2y(k) + ln2. \qquad (E.2)$
Where $y_0 = 1$
We will take Z-transform of equations $(E.2)$

$$Z\{y_{k+1}\} = 2Z\{y_k\} + lnZ\{2\}.$$

$$zZ\{y_k\} - zy_0 - 2Z\{y_k\} = ln\frac{2}{z-2}$$

$$\{z-2\}Z\{y_k\} = ln\frac{2}{z-2} + z$$

$$Z\{y_k\} = \frac{z}{z-2} + ln2$$

$$y(k) = C2^k + ln2,$$

And finally

$$u(k) = \frac{1}{2}(1 - e^{2^k})$$

Example 6. Consider the nonlinear difference equation

$u_{k+1}^2 = u_k \qquad (e.1)$

Now, we will convert the nonlinear difference equation to linear difference equation

If we set $x_k = logu_k$, then x_k satisfies the $eq\ (e.1)$

$[\ 2x_{k+1} = x_k\ \ ] \div 2$

$$x_{k+1} - \frac{1}{2}x_k = 0 \qquad (e.2)$$

Where $x_0 = 1$

Now, can be taking Z-transform of $eq\ (e.2)$

$$Z\{x_{k+1}\} - \frac{1}{2}Z\{x_k\} = 0$$

$$zZ\{x_k\} - zx_0 - \frac{1}{2}Z\{x_k\} = 0$$

$$\left(z - \frac{1}{2}\right)Z\{x_k\} = z$$

$$Z\{x_k\} = \frac{z}{z+1},$$

$$x_k = Z^{-1}\{\frac{z}{z-\frac{1}{2}}\}$$

$$x_k = \left(\frac{1}{2}\right)^k.$$

Since $x_k = logu_k$

$$u_k = e^{\left(\frac{1}{2}\right)^k}.$$

CONCLUSIONS

In this paper, we discussed how to solve nonlinear difference equations using Z-transform, but it turns out that nonlinear difference equations cannot be solved directly by Z-transform. For this reason, we present methods for converting nonlinear difference equations into linear difference equations that can be solved directly by Z transformation.

REFERENCES

[1] Roy, S.C.D., Difference Equations, Z-transforms and Resistive Ladders, IETE Journal of Equation, Vol. 52, Issue 1, 2011. https://doi.org/10.1080/09747338.2011.10876077

[2] Cochran, W.T., James W. Cooley, D.L.Favin, H. D. Helms, R. A. Kaenel, W .W. Lang, Jr. Maling G. C., D. E. Nelson,C. M. Rader, and Peter D. Welch. 1967. "What Is the Fast Fourier Transform?" Proceedings of the IEEE 55 (10): 1664-74. https://doi.org/10.1109/PROC.1967.5957.

[3] Reabiner, L. R., Schafer, R. W. , Rader, C. M.: The Chirp Z- transform Algorithm ,June 1969, IEEE Transactions On Audio And Electroacoustics Vol. Au-17, No. 2. https://doi.org/10.1109/TAU.1969.1162034

[4] L. Brand, A sequence defined by a difference equation, Amer. Math. Monthly 62 (1955), no. 7, 489–492. https://doi.org/10.2307/2307362

[5] S. N. Elaydi, An Introduction to Difference Equations, second edition, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 1999. https://doi.org/10.1007/978-1-4757-3110-1

[6] K. S. Berenhaut and S. Stević, The behaviour of the positive solutions of the difference equation $x_n = A + \left(\frac{x_{n-2}}{x_{n-1}}\right)^p$, J. Difference Equ. Appl. 12 (2006), no. 9, 909–918. https://doi.org/10.1080/10236190600836377

[7] I. Bajo and E. Liz, Global behaviour of a second-order nonlinear difference equation, J. Difference Equ. Appl. 17 (2011), no. 10, 1471–1486 https://doi.org/10.1080/10236191003639475

[8] M. Dehghan, R. Mazrooei-Sebdani, and H. Sedaghat, Global behaviour of the Riccati difference equation of order two, J. Difference Equ. Appl. 17 (2011), no. 4, 467–477. https://doi.org/10.1080/10236190903049017

[9] Stević, S.: Solvable subclasses of a class of nonlinear second-order difference equations. Adv. Nonlinear Anal. 5(2), 147–165 (2016). https://doi.org/10.1515/anona-2015-0077

[10] Stević, S.: Solvability of a one-parameter class of nonlinear second-order difference equations by invariants. Adv. Differ. Equ. 2019, Article ID 151 (2019). https://doi.org/10.1186/s13662-019-2091-7

[11] S. Stevi´c, M. A. Alghamdi, N. Shahzad, and D. A. Maturi, On a class of solvable difference equations, Abstr. Appl. Anal. 2013 (2013), Art. ID 157943. https://doi.org/10.1155/2013/157943

[12] On a solvable nonlinear difference equation of higher order, Turkish J. Math. 42 (2018), no. 4, 1765–1778. https://doi.org/10.3906/mat-1705-33

[13] D. T. Tollu, Y. Yazlik, and N. Taskara, On the solutions of two special types of Riccati difference equation via Fibonacci numbers, Adv. Difference Equ. 2013 (2013), 174, 7 pp. https://doi.org/10.1186/1687-1847-2013-174

[14] Y. Halim and M. Bayram, On the solutions of a higher-order difference equation in terms of generalized

Fibonacci sequences, Math. Methods Appl. Sci. 39 (2016), no. 11, 2974–2982.
https://doi.org/10.1002/mma.3745