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Analysis of Composite Plate Subjected to Impact Load (Part I: Analytical Solution)*

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Abstract

Composite laminated plates are found in many types of structural and aeronautical weight sensitive applications, such as, ships, automobiles. This work includes developing the equations of motion of composite laminated plates to obtain deformations and stresses under impact loading.

The Hertzian impact law is modeled to describe the contact force between the projectile and the laminated plates. Analytical solution using (Navier solution) and theories (First Order Shear Deformation Theory 'FSDT' and Higher Order Shear Deformation Theory 'HOST 5') is presented. The results show the parametric effect of number of layers, lamination angle, and orthotropy ratio on the behavior of the laminated plates.

الخلاصـــــة

1. Introduction

Deformation engineering of modern composite materials has had a significant impact on the technology of design and construction. The advanced composite materials are lighter, stiffer and stronger than any other structural material man has ever used. Composite materials are ideal for structural applications where high strength-to-weight and stiffness-to-weight ratios are required. Aircraft and spacecraft are typical weight-sensitive structures in which composite materials are cost-effective.

Although the response of composite material to particle or foreign-body impact could be studied using empirical or semiempirical approaches, this appears undesirable because of the large and costly efforts that would be required to cover the various combinations of constituent material, layups, stacking sequences and constructions. The determination of the available fibers for impact response would constitute a monumental task. When designing for impact response, it appears desirable to have a criterion for determining how the various properties of the target and the impact parameters influence target damage. The analytical approach described here is oriented towards that goal.

In early days, Classical Lamination Theory (CLT) based on the Kirchhoff hypothesis was adopted for the analysis of laminated composite plates. It was soon realized that this theory which neglects shear strain and transverse normal strain and stress is inadequate for the analysis of laminated composite plates as transverse shear effects are more pronounced, even in thin composite plates. This realization was the starting point in the development of the First Shear Plate Theory, in which the displacements are taken linearly over the thickness of the entire laminate of each layer with the assumption that the normal to the middle surface need not remain normal after deformation. Reissner [1] and Mindlin [2] initiated this work. However, both of these theories [1,2] neglected the effects of transverse normal strain and stress and were based on a non-realistic (constant) variation of the transverse shear strain and stress through the plate thickness. This necessitated the introduction of a shear correction factor. Later, these discrepancies were rectified by introducing higher-order functions in the displacement model leading to the higher-order plate theories. Yang, P. C. et. al. [3] extended the Rissner-Mindlin shear deformation theory of isotropic homogeneous plates to laminates composite plate. Whitney and Pagano [4] incorporated plane Stress State into the theory of Yang [3] in connection with the analysis of thick laminated plates. Whitney [5] applied this theory to calculate the in-plane stresses, which in turn are used in equilibrium equations to evaluate the inter-laminar transverse stresses. A generalized Le'vy-type solution in conjunction with the closed form solution was developed for the bending, buckling and vibration of antisymmetric angle-ply laminated plates by A. A. Khdeir [6]. A higher-order shear deformation theory of laminated composite plates was developed by J. N. Reddy [7]. This theory contains the same dependent unknowns as in the first-order shear deformation theory of Whitney and Pagano [4]. P. Bose and J. N. Reddy [8] presented a unified third-order laminated plate theory that contains classical, first and third-order theories as special cases. Analytical solution using the Navier solution procedure was presented.

2. First Order Shear Deformation Theory (FSDT)

The displacement in *x*, *y* and *z* directions can take the following form [9]:

$$\begin{aligned}
u(x,y,z)&=u_o(x,y)+z\phi_x(x,y)\\
v(x,y,z)&=v_o(x,y)+z\phi_y(x,y) \quad \text{.......... (1)}\\
w(x,y,z)&=w_o(x,y)
\end{aligned}$$

The strain components will be derived, based on the displacement, as:

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = \frac{\partial u_o}{\partial x} + z\frac{\partial \phi_x}{\partial x}\\
\varepsilon_y &= \frac{\partial v}{\partial y} = \frac{\partial v_o}{\partial y} + z\frac{\partial \phi_y}{\partial y}\\
\varepsilon_z &= \frac{\partial w}{\partial z}\\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(\frac{\partial u_o}{\partial y} + \frac{\partial v_o}{\partial x}\right) + z\left(\frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}\right) \quad \text{.......... (2)}\\
\gamma_{yz} &= \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \phi_y + \frac{\partial w_o}{\partial y}\\
\gamma_{xz} &= \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \phi_x + \frac{\partial w_o}{\partial x}
\end{aligned}$$

Substituting eq.(2) in the stress-strain relation of the lamina, the constitutive relations for any layer in the (x , y) can be expressed in the form [10]:

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{Bmatrix} = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}_k \left\{\begin{Bmatrix}\varepsilon_x^o\\ \varepsilon_y^o\\ \gamma_{xy}^o\end{Bmatrix} + z\begin{Bmatrix}R_x\\ R_y\\ R_{xy}\end{Bmatrix}\right\} \quad \begin{Bmatrix}\tau_{yz}\\ \tau_{xz}\end{Bmatrix}_k = \begin{bmatrix}\bar{Q}_{44} & \bar{Q}_{45}\\ \bar{Q}_{45} & \bar{Q}_{55}\end{bmatrix}_k \begin{Bmatrix}\phi_y + \dfrac{\partial w_o}{\partial y}\\ \phi_x + \dfrac{\partial w_o}{\partial x}\end{Bmatrix} \quad \text{... (3)}$$

where:

$$\begin{Bmatrix}\varepsilon_x^o\\ \varepsilon_y^o\\ \gamma_{xy}^o\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial u_o}{\partial x}\\ \dfrac{\partial v_o}{\partial y}\\ \dfrac{\partial u_o}{\partial y} + \dfrac{\partial v_o}{\partial x}\end{Bmatrix}, \quad \begin{Bmatrix}R_x\\ R_y\\ R_{xy}\end{Bmatrix} = \begin{Bmatrix}\dfrac{\partial \phi_x}{\partial x}\\ \dfrac{\partial \phi_y}{\partial y}\\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x}\end{Bmatrix} \quad \text{.......... (4)}$$

$$[\bar{Q}] = [T]^T[Q][T] \quad \text{.......... (5)}$$

[T]: Transformation matrix given by:

$$[T]=\begin{bmatrix} c^2 & s^2 & sc & 0 & 0 \\ s^2 & c^2 & -sc & 0 & 0 \\ -2sc & 2sc & c^2-s^2 & 0 & 0 \\ 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & s & c \end{bmatrix}$$

where:

c: cos θ,

s =sin θ

$Q_{11}=E_1/(1-\nu_{12}\nu_{21})$, $Q_{12}=\nu_{12}E_1(1-\nu_{12}\nu_{21})$, $Q_{22}=E_2/(1-\nu_{12}\nu_{21})$, $Q_{33}=G_{12}$, $Q_{44}=G_{23}$, $Q_{55}=G_{13}$

The transformation eq.(5) can be represented in the following form:

$$\begin{aligned} \bar{Q}_{11}&=Q_{11}c^4+2(Q_{12}+2Q_{33})s^2c^2+Q_{22}s^4 \\ \bar{Q}_{12}&=(Q_{11}+Q_{22}-4Q_{33})s^2c^2+Q_{12}(s^4+c^4) \\ \bar{Q}_{22}&=Q_{11}s^4+2(Q_{12}+2Q_{33})s^2c^2+Q_{22}c^4 \\ \bar{Q}_{16}&=(Q_{11}-Q_{12}-2Q_{33})sc^3+(Q_{12}-Q_{22}+2Q_{33})s^3c \quad \ldots\ldots (6) \end{aligned}$$

$\bar{Q}_{26}=(Q_{11}-Q_{12}-2Q_{33})cs^3+(Q_{12}-Q_{22}+2Q_{33})sc^3$

$\bar{Q}_{66}=(Q_{66}+Q_{22}-2Q_{12}-2Q_{33})s^2c^2+Q_{33}(s^4+c^4)$

$\bar{Q}_{44}=Q_{44}c^2+Q_{55}s^2$, $\bar{Q}_{45}=(Q_{55}-Q_{44})sc$, $\bar{Q}_{55}=Q_{44}s^2+Q_{55}c^2$

all other elements of $[Q_{ij}]$ and $[\bar{Q}_{ij}]$ are zero.

The entire collection of forces and moments resultants for N-layered laminate is defined as:

$$\begin{Bmatrix} N_x & M_x \\ N_y & M_y \\ N_{xy} & M_{xy} \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k (1\,,z)dz \qquad \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k dz \quad \ldots\ldots (7)$$

After substituted eq.(3) into eq.(7) yielding:

$$\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_x^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} R_x \\ R_y \\ R_{xy} \end{Bmatrix} \quad \ldots\ldots (8a)$$

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_x^o \\ \gamma_{xy}^o \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} R_x \\ R_y \\ R_{xy} \end{Bmatrix} \quad \text{(8b)}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \dfrac{\partial w_o}{\partial y} \\ \phi_x + \dfrac{\partial w_o}{\partial x} \end{Bmatrix} \quad \text{(8c)}$$

where:

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k (z_k - z_{k-1}), \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{N} \left(\overline{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right) \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{N} \left(\overline{Q}_{IJ}\right)_k \left(z_k^3 - z_{k-1}^3\right) \end{aligned}\right\} i,j = 1,2,6 \quad \text{(9)}$$

$$A_{ij} = K_{ij}^2 \sum_{k=1}^{N} \left(Q_{ij}\right)_k (z_k - z_{k-1}) \qquad i,j = 4,5$$

and K_{ij} are the transverse shear correction factors. $K_{ij}=\pi^2/12$ [11].

3. Differential Equations of Equilibrium of Laminated Plates

The equilibrium differential equations in terms of the moments and forces resultants for a plate are [9]:

$$\begin{aligned} & N_{x'x} + N_{xy'y} = I_1 \ddot{u} + I_2 \ddot{\phi}_x \\ & N_{xy'x} + N_{y'y} = I_1 \ddot{v} + I_2 \ddot{\phi}_y \\ & Q_{x'x} + Q_{y'y} = I_1 \ddot{w} + q(x,y,t) \quad \text{(10)} \\ & M_{xy'y} - Q_x = I_2 \ddot{u} + I_3 \ddot{\phi}_x \\ & M_{xy'x} + M_{y'y} - Q_y = I_2 \ddot{v} + I_3 \ddot{\phi}_y \end{aligned}$$

where, a comma denotes differentiation of principal symbol with respect to the subscript, q(x,y,t) is the applied load and I_1, I_2, I_3 ,are the normal, coupled normal- rotary and rotary inertia coefficients, defined as:

$$(I1, I2, I3) = \int_{-h/2}^{h/2} \rho\left(1, z, z^2\right) dz = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \rho^{(k)}\left(1, z, z^2\right) dz \quad \text{(11)}$$

4. Higher Order Shear Deformation Theory (HOST 5)

A higher-order laminated plate theory involves, the in plane displacement (*u*,*v*) expanded up to the cubic term in the thickness term z, and the transverse displacement *w* which is constant through plate thickness is considered. This is done to take into account the parabolic variation of the transverse shear stresses through the thickness of the plate [8].

$$\begin{aligned}
u(x,v,z,t) &= u_o(x,y,t) + z\left[\phi_x(x,y,t) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_x(x,y,t) + \frac{\partial w}{\partial x}(x,y,t)\right)\right] \\
v(x,v,z,t) &= v_o(x,y,t) + z\left[\phi_y(x,y,t) - \frac{4}{3}\left(\frac{z}{h}\right)^2\left(\phi_y(x,y,t) + \frac{\partial w}{\partial y}(x,y,t)\right)\right] \quad \text{....... (12)} \\
w(x,y,z,t) &= w_o(x,y,t)
\end{aligned}$$

The constitutive relations for any layer in the (x,y) can be expressed in the form [10]:

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix}_k \left\{ \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} + z \begin{Bmatrix} R_x \\ R_y \\ R_{xy} \end{Bmatrix} + z^3 \begin{Bmatrix} S_x \\ S_y \\ S_{xy} \end{Bmatrix} \right\} \quad \text{.......................... (13 a)}$$

$$\begin{Bmatrix} \tau_{yz} \\ \tau_{xz} \end{Bmatrix}_k = \begin{bmatrix} \overline{Q}_{44} & \overline{Q}_{45} \\ \overline{Q}_{45} & \overline{Q}_{55} \end{bmatrix}_k \left\{ \begin{Bmatrix} \phi_y + \dfrac{\partial w_o}{\partial y} \\ \phi_x + \dfrac{\partial w_o}{\partial x} \end{Bmatrix} + 3z^2 \begin{Bmatrix} \theta_y \\ \theta_x \end{Bmatrix} \right\} \quad \text{.. (13 b)}$$

where:

$$\begin{Bmatrix} \theta_x \\ \theta_y \end{Bmatrix} = \frac{-4}{3h^2} \begin{Bmatrix} \dfrac{\partial w}{\partial x} + \phi_x \\ \dfrac{\partial w}{\partial y} + \phi_y \end{Bmatrix} \quad \text{.. (14 a)}$$

$$\begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_o}{\partial x} \\ \dfrac{\partial v_o}{\partial y} \\ \dfrac{\partial u_o}{\partial y} + \dfrac{\partial v_o}{\partial x} \end{Bmatrix}, \begin{Bmatrix} R_x \\ R_y \\ R_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}, \begin{Bmatrix} S_x \\ S_y \\ S_{xy} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix} \quad \text{........... (14 b)}$$

The following definition for stress-strain is resultant expressions appropriate to the higher order shear deformation theory:

$$\begin{bmatrix} N_x & M_x & M_x^* \\ N_y & M_y & M_y^* \\ N_{xy} & M_{xy} & M_{xy}^* \end{bmatrix} = \sum_{L=1}^{N} \int_{h_l}^{h_{L+1}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{bmatrix} (1 \quad , \quad z \quad , \quad z^3) dz \quad \text{.................................... (15 a)}$$

$$\begin{bmatrix} Q_x & Q_x^* \\ Q_y & Q_y^* \end{bmatrix} = \sum_{L=1}^{N} \int_{h_l}^{h_{L+1}} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} (1 \quad , \quad z^2) dz \quad \text{.. (15 b)}$$

after substituted eq.(13) into eq.(15) yielding:

$$\begin{Bmatrix} \begin{Bmatrix} N_x \\ N_y \\ N_{xy} \end{Bmatrix} \\ \begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} \\ \begin{Bmatrix} M_x^* \\ M_y^* \\ M_{xy}^* \end{Bmatrix} \end{Bmatrix} = \begin{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ & A_{22} & A_{26} \\ SYM & & A_{66} \end{bmatrix} & \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ & B_{22} & B_{26} \\ SYM & & B_{66} \end{bmatrix} & \begin{bmatrix} E_{11} & E_{12} & E_{16} \\ & E_{22} & E_{26} \\ SYM & & E_{66} \end{bmatrix} \\ & \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ & D_{22} & D_{26} \\ SYM & & D_{66} \end{bmatrix} & \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ & F_{22} & F_{26} \\ SYM & & F_{66} \end{bmatrix} \\ SYM & & \begin{bmatrix} H_{11} & H_{12} & H_{16} \\ & H_{22} & H_{26} \\ SYM & & H_{66} \end{bmatrix} \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \varepsilon_{xy}^o \end{Bmatrix} \\ \begin{Bmatrix} R_x \\ R_y \\ R_{xy} \end{Bmatrix} \\ \begin{Bmatrix} S_x \\ S_y \\ S_{xy} \end{Bmatrix} \end{Bmatrix}$$

$$\begin{Bmatrix} Q_x \\ Q_y \\ Q_x^* \\ Q_y^* \end{Bmatrix} = \begin{bmatrix} A_{55} & A_{45} & D_{55} & D_{45} \\ & A_{44} & D_{45} & D_{44} \\ & & F_{55} & F_{45} \\ SYM & & & F_{44} \end{bmatrix} \begin{Bmatrix} \phi_x + \dfrac{\partial w_o}{\partial x} \\ \phi_y + \dfrac{\partial w_o}{\partial y} \\ 3\theta_x \\ 3\theta_y \end{Bmatrix} \quad \text{..... (16)}$$

where: A_{ij}, B_{ij}, etc., are the plate stifnesses, defined by:

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, H_{ij}) = \int_{-h/2}^{h/2} \overline{Q}_{ij}^{(k)} (1, z, z^2, z^3, z^4, z^6) dz \qquad (i, j = 1,2,6)$$

$$(A_{ij}, D_{ij}, F_{ij}) = \int_{-h/2}^{h/2} \overline{Q}_{ij}^{(k)} (1, z^2, z^4) dz \qquad (i, j = 4,5) \quad \text{........ (17)}$$

5. Differential Equations of Equilibrium of Laminated Plates

The equilibrium differential equations in terms of the moments and forces resultants for a plate are [12]:

$$* \quad N_{x,x} + N_{xy,y} = I_1 \ddot{u} + \Gamma_2 \ddot{\phi}_x - \frac{4}{3h^2} I_4 \ddot{w}_{,x}$$

$$* \quad N_{xy,x} + N_{y,y} = I_1 \ddot{v} + \Gamma_2 \ddot{\phi}_y - \frac{4}{3h^2} I_4 \ddot{w}_{,y}$$

$$* \quad Q_{x,x}+Q_{y,y}-\frac{4}{h^2}\left(Q^*_{x,x}+Q^*_{y,y}\right)+\frac{4}{3h^2}\left(M^*_{x,xx}+2M^*_{xy,xy}+M^*_{y,yy}\right)$$
$$=I_1\ddot{w}-\frac{16}{9h^4}I_7\left(\ddot{w}_{,xx}+\ddot{w}_{,yy}\right)+\frac{4}{3h^2}I_4\left(\ddot{u}_{,x}+\ddot{v}_{,y}+\frac{4}{3h^2}\Gamma_5\left(\ddot{\phi}_{x,x}+\ddot{\phi}_{y,y}\right)\right)-q(x,y,t) \quad \ldots (18)$$
$$* \quad M_{x,x}+M_{xy,y}-Q_x+\frac{4}{h^2}Q^*_x-\frac{4}{3h^2}\left(M^*_{x,x}+M^*_{xy,y}\right)=\Gamma_2\ddot{u}+\Gamma_3\ddot{\phi}_y-\frac{4}{2h^2}\Gamma_5\ddot{w}_{,x}$$
$$* \quad M_{xy,x}+M_{y,y}-Q_y+\frac{4}{h^2}Q^*_y-\frac{4}{3h^2}\left(M^*_{xy,x}+M^*_{y,y}\right)=\Gamma_2\ddot{v}+\Gamma_3\ddot{\phi}_y-\frac{4}{2h^2}\Gamma_5\ddot{w}_{,x}$$

where:

$$\Gamma_2=I_2-\frac{4}{3h^2}I_4 \quad , \quad \Gamma_5=I_5-\frac{4}{3h^2}I_7 \quad , \quad \Gamma_3=I_3-\frac{8}{3h^2}I_5+\frac{16}{9h^4}I_7 \quad \ldots\ldots (19\text{ a})$$

$$\left(I_1,I_2,I_3,I_4,I_5,I_7\right)=\sum_{k=1}^{N}\int_{z_k}^{z_{k+1}}\rho^{(k)}\left(1,z,z^2,z^3,z^4,z^6\right)dz \quad \ldots\ldots (19\text{ b})$$

6. Exact Solution for Simply Supported Rectangular Plates

The exact analytical solution of the differential equations (10) (FSDT), and (18) (HOST 5) for a general laminate plate under arbitrary boundary conditions is impossible task. However, closed-form solution for 'simply-supported' rectangular plates is to be considered. The following simply supported boundary conditions are assumed, see **Fig.(1)**.

u(x,0)=u(x,b)=v(0,y)=v(a,y)=0 cross-ply

$N_y(x,0)=N_y(x,b)=N_x(0,y)=N_x(a,y)=0$

v(x,0)=v(x,b)=u(0,y)=u(a,y)=0 angle-ply ……………………… (20)

$N_{xy}(x,0)=N_{xy}(x,b)=N_{xy}(0,y)=N_{xy}(a,y)=0$

w(x,0)=w(x,b)=w(0,y)=w(a,y)=0

$M^*_y(x,0) = M^*_y(x,b) = M^*_x(0,y) = M^*_x(a,y)=0$ cross-ply & angle-ply

$M_y(x,0) = M_y(x,b) = M_x(0,y) = M_x(a,y)=0$

$\phi_x(x,0)= \phi_x(x,b)= \phi_y(0,y)= \phi_y(a,y)=0$

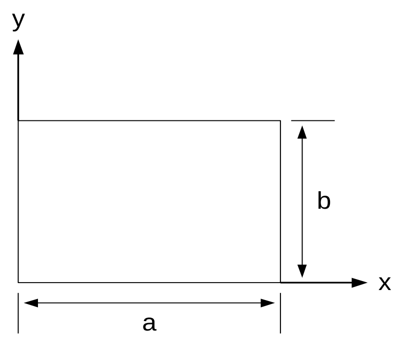

Figure (1) Geometry and the co-ordinate system of a rectangular plate

The analytical solution can be obtained by using the equations of (FSDT) as in the following:

Substituting eq.(8) into eq.(10) yields:

$$\begin{aligned}
&A_{11}u,_{xx}+(A_{12}+A_{66})v,_{xy}+2A_{16}u,_{xy}+A_{16}v,_{xx}+B_{11}\phi_{x,xx}+(B_{12}+B_{66})\phi_{y,xy}\\
&+2B_{16}\phi_{x,xy}+B_{16}\phi_{y,xx}+A_{26}v,_{yy}+A_{66}u,_{yy}+B_{26}\phi_{y,yy}+B_{66}\phi_{x,yy}=I_1u,_{tt}+I_2\phi_{x,tt}\\[1em]
&A_{16}u,_{xx}+2A_{26}v,_{xy}+(A_{66}+A_{12})u,_{xy}+A_{66}v,_{xx}+B_{16}\phi_{x,xx}+2B_{26}\phi_{y,xy}\\
&+(B_{66}+B_{12})\phi_{x,xy}+B_{66}\phi_{y,xx}+A_{22}v,_{yy}+A_{26}u,_{yy}+B_{22}\phi_{y,yy}+B_{26}\phi_{x,yy}=I_1v,_{tt}+I_2\phi_{y,tt}\\[1em]
&A_{45}(\phi_{y,x}w,_{yx})+A_{55}(\phi_{x,x}+w,_{xx})+A_{44}(\phi_{y,y}+w,_{yy})\\
&+A_{45}(\phi_{x,y}+w,_{xx})=I_1w,_{tt}+q(x,y)\\[1em]
&B_{11}u,_{xx}+(B_{12}+B_{66})v,_{xy}+2B_{16}u,_{xy}+B_{16}v,_{xx}+D_{11}\phi_{x,xx}+(D_{12}+D_{66})\phi_{y,xy}\\
&+2D_{16}\phi_{x,xy}+D_{16}\phi_{y,xx}+B_{26}v,_{yy}+B_{66}u,_{yy}+D_{26}\phi_{y,yy}+D_{66}\phi_{x,yy}-A_{45}(\phi_y+w,_y)\\
&-A_{55}(\phi_x+w,_x)=I_3\phi_{x,tt}+I_2u,_{tt}\\[1em]
&B_{16}u,_{xx}+2B_{26}v,_{xy}+(B_{66}+B_{12})u,_{xy}+B_{66}v,_{xx}+D_{16}\phi_{x,xx}+2D_{26}\phi_{y,xy}\\
&+(D_{66}+D_{12})\phi_{x,xy}+D_{66}\phi_{y,xx}+B_{22}v,_{yy}+B_{26}u,_{yy}++D_{22}\phi_{y,yy}+D_{26}\phi_{x,yy}-A_{44}(\phi_x+w,_x)\\
&-A_{45}(\phi_y+w,_y)=I_3\phi_{y,tt}+I_2v,_{tt}
\end{aligned} \qquad \text{... (21)}$$

7. Static Solution

The exact static solution exists for antisymmetric cross-ply and antisymmetric angle-ply rectangular plates, when the inertial loads on the right hand side of eq.(21) are set to zero.

According to the Navier solution, the following form of spatial variation of (w,ϕ_x,ϕ_y) that satisfies the differential equations, eq.(21) and the boundary conditions in eq.(20) can be assumed:

$$\begin{aligned}
&w=\sum_{m,n=1}^{\infty}W_{mn}\sin\alpha x\cos\alpha y\ ,\\
&\phi_x=\sum_{m,n=1}^{\infty}X_{mn}\cos\alpha x\sin\beta y\ , \quad \text{.......... (22 a)}\\
&\phi_y=\sum_{m,n=1}^{\infty}Y_{mn}\sin\alpha x\cos\beta y
\end{aligned}$$

where: $\alpha = m\pi/a$ *and* $\beta = n\pi/b$. The variation of *u* and *v* is different for antisymmetric cross-ply and antisymmetric angle-ply laminates.

$$u = \sum_{m,n=1}^{\infty} U_{mn} \cos\alpha x \sin\beta y\ ,\qquad v = \sum_{m,n=1}^{\infty} V_{mn} \sin\alpha x \cos\beta y$$

antisymmetric cross-ply

.... (22 b)

$$u = \sum_{m,n=1}^{\infty} U_{mn} \sin\alpha x \cos\beta y\ ,\qquad v = \sum_{m,n=1}^{\infty} V_{mn} \cos\alpha x \sin\beta y$$

antisymmetric angle-ply

By substituting eq.(22) into eq.(21), the solution to these equations exists when the transverse loading is:

$$q(x,y) = \sum_{m,n=1}^{\infty} Q_{mn} \sin\alpha x \sin\beta y \quad \text{.......... (23)}$$

where:

Q_{mn}: can be evaluated for different types of loading conditions as [8]*:*

$$Q_{mn} = (4p/ab)\sin(m\pi/2)\sin(n\pi/2) \quad \text{for point load at the center (PL)} \quad \text{..... (24)}$$

Under these conditions eq.(22) becomes:

$$[K]\{\Delta\} = \{F\} \quad \text{.......... (25)}$$

where:

$$\{\Delta\} = \{U_{mn} \quad V_{mn} \quad W_{mn} \quad X_{mn} \quad Y_{mn}\}$$
$$\{F\} = (0 \quad 0 \quad Q_{mn} \quad 0 \quad 0)$$

The elements of the coefficient matrix [K] (*Stiffness Matrix*) are given in [10]. When the (HOST 5) is employed, using the same above procedure, the [K] matrix is calculated and given in [10].

8. Theory Development of Impact

The rate of change of velocity during impact (as the two bodies come in contact) is:

$$m_1 \frac{dV_1}{dt} = -p \quad \text{.......... (26)}$$

If we denote by the same distance that the impactor and target approach one another because of local compression at the point of contact, the velocity of this approach is:

$$\dot{\mu} = V_1 + V_2 \quad \text{.......} \quad (27)$$

If the contact duration between the impactor and the target is very long in comparison with their natural periods, vibrations of the system can be neglected. Therefore, the Hertzian law is applicable.

$$p = n_1 \mu^{3/2} \quad \text{.......} \quad (28)$$

The term n_1 is defined as:

$$n_1 = \frac{4\sqrt{R_1}}{3\pi(k_1 + k_2)} \quad \text{.......} \quad (29)$$

where:

k_1,k_2: depend on properties of impactor and target and defined in [10]:

Differentiating (27), combining it with (26), and substituting of (28) into the resultant equation yields:

$$\ddot{\mu} = \frac{-n_1}{m_1} \mu^{3/2} \quad \text{.......} \quad (30)$$

If both sides of eq.(30) are multiplied by $\dot{\mu}$ and the resultant equation is integrated yield:

$$\left(\dot{\mu}^2 - V^2\right) = -\frac{4}{5} \frac{n_1 \mu^{5/2}}{m_1} \quad \text{.......} \quad (31)$$

where:

V: is the approach velocity of the two bodies at t = 0, that is, at the beginning of impact.

Maximum deformation, μ_1, occurs when $\dot{\mu}$ =0 and is

$$\mu_1 = \left(\frac{5m_1 V^2}{4n_1}\right)^{2/5} \quad \text{.......} \quad (32)$$

Substituting of eq.(32) in to eq.(28) gives the following final relationship:

$$p = n_1^{2/5} \left(\frac{5m_1 V^2}{4}\right)^{3/5} \quad \text{.......} \quad (33)$$

For the case of the Hertzian contact problem involving a sphere pressed onto a flat surface by a force P, the area of contact is very small, therefore we assumed that the impact force is concentrated at the point of contact.

9. Result and Discussion

In the following it is assumed that the material is fiber-reinforced and remains in the elastic range. The boundary conditions are SSSS, and the analytical procedure (HOST 5) is used in this work.

The material properties are:

$E_2 = 6.92\ x10^9\ N/m^2$, $E_1 = 40E_2$, $G_{12} = G_{13} = 0.5E_2$, $G_{23} = 0.6E_2$, $\nu_{12} = 0.25$

Dimensions of plate:

$a = 1\ m$, $b = 1\ m$, $h = 0.02\ m$

Properties of impactor:

$E = 200\ x10^6\ N/m^2$, $v = 0.3$, $mass = 0.1\ kg$, $Radius = 0.01\ m$

From **Figs.(2)** and **(3)** it can be observed that the effect of coupling between bending and extension on deflections is significant for all modulus ratios except those quite close to $E_1/E_2 = 1$.

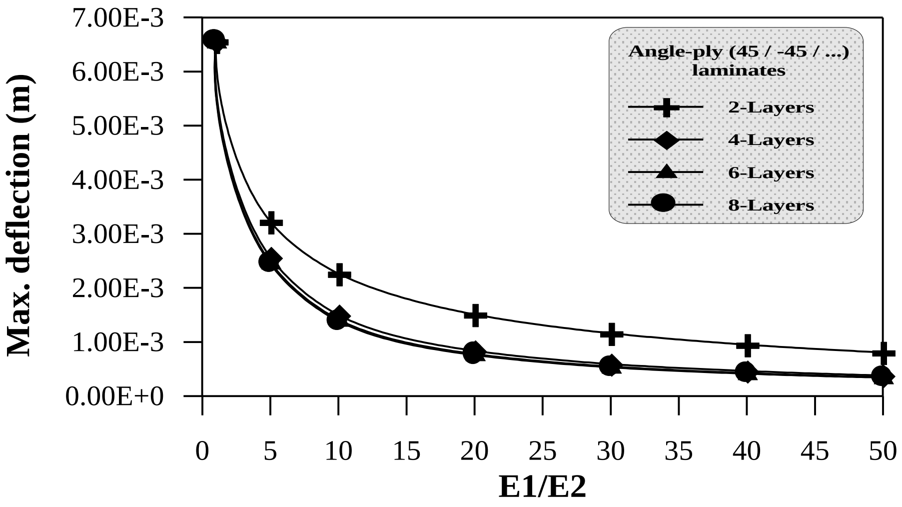

Figure (2) Effect of orthotropic ratio on a maximum deflection of a square antisymmetric angle-ply laminated plate (V=10 m/sec)

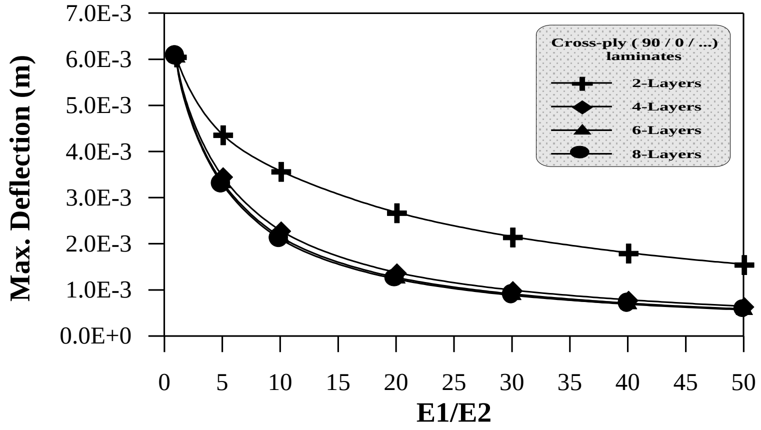

Figure (3) Effect of orthotropic ratio on a maximum deflection of a square antisymmetric cross-ply laminated plate (V=10 m/sec)

Figures (4) and **(5)** show the relation between similar calculations, which is repeated for the case of (20 m/sec) velocity of impactor, respectively.

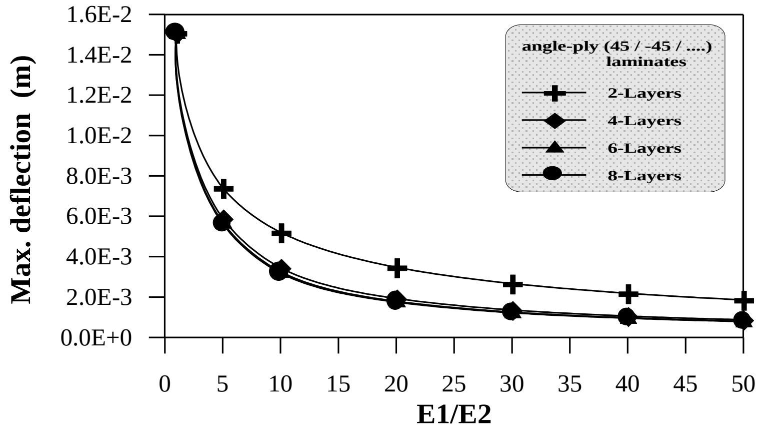

Figure (4) Effect of orthotropic ratio on a maximum deflection of a square antisymmetric angle-ply laminated plate (V=20 m/sec)

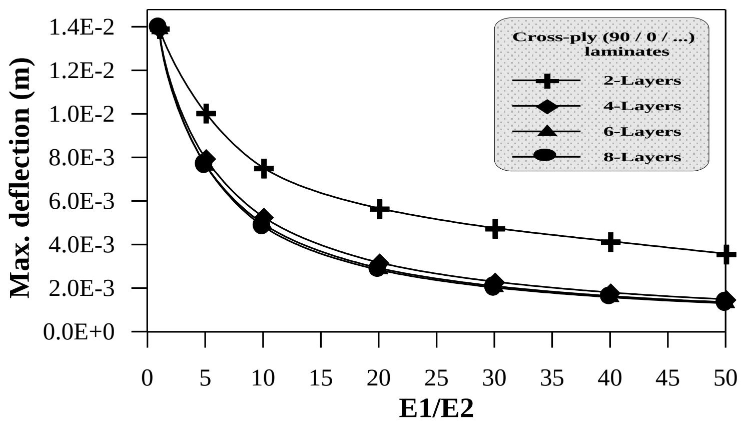

Figure (5) Effect of orthotropic ratio on a maximum deflection of a square antisymmetric cross-ply laminated plate (V=20 m/sec)

In general, The central deflection decreases with the increase in orthotropic ratio (E1/E2) and the number of layers due to increasing the stiffness of the laminate, but it increases with the increase in the velocity of impactor due to the increase in impact loading. The difference in deflections between the 2 and 6 or 8 layers is quite substantial due to the bending-stretching coupling which is vary according to layer numbers.

Figure (6) shows the relation between the lamination angle (θ°) and the central deflection of a square antisymmetric laminated plates, which consist of 2, 4, 6 and ∞ layers. Clearly, coupling is quite significant for two-layered laminates, which decreases as the number of layers increases. Increasing N more than 8 for antisymmetric laminate has no effect on the laminate stiffness because B_{ij} die out when N=8 Moreover the rate of change of deflection with θ° is large in the range of θ =5° to θ=30°, therefore, for design purpose θ is recommended to be between 30° and 45°.

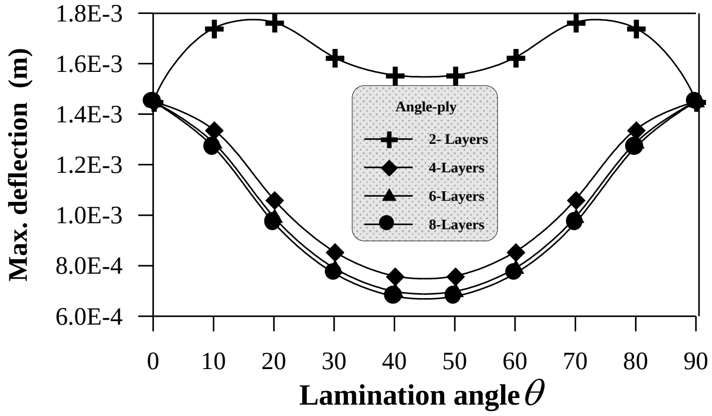

Figure (6) Effect of lamination angle on a maximum deflection of a square antisymmetric angle-ply laminated plate (V=15 m/sec)

As it is indicated from **Fig.(7)** the central deflection reduces with the increase of (E_1/ E_2) ratio due to the increase of laminate stiffness. Furthermore, the minimum deflection occurs when the lamination angle is equal to 45°.

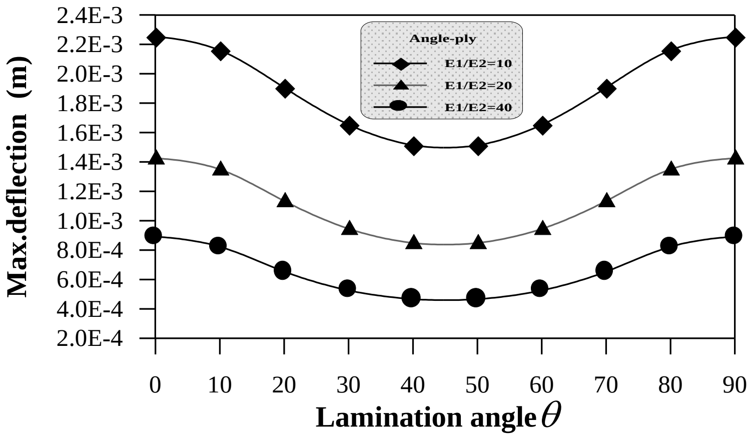

Figure (7) Effect of lamination angle and orthotropic ratio on a maximum deflection of a square antisymmetric angle-ply laminated plate (V=10 m/sec)

Figure (8) shows the effect of lamination angle on the stiffness-to-weight ratio, for antisymmetric angle-ply laminates with 2, 4, and 6 layers. This figure shows that increasing lamination angle from (^{o} □□to □45^{o}), increases stiffness-to-weight ratio by: 85□□□□ and 102□□ for 4, and 6 layers respectively.

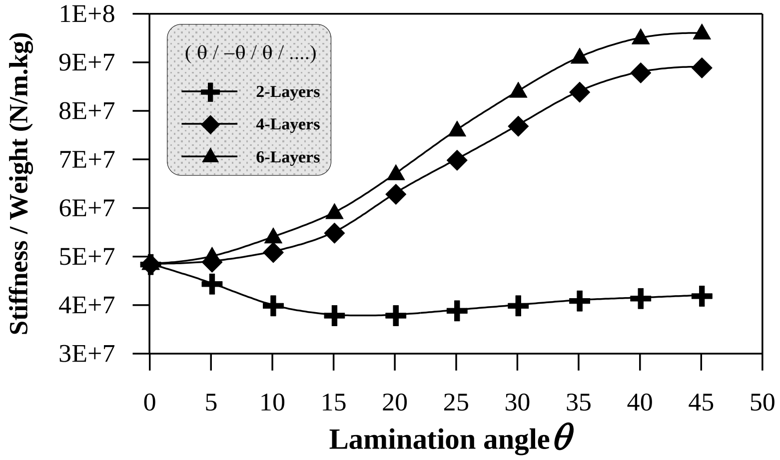

Figure (8) Effect of lamination angle on the stiffness-to-weight ratio of antisymmetric angle-ply laminates

Figures (9) and **(10)** show the variation of σ_x and σ_y respectively through the thickness with respect to the point of impact. The distribution of stresses through the thickness is discontinuous due to change of layers properties. The stress vanishes at the center of plate (neutral axis of plate). Also, the maximum stress value occurs at the top and the bottom of the plate due to tension and compression states of plate.

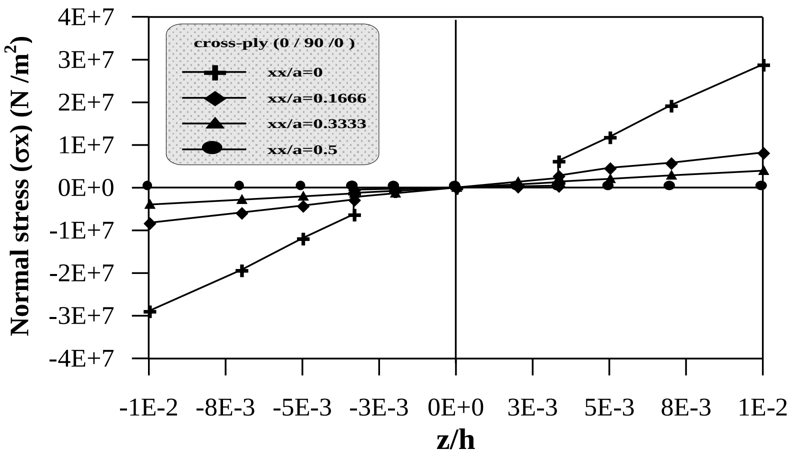

Figure (9) Stress distribution σ_x through the thickness

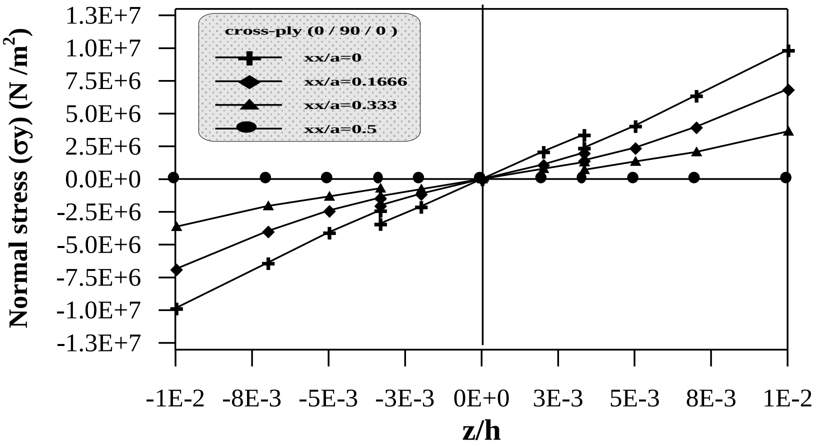

Figure (10) Stress distribution σ_y through the thickness

10. Conclusions

The main conclusions of this work for static analyses are the coupling between bending and extension B_{ij} decreases the effective stiffness.

The effect of degree of orthotropy (E_1/E_2) becomes more pronounced as the number of layers increases (for the same laminate thickness). Increasing (E_1/E_2) decreases the maximum deflection.

The number of layers N in the laminated plates affects the laminated plate stiffness, in two different manners (for symmetric and antisymmetric laminates). For symmetric laminate (coupling stiffness $B_{ij} = 0$) increasing N increases the extensional stiffness A_{ij} and bending stiffness D_{ij}, while for antisymmetric laminate, increasing N decreases the coupling stiffness B_{ij}. For antisymmetric laminates (which consist of equal thickness layers), increasing N more than 8 does not affect the laminate stiffness, that is at ($N = 8$) B_{ij} vanish.

For an angle-ply laminated plate, it is found that, ($\theta = 45^o$) represents the best lamination angle at which minimum deflection and maximum stiffness-to-weight ratio are achieved.

Maximum deflection and stresses occurs at the point of contact on the top and bottom surface of the plate.

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List of Symbols

A_{ij}	Extension stiffness element (N/m).
a, b	Dimensions of rectangular plate in x and y directions (m).
B_{ij}	Bending-extension coupling stiffness element (N).
D_{ij}	Bending stiffness element (N. m).
E_{ij}, F_{ij}, H_{ij}	Higher-Order stifnesses $N.m^2$, $N.m^3$ and $N.m^5$ respectively.
m, n	Longitudinal and Transverse mode shape.
m_1	Mass of impactor (kg).
M_x, M_y, M_{xy}	Resultant Moments per unit length (N. m / m) respectively.
M_x^*, M_y^*, M_{xy}^*	High-order stress-resultants (N m).
N	Number of laminate's layers.
N_x, N_y, N_{xy}	Resultant forces per unit length (N/m).
P	Impact load (N).
Q_{ij}	Element of elasticity matrix (N/m^2).
$\overline{Q}_{ij}$	Transformed stress-strain relation (N/m^2).
Q_x, Q_y	Shear forces per unit length (N/m).
Q_x^*, Q_y^*	High-order shear forces (N. m)
R_1	Radius of a spherical impactor (m).
u, v, w	Displacement in the x, y and z directions (m) respectively.
V_1, V_2	Initial velocity of impactor and target (m/sec) respectively.
xx	Distance from impact point (m).
ϕ_x , ϕ_y	Rotations of the transverse normal in xz and yz plane.
θ_i	High-order transverse cross section deformation mode