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Topological Structure of Generalized Rough Graphs

Samah Sarmad **Yousif Yaqoub Yousif**

Department of Mathematics, College of Education for Pure Sciences (Ibn Al-Haitham), University of Baghdad, Baghdad, Iraq.

samahsarmad0@gmail.com yoyayousif@yahoo.com

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Abstract

The main purpose of this paperis to introduce a topological space(D, τ_D), which is induced by reflexive graph and tolerance graph D, such that D may be infinite. Furthermore, we offer some properties of (D, τ_D) such as connectedness, compactness, Lindelöf and separate properties. We also study the concept of approximation spaces and get the sufficient and necessary condition that topological space is approximation spaces.

Keywords. Reflexive graph, tolerance graph, transmitting expression, approximation spaces.

1. Introduction

Graph theory [1] is a tool for optimization and solving practical application in all fields such as engineering study and representation of economic and social networks, complex general systems, information theory and others. In particular, graphs are one of the prime objects of study in mathematics.

Rough set was offered by Pawlak [2] as a method for dealing with uncertainly of imprecise data, the equivalence relation is the cornerstone of Pawlak's theory of rough set. Topology is a major mathematics branch with independent theoretic frame work and wide applications.

Z. Li [3] offered the concept of transmitting expression of relation and produced several important results of rough sets topological properties. We can apply topological approaches to the theory of rough set and search the connection between rough set theory and topological theory. The topological properties of various rough operators have been debated in [4]. We built on some of the results in [5-10], [11-15] and [16].

2. Generalized Rough Graphs Generated By Graphs.

We will remember several fundamental concepts of the theory of rough set. In this article,$D = (V(D), E(D))$ is a graph where$V(D)$ implies the universe which may be infinite, the power set of $V(D)$ symbolized by $P(V(D))$ and the closure of subgraph Q in D symbolized by $\overline{Q}$ wherever $V(D)$ is a topological space.

Let $D = (V(D), E(D))$ be a graph. For each subgraph Q of D, we will define operators D_- and D_+ from $P(V(D))$to itself as the following:

$$D_-(Q) = \{\mathfrak{r} \in V(D)\text{: if } (\mathfrak{r}, u) \in E(D)\text{, then } u \in V(Q)\},$$

$$D_+(Q) = \{\mathfrak{r} \in V(D)\text{: there exists } u \in V(Q) \text{ such that } (\mathfrak{r}, u) \in E(D)\}.$$

$D_-(Q)$ is named lower approximation of Q and $D_+(Q)$ is named upper approximation of $V(Q)$. The pair $(V(D), E(D))$ is named generalized rough graph or generalized approximation space. Q is named generalized exact graph or definable graph if $D_-(Q) = D_+(Q)$.While Q is called undefinable graph if $D_-(Q) \neq D_+(Q)$. If D is an equivalence graph, a generalized rough graph $(V(D), E(D))$means the rough graph in the Pawlak's sense.

Definition 2.1. Let $D = (V(D), E(D))$ be a nonempty graph. We define τ_D, for each $Q \subseteq D$ by

$$\tau_D = \{Q \subseteq D: D_-Q = V(Q)\}.$$

If D is a reflexive graph, then τ_D constitutes a topology on $V(D)$, $\tau_D$can be named the topology produced by D.

Definition 2.2. If$D$is a reflexive graph, then (D, τ_D) is named the topological space produced by D.

Definition 2.3. Let$D$ be a graph, if D is both reflexive and symmetric graph then D is called tolerance graph.

Definition 2.4. Let D_α and D_β be two graphs on $V(D_\alpha) = V(D_\beta) = V(D)$. D_β is named transmitting expression of D_α, if for every$\mathfrak{r}$, $u \in V(D)$,$(\mathfrak{r}, u) \in E(D_\beta)$if and only if $(\mathfrak{r}, u) \in E(D_\alpha)$ or there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where$(\mathfrak{r}, v_1) \in E(D_\alpha)$, $(v_1, v_2) \in E(D_\alpha)$, … , $(v_n, u) \in E(D_\alpha)$.

Proposition 2.5. Let D_α be a graph and D_β the transmitting expression of D_α, then D_β is a transitive graph, furthermore,

(1) If D_α is reflexive, then D_β is also reflexive,
(2) If D_α is symmetric, then D_β is also symmetric,
(3) If D_α is transitive, then $D_\beta = D_\alpha$.

Proof. (1) Let D_α is reflexive graph, then for each $\mathfrak{r} \in V(D)$,$(\mathfrak{r}, \mathfrak{r}) \in E(D_\alpha)$, since D_β is a transmitting expression of D_α, then $(\mathfrak{r}, \mathfrak{r}) \in E(D_\beta)$, so D_β is a reflexive graph.

(2) Let $(ɍ, u) \in D_\beta$, since D_β the transmitting expression of D_α, then $(ɍ, u) \in D_\alpha$ or there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where $(ɍ, v_1) \in D_\alpha$, $(v_1, v_2) \in D_\alpha$, … , $(v_n, u) \in D_\alpha$ if and only if $(ɍ, u) \in D_\beta$, because D_α is symmetric, so $(u, ɍ) \in D_\alpha$ or there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where,$(u, v_n) \in D_\alpha$, … , $(v_2, v_1) \in D_\alpha$, $(v_1, ɍ) \in D_\alpha$ if and only if $(u, ɍ) \in D_\beta$, which implies to D_β is symmetric.

(3) Let $(ɍ, u) \in E(D_\beta)$, we have to show that $(ɍ, u) \in E(D_\alpha)$.Since D_β the transmitting expression of D_α, then$(ɍ, u) \in E(D_\beta)$if and only if $(ɍ, u) \in D_\alpha$ or there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where $(ɍ, v_1) \in E(D_\alpha)$, $(v_1, v_2) \in E(D_\alpha)$, $(v_2, v_3) \in E(D_\alpha)$ … , $(v_n, u) \in E(D_\alpha)$.

(i) If $(ɍ, u) \in E(D_\alpha)$ the prove is complete.

(ii) If there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where $(ɍ, v_1) \in E(D_\alpha)$, $(v_1, v_2) \in E(D_\alpha)$, $(v_2, v_3) \in E(D_\alpha)$ … , $(v_n, u) \in E(D_\alpha)$, and we have that D_α is transitive, so $(ɍ, u) \in E(D_\alpha)$, which means $D_\beta = D_\alpha$.

Definition2.6. (1) Let (D, τ_D) be a topological space and $\mathcal{B}_D$ a base of (D, τ_D), where (D, τ_D) is induced by a reflexive graph D. Then B $\in \mathcal{B}_D$ is called maximal element of $\mathcal{B}_D$ if does not exist $B^\iota \in \mathcal{B}_D \backslash \{B\}$ such that $B \subseteq B^\iota$.

(2)The set of all maximal elements of $\mathcal{B}_D$ symbolized by $\mathcal{B}_D^*$. Because $\bigcup \mathcal{B}_D = V(D)$, $\mathcal{B}_D^*$ is referred to as the minimal complete cover of (D, τ_D) according to the base $\mathcal{B}_D$.

We will define a pseudo-metric map on graph D.

Definition2.7. Let $D = (V(D), E(D))$ be a nonempty graph, then $d: V(D) \times V(D) \longrightarrow [0, +\infty)$ is called pseudo-metric map on D, if for all $ɍ, v, u \in V(D)$,

(a) $d(ɍ, ɍ) = 0$,
(b) $d(ɍ, v) = d(v, ɍ)$,
(c) $d(ɍ, v) \leq d(ɍ, u) + d(u, v)$.

For each $ɍ \in V(D)$, $Q \subseteq D$, $\epsilon > 0$,

$$B(ɍ, \epsilon) = \{v \in V(D): d(ɍ, v) < \epsilon\}, d(ɍ, Q) = inf\{d(ɍ, v): v \in V(Q)\}.$$

If there exists pseudo-metric map$d$on $D$where$\{B(ɍ, \epsilon): ɍ \in V(D), \epsilon > 0\}$ configures a base of D, then a topological space (D, τ_D) is referred to as pseudo-metrizable space.

Proposition 2.8. Let D be pseudo-metrizable space. If$Q \subseteq D$and d is pseudo-metric map on D, then $ɍ \in \bar{Q}$ if and only if $d(ɍ, Q) = 0$.

Proof. $ɍ \in \bar{Q}$ if and only if for each $\epsilon > 0$, $B(ɍ, \epsilon) \cap V(Q) \neq \emptyset$ if and only if for each $\epsilon > 0$, there exists $u \in V(Q)$ such that $d(ɍ, u) < \epsilon$ if and only if $inf\{d(ɍ, u): u \in V(Q)\}$ if and only if $d(ɍ, Q) = 0$.

Definition 2.9. Let (D, τ_D) be a topological space. D is named a pseudo-discrete space if $Q \subseteq D$ is open in D if and only if Q is closed in D.

3. The Properties of Topological Spaces Induced by a Reflexive Graph

We will study through this part, the properties of the topological space (D, τ_D), where (D, τ_D) is produced by a reflexive graph D.

Lemma3.1. Let D_α be a reflexive graph and D_β the transmitting expression of D_α, for every $ɍ \in V(D)$, chose $L_ɍ = \{v \in V(D): (ɍ, v) \in E(D_\beta)\}$, then

(1) $L_ɍ \in \tau_{D_\alpha}$,
(2) $\{L_ɍ\}$ is an open neighborhood base of $ɍ$,
(3) $L_ɍ$ is compact subset of $(D_\alpha, \tau_{D_\alpha})$,
(4) $\mathcal{B}_{D_\alpha} = \{L_ɍ: ɍ \in V(D)\}$ is a base for $(D_\alpha, \tau_{D_\alpha})$.

Proof. (1) It is sufficient to show that$L_ɍ \subseteq Int(L_ɍ)$. Let $u \in L_ɍ$,so $(ɍ, u) \in E(D_\beta)$, then there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where $(ɍ, v_1) \in E(D_\alpha)$, $(v_1, v_2) \in E(D_\alpha)$, … , $(v_n, u) \in E(D_\alpha)$, so $u \in [v_n]_{D_\alpha}$. For $y \in V(D)$ such that $y \in L_ɍ$, $(v_n, y) \in E(D_\alpha)$, so$(ɍ, v_1) \in E(D_\alpha)$, $(v_1, v_2) \in E(D_\alpha)$, … , $(v_n, y) \in E(D_\alpha)$, then $(ɍ, y) \in E(D_\beta)$, so $y \in [ɍ]_{D_\beta}$, then $y \in L_ɍ$, $[v_n]_{D_\alpha} \subseteq L_ɍ$, so$u \in Int(L_ɍ)$, which implies to $L_ɍ \subseteq Int(L_ɍ)$. Hence, $L_ɍ \in \tau_{D_\alpha}$.

(2) Let $B \in \tau_D$ such that $ɍ \in B$, we will show that $L_ɍ \subseteq B$. Let $u \in L_ɍ$, then $(ɍ, u) \in E(D_\alpha)$ or there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where $(ɍ, v_1) \in E(D_\alpha)$, $(v_1, v_2) \in E(D_\alpha)$, … , $(v_n, u) \in E(D_\alpha)$.

(i) If $(ɍ, u) \in E(D_\alpha)$, then we claim $u \in B$. For otherwise, $u \in B^c$, then, $L_ɍ \subseteq B^c$, but D_α is reflexive, so $ɍ \in L_ɍ$ and $L_ɍ \subseteq B^c$, then $ɍ \in B^c$, which is a contradiction. Hence, $u \in B$.

(ii) If there exists $\{v_1, v_2, v_3, \dots, v_n\} \subseteq V(D)$ where $(ɍ, v_1) \in E(D_\alpha)$, $(v_1, v_2) \in E(D_\alpha)$, … , $(v_n, u) \in E(D_\alpha)$, then by (i) we get $v_1 \in B$, $v_2 \in B$, … , $u \in B$.

So $L_ɍ \subseteq B$, Which implies to $\{L_ɍ\}$constitutes an open neighborhood base of $ɍ$.

(3) Let $\{K_\lambda | \lambda \in \Lambda\}$ be an open cover of $L_ɍ$ then $ɍ \in K_{\lambda_i}$ for some $\lambda_i \in \Lambda$, then by (2) $L_ɍ \subseteq K_{\lambda_i}$. Therefore, $L_ɍ$ is a compact subset of (D, τ_D).

(4) It is obvious by (2)

Remark 3.2. (1) Let $D = \big(V(D), E(D)\big)$ be a graph, for each $ɍ, u \in V(D)$, if $(ɍ, u) \in E(D)$and$(u, ɍ) \in E(D)$, then $L_ɍ = L_u$.

(2) For all $B \in \mathcal{B}_D$, B cannot be represented as the union of some elements of $\mathcal{B}_D \backslash \{B\}$. Otherwise, there exists $\mathcal{O}_D \subseteq \mathcal{B}_D \backslash \{B\}$ such that $B = \cup\, \mathcal{O}_D$. By $B \in \mathcal{B}_D$, there exists $ɍ \in V(D)$ where $B = L_ɍ$. Because $ɍ \in B$, there exists $Q \in \mathcal{O}_D$ such that $ɍ \in Q \subseteq B$. By Lemma 3.1, $L_ɍ \subseteq Q$. Then $B = Q$, so we obtain a contradiction. Hence, H cannot be represented as the union of some elements of $\mathcal{B}_D \backslash \{B\}$.

(3) Let $\mathcal{H}_D$ form a base for (D,τ_D)**.** Then $\mathcal{B}_D \subseteq \mathcal{H}_D$. Otherwise, there exists $B \in \mathcal{B}_D$ but $B \notin \mathcal{H}_D$. Notice that $B \in \mathcal{B}_D$, there exists $ɟ \in B$ where $B = L_ɟ$. Because $\mathcal{H}_D$ is a base for (D,τ_D), there exists $\mathcal{H}_D^\iota \subseteq \mathcal{H}_D$ such that $B = \cup \mathcal{H}_D^\iota$. Thus $ɟ \in H \subseteq B$ for some $H \in \mathcal{H}_D^\iota$. By using Lemma 3.1, $B \subseteq H$. So $B = H \in \mathcal{H}_D$ and that means a contradiction. Therefore, $\mathcal{B}_D \subseteq \mathcal{H}_D$.

Theorem 3.3. Let (D,τ_D) be a topological space generated by a reflexive graph D, then

(1) (D,τ_D) is a first countable space,
(2) (D,τ_D) is a locally compact space,
(3) If D is countable, then (D,τ_D) is second countable space.

Proof. (1) By lemma 3.1(2) $\{L_ɟ\}$ is an open neighborhood base of $ɟ$, then $(D,\tau_D)$is a first countable space.

(2) By lemma 3.1(3), we have for each $ɟ \in V(D)$, $ɟ$ has compact neighborhood. Hence (D,τ_D) is locally compact space.

(3) By lemma 3.1(4),$\mathcal{B}_D = \{L_ɟ : ɟ \in V(D)\}$ is a base for (D,τ_D)**,** which implies to there exists a countable base for τ_D, so (D,τ_D) is a second countable space.

Theorem 3.4. If $D = (V(D),E(D))$ is a reflexive graph and D_β the transmitting expression of D, then $(D,\tau_D) = (D,\tau_{D_\beta})$.

Proof. According to Lemma 3.1(4), $\mathcal{B}_D = \{L_ɟ : ɟ \in V(D)\}$ is a base for (D,τ_D), we will prove that $\mathcal{B}_D = \{L_ɟ : ɟ \in V(D)\}$ is also a base for (D,τ_{D_β}).

By definition $L_ɟ = \{u\, V(D) : (ɟ,u) \in D_\beta\} \in \tau_{D_\beta}$. Let $ɟ \in K \in \tau_{D_\beta}$, for any $u \in L_ɟ$, then $(ɟ,u) \in D_\beta$, since $ɟ \in K$, by Lemma 3.1(2) $ɟ \in L_ɟ \subseteq K$. So $\mathcal{B}_D$ is also base for τ_{D_β}, hence $(D,\tau_D) = (D,\tau_{D_\beta})$.

Lemma 3.5. Let (D,τ_D)be atopological space generated by a reflexive graph D, if $\mathcal{B}_D^*$ the minimal complete cover of (D,τ_D) according to the base $\mathcal{B}_D$, then for all $F \in \mathcal{B}_D^*$, $\cup(\mathcal{B}_D \backslash \{F\}) \neq V(D)$ and $\cup(\mathcal{B}_D^* \backslash \{F\}) \neq V(D)$.

Proof. Suppose that $\cup(\mathcal{B}_D \backslash \{F\}) = V(D)$, then there exists $\mathcal{B}_D^\iota \subseteq \mathcal{B}_D \backslash \{F\}$ such that $F \subseteq \cup \mathcal{B}_D^\iota$, Since $F \in \mathcal{B}_D^* \subseteq \mathcal{B}_D$, there exists $ɟ \in V(D)$ such that $F = L_ɟ$, so $ɟ \in F^\iota$ for some $F^\iota \in \mathcal{B}_D^\iota$. By Lemma 3.1(2), $F = L_ɟ \subseteq F^\iota$. Consequently F is not a maximal element of $\mathcal{H}_G$ which implies a contradiction. So, $\cup(\mathcal{B}_D \backslash \{F\}) \neq V(D)$. Since $\cup(\mathcal{B}_D \backslash \{F\}) \neq V(D)$, $\cup(\mathcal{B}_D^* \backslash \{F\}) \neq V(D)$.

Lemma 3.6. Let (D,τ_D) be a topological space generated by a reflexive graph D, If $\mathcal{B}_D^*$ the minimal complete cover of (D,τ_D) according to the base $\mathcal{B}_D$ and $\mathcal{H}_D$ an open cover of (D,τ_D)**.** Then for all $F \in \mathcal{B}_D^*$, there exists $H \in \mathcal{H}_D$ where $F \subseteq H$**.**

Proof. Since $\mathcal{H}_D$ an open cover of (D,τ_D), for any $F \in \mathcal{B}_D^*$, there exists $\mathcal{H}_D^\iota \subseteq \mathcal{H}_D$ such that $F \subseteq \cup \mathcal{H}_D^\iota$. Because $F \in \mathcal{B}_D^* \subseteq \mathcal{B}_D$, then $F = L_ɟ$ for some $ɟ \in F$, so there exists $H \in \mathcal{H}_D^\iota \subseteq \mathcal{H}_D$ such that $ɟ \in H$**.** By Lemma 3.1, $F \subseteq H$**.**

Lemma 3.7. Let (D, τ_D) be a topological space generated by a reflexive graph D. If $\mathcal{B}_D^*$ is the minimal complete cover of (D, τ_D) according to the base $\mathcal{B}_D$ and $\mathcal{H}_D$ an open cover of (D, τ_D), which is made up of some elements of $\mathcal{B}_D$, then $\mathcal{B}_D^* \subseteq \mathcal{O}_D$.

Proof. For each $B \in \mathcal{B}_D^*$, we claim that $B \in \mathcal{O}_D$. If not, $B \notin \mathcal{O}_D$. Since $\bigcup \mathcal{O}_D = V(D)$, $\bigcup(\mathcal{O}_D \backslash \{B\}) = V(D)$, So $\bigcup(\mathcal{B}_D \backslash \{B\}) = V(D)$. By using Lemma3.5,$\bigcup(\mathcal{B}_D \backslash \{B\}) \neq V(D)$, which implies a contradiction. So $\mathcal{B}_D^* \subseteq \mathcal{O}_D$.

Theorem 3.8. Let (D, τ_D)be a topological space generated by a reflexive graph D, $\mathcal{B}_D^*$ the minimal complete cover of (D, τ_D)according to the base $\mathcal{B}_D$**.** Then$\mathcal{H}_G^*$ is a finite set if and only if(D, τ_D) is compact space.

Proof. The only if part clear by Lemma (4.6).Conversely, suppose that (D, τ_D) is compact, as$\mathcal{H}_G$ is an open cover of (D, τ_D) then $\mathcal{H}_G$ has a finite subcover $\mathcal{H}_G^\iota$. By using Lemma 3.7, $\mathcal{B}_D^* \subseteq \mathcal{B}_D^\iota$, thus $|\mathcal{B}_D^*| \leq |\mathcal{B}_D^\iota|$. Hence $\mathcal{B}_D^*$ is a finite set.

4. The Properties of Topological Spaces generated by a Tolerance Graph

Through this part, we will achieve the properties of (D, τ_D), where (D, τ_D) is a topological space induced by tolerance graph D.

Lemma 4.1 If (D, τ_D) is a topological space generated by a tolerance graph D, then for all $Q \subseteq D$, Q is open if and only if Q is closed.

Proof. Q is open $\Leftrightarrow Q = Int(Q) \Leftrightarrow Q^c = Int(Q^c) \Leftrightarrow Q^c$ is open graph $\Leftrightarrow Q$ is closed.

Theorem 4.2. If$(D, \tau_D)$is a topological space generated by a tolerance graph D. Then (D, τ_D) is discrete if and only if (D, τ_D) is $T_0 - space$.

Proof. The only if part is clear. We are going to prove the if part**.** Let (D, τ_D) be $T_0 - space$. Depending on the Lemma 3.1(4), we have if D is reflexive, then $\{L_ɍ: ɍ \in V(D)\}$ is a base for (D, τ_D). We claim that $L_ɍ = \{ɍ\}$ for any $ɍ \in V(D)$. Suppose that $L_ɍ \neq \{ɍ\}$ for some $ɍ \in V(D)$. By Proposition 2.5, D_β is an equivalent graph on $V(D)$, so $L_ɍ = [ɍ]_{D_\beta}$. Chose $u \in [ɍ]_{D_\beta}$ such that $u \neq ɍ$. Since (D, τ_D) is $T_0 - space$, there exists an open subgraph O where $ɍ \in V(O)$ and $u \notin V(O)$, or there exists an open subgraph U where $u \in V(U)$ and $ɍ \notin V(U)$. If there exists an open subgraph $V(O)$ where $ɍ \in V(O)$ and $ɍ \notin V(U)$, then $ɍ \in L_v \subseteq V(O)$ for some $v \in V(D)$depending on the Lemma 3.1(4). It follows $u \notin L_v$. As D_β is an equivalence graph on $V(D)$, $[ɍ]_{D_\beta} = [v]_{D_\beta} = L_v$. Thus $u \in [ɍ]_{D_\beta} = L_v$ means a contradiction. Similarly if there exists an open subgraph $U$where$u \in V(U)$ and $ɍ \notin V(U)$. Hence, $\{ɍ\}$ is open for all $ɍ \in V(D)$. Therefore, all subgraphs of $V(D)$ are open which means that (D, τ_D) is discrete.

Theorem 4.3. Let (D, τ_D) be a topological space generated by a tolerance graph D. Then, the statements are equivalent:

(1) $V(D) \,/\, E(D_\beta)$ is countable,
(2) (D, τ_D) is a second countable space,
(3) (D, τ_D) is a separable space,

(4) (D,τ_D) is a lindelöf space.

Proof. (1)⟹(2). Since $D_\beta$is an equivalence graph on $V(D)$, $\{L_ɍ : ɍ \in V(D)\} = V(D) / E(D_\beta)$. By Lemma 3.1(4), (D,τ_D) is second countable space.

(2)⟹(1) Suppose that $\mathcal{B}$ is a countable base for (D,τ_D), then for $ɍ \in V(D)$, there exists $B_ɍ \in \mathcal{B}$ such that $ɍ \in B_ɍ \subseteq L_ɍ$. By Lemma 3.1(4), $ɍ \in L_u \subseteq B_ɍ$ for some $u \in V(D)$. Since $L_ɍ = [ɍ]_{D_\beta} = [u]_{D_\beta} = L_u$, $B_ɍ = [ɍ]_{D_\beta}$, we define $f: V(D) / E(D_\beta) \longrightarrow \mathcal{B}$ by $f\left([ɍ]_{D_\beta}\right) = B_ɍ$, then f is injective. So$|V(D) / E(D_\beta)| \leq |\mathcal{B}|$. Hence $V(D) / E(D_\beta)$ is countable.

(2) ⟹ (3) and (2) ⟹ (4) are clear.

(3) ⟹ (2). Suppose that C is a countable dense subgraph of (D,τ_D). Put $\lambda = \{L_ɍ : ɍ \in V(C)\}$, then λ is countable. By Lemma 3.1(4), for all $ɍ \in V(D)$ and open subgraph O with $ɍ \in V(O)$, we have $ɍ \in L_u \subseteq V(O)$ for some $u \in V(D)$. Since C is dense, $L_u \cap V(C) \neq \emptyset$, Chose $v \in L_u \cap V(C)$, then $L_v \in \lambda$. Since D_β is an equivalence graph on $V(D)$, $L_v = [v]_{D_\beta} = [u]_{D_\beta} = L_u$. It follows $ɍ \in L_v \subseteq V(O)$. Therefore, λ is a base for (D,τ_D). Hence (D,τ_D) is a second countable space.

(4) ⟹ (2). Suppose that $V(D) / E(D_\beta)$ is not countable. Since D_β is an equivalence graph on $V(D)$, $\{L_ɍ : ɍ \in V(D)\} = V(D) / E(D_\beta)$. It is obvious that $\{L_ɍ : ɍ \in V(D)\}$ is an open cover of (D,τ_D) but $\{L_ɍ : ɍ \in V(D)\}$ does not have any countable subcover Hence we get a contradiction.

Theorem 5.4 Let (D,τ_D) be a topological space generated by a tolerance graph D. Then (D,τ_D) is a connected space if and only if $E(D_\beta) = V(D) \times V(D)$.

Proof. Suppose that (D,τ_D) is connected, If $E(D_\beta) \neq V(D) \times V(D)$, then $V(D) \times V(D) / E(D_\beta) \neq \emptyset$. Chose $(ɍ, u) \in \big(V(D) \times V(D)\big) \backslash E(D_\beta)$, then $u \notin [ɍ]_{D_\beta} = L_ɍ$. So $L_ɍ \neq V(D)$ and $L_ɍ \neq \emptyset$. By Lemma 4.1, $L_ɍ$ is both open and closed, so we obtain a contradiction. Conversely, Suppose that $E(D_\beta) = V(D) \times V(D)$, then $V(D) / E(D_\beta) = [V(D)]$. So $\tau_D = \{V(D), \emptyset\}$, thus (D,τ_D) is connected.

Theorem 4.6. Let (D,τ_D) be a topological space generated by a tolerance graph D. Then

(1) (D,τ_D) is a locally connected space
(2) (D,τ_D) is a locally separable space,
(3) (D,τ_D) is a regular space,
(4) (D,τ_D) is a normal space,
(5) $(D,\tau_D)$is a pseudo-metrizable space.

Proof.(1) By lemma 3.1(2) every open neighborhood of $ɍ$ contains $L_ɍ$ which is connected.

(2)Since $\{L_ɍ\}$ is an open neighborhood base of $ɍ$, we just need to show that $L_ɍ$ is a separable subset of (D,τ_D). Let $\overline{\{ɍ\}}$ be the closure of $\{ɍ\}$ and suppose that there exists $u \in \overline{\{ɍ\}}$ such that $u \notin L_ɍ$, so $[ɍ]_{D_\beta} \cap [u]_{D_\beta} = \emptyset$. For an open neighborhood L_u of u, $\{ɍ\} \cap L_u = \emptyset$,so $u \notin \overline{\{ɍ\}}$

which is a contradiction, hence, $u \in L_{\mathfrak{r}}$ then $\overline{\{\mathfrak{r}\}} \subseteq L_{\mathfrak{r}}$. On the other hand, let $u \in L_{\mathfrak{r}}$ then $u \in [\mathfrak{r}]_{D_\beta}$, then $L_{\mathfrak{r}} = L_u$. Suppose O is an open neighborhood of u, so $L_u \subseteq V(O)$ then $L_u \cap V(O) \neq \emptyset$, so $L_{\mathfrak{r}} \cap V(O) \neq \emptyset$, then, $\{\mathfrak{r}\} \cap V(O) \neq \emptyset$ then $u \in \overline{\{\mathfrak{r}\}}$, so $L_{\mathfrak{r}} \subseteq \overline{\{\mathfrak{r}\}}$. Hence, $L_{\mathfrak{r}} = \overline{\{\mathfrak{r}\}}$, and we obtained that $\{\mathfrak{r}\}$ is countable dense subset of $L_{\mathfrak{r}}$ which implies to $L_{\mathfrak{r}}$ is separable subset of (D, τ_D). Hence, (D, τ_D) is locally separable space.

(3) Let Q be closed subgraph of D and $\mathfrak{r} \in V(Q)^c$, by Lemma 4.1 Q is open if and only if Q is closed, so$Q$ and Q^c are two open disjoint subgraph of D such that $V(Q) \subseteq V(Q)$ and $\mathfrak{r} \in V(Q)^c$. Hence (D, τ_D) is a regular space.

(4) Let Q, M are two disjoint closed subgraphs of D, then by Lemma 4.1 they are also disjoint closed subgraphs of $V(D)$. But we have $V(Q) \subseteq V(Q)$and $V(M) \subseteq V(M)$. Hence (D, τ_D) is a normal space.

(5) Since there exists the trivial pseudo-metrizable map d induced by the pseudo-metrizable space, where

$$d: V(D) \times V(D) \longrightarrow [0, \infty), \text{ such that } d = \begin{cases} 1 & if\ \mathfrak{r} = u \\ 0 & if\ \mathfrak{r} \neq u \end{cases}$$

For any $\mathfrak{r} \in V(D)$ and $\epsilon > 0$,

$$B(\mathfrak{r}, \epsilon) = \begin{cases} \{\mathfrak{r}\} & if \epsilon < 1 \\ V(D) & if\ \epsilon \geq 1 \end{cases}$$

Then, $\{\mathfrak{r}\} \in \tau_D$, so (D, τ_D) is pseudo-discrete, then $B(\mathfrak{r}, 1) = \{u \in V(D): d(\mathfrak{r}, u) < 1\} = \{u \in V(D): d(\mathfrak{r}, u) = 0\} = \{\mathfrak{r}\}$. Thus $\{B(\mathfrak{r}, \epsilon): \mathfrak{r} \in V(D) and\ \epsilon > 0\}$ forms a base for (D, τ_D). Hence (D, τ_D) is pseudo-metrizable.

5. Approximation spaces on digraph

We will present the concept of approximation spaces in this part; furthermore, we will get their characterizations and properties.

Definition 5.1. Let (D, ρ) be a topological space, then (D, ρ) is called an approximation space if there exists an equivalence graph $D = (V(D), E(D))$such that $\tau_D = \rho$.

According to Lemma 4.1, we get that approximating spaces are pseudo-discrete spaces. But the question, would pseudo-discrete spaces are approximating space? This problem is certainly answered by the following theorem.

Theorem 5.2. 6.2 If (D, ρ) is a topological space, we have the next equivalence:

(1) (D, ρ) is an approximating space,
(2) (D, ρ) is both pseudo-metrizable and pseudo-discrete,
(3) (D, ρ) is pseudo-discrete space.

Proof. (1) $\Longrightarrow$ (2). It holds depending on Lemma 4.1 and Theorem 4.6.

(2) $\Longrightarrow$ (1). Let(D,ρ)be both pseudo-metrizable and pseudo-discrete, then there exists a pseudo-metric map d on $V(D)$ where $\{B(\mathfrak{r},\epsilon): \mathfrak{r} \in V(D) and\ \epsilon > 0\}$ is a base for (D,ρ). We define a graph D on $V(D)$ as thereinafter:

$$\text{For all } \mathfrak{r}, u \in V(D), (\mathfrak{r}, u) \in E(D) \text{ if and only if } d(\mathfrak{r}, u) = 0.$$

Since d is pseudo-metric on $V(D)$, so D is an equivalence graph. We will prove that $\tau_D = \rho$. Let $Q \in \rho$, by Proposition 2.8, $\bar{Q} = \{\mathfrak{r} \in V(D): d(\mathfrak{r}, Q) = 0\}$. Since $(D,\rho)$is pseudo-discrete, Q is closed in (D,ρ), so $V(Q) = \{\mathfrak{r} \in V(D): d(\mathfrak{r}, Q) = 0\}$. It is obvious that $V(Q) \subseteq \bigcup\{[\mathfrak{r}]_D: \mathfrak{r} \in V(Q)\}$. If $u \in [\mathfrak{r}]_D$ with $\mathfrak{r} \in V(Q)$, then, $d(\mathfrak{r}, Q) \leq d(u, \mathfrak{r}) = 0$. So $u \in \{\mathfrak{r} \in V(D): d(\mathfrak{r}, Q) = 0\} = V(Q)$, that is mean$V(Q) \supseteq \bigcup\{[\mathfrak{r}]_D: \mathfrak{r} \in V(Q)\}$. Thus $V(Q) = \bigcup\{[\mathfrak{r}]_D: \mathfrak{r} \in V(Q)\}$, it follows that $V(Q) \in \tau_D$, so$\rho \subseteq \tau_D$.

On the other side, let $\mathfrak{r} \in V(D)$, by Proposition 2.8, $\overline{\{\mathfrak{r}\}} = \{u \in V(D): d(u, \mathfrak{r}) = 0\}$, then $\overline{\{\mathfrak{r}\}} = \{\mathfrak{r}\}_D$. Now $[\mathfrak{r}]_D$ is closed in (D,ρ), since $(D,\rho)$is pseudo-discrete,$[\mathfrak{r}]_D \in \rho$. Since $\{[\mathfrak{r}]_D: \mathfrak{r} \in V(D)\}$ is a base for (D,τ_D), $\tau_D \subseteq \rho$.

Hence $\tau_D = \rho$. This means that (D,ρ) are an approximation space.

(2) $\Longrightarrow$ (3). Clear.

(3) $\Longrightarrow$ (2). Let(D,ρ)be pseudo-discrete. For each$\mathfrak{r} \in V(D)$, $C(\mathfrak{r})$ denoted a connected component with $\mathfrak{r} \in C(\mathfrak{r})$, then $C(\mathfrak{r})$ is closed in (D,ρ). So $\overline{\{\mathfrak{r}\}} \subseteq C(\mathfrak{r})$. Let $u \in C(\mathfrak{r})$, since $C(\mathfrak{r})$ is a connected component with $\mathfrak{r} \in C(\mathfrak{r})$, there exists a connected subgraph Q of $D$where$\mathfrak{r}, u \in V(Q)$. Since D is pseudo-discrete, $\overline{\{\mathfrak{r}\}}$ is both open and closed in (D,ρ). Note that $\overline{\{\mathfrak{r}\}} \cap V(Q)$ is both open and closed in the subspace Q and Q is connected. Then $\overline{\{\mathfrak{r}\}} \cap V(Q) = V(Q)$, so $u \in \overline{\{\mathfrak{r}\}}$. This indicates that $C(\mathfrak{r}) \subseteq \overline{\{\mathfrak{r}\}}$. Thus $C(\mathfrak{r}) = \overline{\{\mathfrak{r}\}}$.

We define $d: V(D) \times V(D) \longrightarrow [0, \infty)$ as follows:

$$d(\mathfrak{r}, u) = \begin{cases} 0 & if\ C(\mathfrak{r}) = C(u), \\ 1 & if\ C(\mathfrak{r}) \neq C(u). \end{cases}$$

The assumption that d is pseudo-metric on $V(D)$can be easily proved. For any $\mathfrak{r} \in V(D)$ and $\epsilon > 0$,

$$B(\mathfrak{r}, \epsilon) = \begin{cases} \overline{\{\mathfrak{r}\}} & if\ \epsilon \leq 1, \\ V(D) & if\ \epsilon > 1. \end{cases}$$

Then $B(\mathfrak{r}, \epsilon)$will be closed in (D,ρ). Because$D$ is pseudo-discrete, $B(\mathfrak{r}, \epsilon) \in \rho$. Let $\mathfrak{r} \in V(D)$ and $V(O) \in \rho$ with $\mathfrak{r} \in V(Q)$. Since $D$is pseudo-discrete, $V(Q)$is closed in (D,ρ). So$\overline{\{\mathfrak{r}\}} \subseteq V(Q)$. By Proposition 2.8,$\overline{\{\mathfrak{r}\}} = \{u \in V(D): d(\mathfrak{r},\ u) = 0\}$. Then $.V(D)\ and\ \epsilon > 0\}$ is a base for (D,ρ). Therefore(D,ρ) is pseudo-metrizable.

Corollary 5.3. Discrete spaces are approximating spaces.

Theorem 5.4. Quotient maps preserve approximating spaces.

Proof. Suppose that the image of an approximating space Dunder a quotient map f is D'. We have to show that D' is an approximating space. Since $f$is a quotient map, $N \subseteq D'$ is open in D' if and only if $f^{-1}(N)$ is open in D. By using Theorem, 5.2, $f^{-1}(N)$ is open in D if and only if $f^{-1}(N)$ is closed in D. Since$f$ is a quotient map, then, $f^{-1}(N)$ is closed if and only if $N \subseteq D'$ is closed in D'. So$N \subseteq D'$is open in D' if and only if N is closed inD'. According to Theorem 5.2,D' is an approximating space.

Corollary5.5. Continuous maps do not preserve an approximating space.

We will explicate Corollary 5.5. in the next example.

Example 5.6. Suppose that$V(D)$ is a real numbers set R given with the usual discrete topology and $V(D')$ is a real numbers Rgiven with the usual Euclidean topology, let $f: V(D) \longrightarrow V(D')$ be the identity map, it is obvious that f is continuous map. According to the Corollary 5.3, $V(D)$ is an approximating space. But $V(D')$ is not an approximating space. Therefore, continuous maps do not preserve approximating spaces.

3.Conclusion

We offer the topological space generated by a reflexive graph and tolerance graph consecutively and discussed the topological structure of generalized rough graph. We have also achieving approximating spaces and get sufficient and necessary conditions that topological spaces are approximating spaces on graphs.

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