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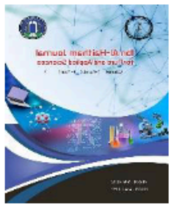

Some Properties for the Restriction of $\mathcal{P}^*-$ field of Sets

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Abstract

The restriction concept is a basic feature in the field of measure theory and has many important properties. This article introduces the notion of restriction of a non-empty class of subset of the power set on a nonempty subset of a universal set. Characterization and examples of the proposed concept are given, and several properties of restriction are investigated. Furthermore, the relation between the P*–field and the restriction of the P*–field is studied, explaining that the restriction of the P*–field is a P*–field too. In addition, it has been shown that the restriction of the P*–field is not necessarily contained in the P*–field, and the converse is true. We provide a necessary condition for the P*–field to obtain that the restriction of the P*–field is included in the P*–field. Finally, this article aims to study the restriction notion and give some propositions, lemmas, and theorems related to the proposed concept .

Keywords: σ −field, σ− ring, field, smallest σ −field and restriction.

1. Introduction

In the real analysis and probability, the σ−field concept is the class $\mathcal{M}$ for a subset of a universal set $\mathcal{U}$ such that $\mathcal{U}\epsilon\mathcal{M}$ and it is closed under the complement, countable union [1] and [2]. The main reason for σ−field is the idea of measure, which is substantial in the real analysis as the basis of Lebesgue integrals, where it exponent as a family of events which may be assigned probability [3] and [4]. In the probability theory, a σ–field is essential in the conditional expected. Also, in statistics, sub σ–field is necessary for an official mathematical definition for sufficient statistic, where a statistic be a map or a random variable. A σ– ring idea was studied by [5] as a class $\mathcal{M}$ such that $B_1 \backslash B_2 \in \mathcal{M}$ and $\bigcup_{n=1}^{\infty} B_n \in \mathcal{M}$ whenever $B_1, B_2, \ldots \in \mathcal{M}$. Many authors were interested in studying σ–field and σ– ring; for example, see [6], [7], and [8]. In this work, we denote a universal set by $\mathcal{U}$.

Preliminaries

In the following, we mention some basic definitions and notations in measure space that will be used in this paper.

Definition 2.1 [9].

Suppose $\mathcal{M}$ is a class of subsets of $\mathcal{U}$. Then, $\mathcal{M}$ is *the* $\mathcal{P}^*$– field of $\mathcal{U}$ if:

1- $\Phi \in \mathcal{M}$.
2- $N, M \in \mathcal{M}$; then, $N \cap M \in \mathcal{M}$.
3- $M_2, \ldots \in \mathcal{M}$; then, $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$.

Example 2.2 [9].

Let $\mathcal{U}$ ={1,2,3,4}. Consider $\mathcal{M} = \{ \Phi, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\} \}$.

Then $\mathcal{M}$ is a $\mathcal{P}^*$– field of $\mathcal{U}$.

Definition 2.3 [5].

The family of all subsets of $\mathcal{U}$ is called a power set and denoted by $P(\mathcal{U})$,
In symbols:
$P(\mathcal{U}) = \{ B : B \text{ is a subset of } \mathcal{U} \}$.

Proposition 2.4 [9] .

If $\{\mathcal{M}_i\}_{i \in I}$ is a family of $\mathcal{P}^*$– field of $\mathcal{U}$, then so is $\bigcap_{i \in I} \mathcal{M}_i$.

Definition 2.5 [9].

Let $\mathcal{J} \subseteq P(\mathcal{U})$. Then, $\mathcal{P}^*(\mathcal{J}) = \bigcap \{\mathcal{M}_i : \mathcal{M}_i$ is a $\mathcal{P}^*$– field of $\mathcal{U}$ and $\mathcal{M}_i \supseteq \mathcal{J}$, $\forall i \in I\}$ is called the $\mathcal{P}^*$– field generated by $\mathcal{J}$.

Proposition 2.6 [9].

If $\mathcal{J} \subseteq P(\mathcal{U})$, then $\mathcal{P}^*(\mathcal{J})$ is the smallest $\mathcal{P}^*$– field of $\mathcal{U}$ that contains $\mathcal{J}$.

Proposition 2.7 [5].

If $\mathcal{M}$ is σ– field, then $\mathcal{M}$ is a σ– ring.

Proposition 2.8 [9].

Every σ– field is $\mathcal{P}^*$– field.

Proposition 2.9 [9].

Every σ− ring is $\mathcal{P}^*$− field.

2. The Main Results

In this section, the basic definitions and facts related to this work are recalled, starting with the following definition:

Definition 3.1

Suppose $\mathcal{M}$ is a $\mathcal{P}^*$− field of $\mathcal{U}$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, then a restriction of $\mathcal{M}$ over $\mathcal{B}$ is defined as:

$\mathcal{M}|_{\mathcal{B}} = \{ N: N = M \cap \mathcal{B}$, for some $M \in \mathcal{M}\}$.

Proposition 3.2

Suppose $\mathcal{M}$ is $\mathcal{P}^*$− field of $\mathcal{U}$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, then $\mathcal{M}|_{\mathcal{B}}$ is $\mathcal{P}^*$− field on $\mathcal{B}$.

Proof.

Since $\Phi \in \mathcal{M}$ and $\Phi = \Phi \cap \mathcal{B}$, then $\Phi \in \mathcal{M}|_{\mathcal{B}}$.

Let $N_1, N_2 \in \mathcal{M}|_{\mathcal{B}}$, then there is $M_1, M_2 \in \mathcal{M}$ such that $N_i = M_i \cap \mathcal{B}$ where i=1,2 which implies that $N_1 \cap N_2 = (M_1 \cap \mathcal{B}) \cap (M_2 \cap \mathcal{B}) = (M_1 \cap M_2) \cap \mathcal{B}$.

Since $\mathcal{M}$ is a $\mathcal{P}^*$− field of $\mathcal{U}$, then, $M_1 \cap M_2 \in \mathcal{M}$. Thus $N_1 \cap N_2 \in \mathcal{M}|_{\mathcal{B}}$

Let $N_1, N_2, \dots \in \mathcal{M}|_{\mathcal{B}}$, then there is $M_1, M_2, \dots \in \mathcal{M}$ such that $N_i = M_i \cap \mathcal{B}$ where i=1, 2… which implies that $\bigcup_{i=1}^{\infty}, N_i = \bigcup_{i=1}^{\infty} (M_i \cap \mathcal{B}) = (\bigcup_{i=1}^{\infty} M_i) \cap \mathcal{B}$..

Since $\mathcal{M}$ is a $\mathcal{P}^*$− field of a set $\mathcal{U}$, then $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$ and hence $\bigcup_{i=1}^{\infty} N_i \in \mathcal{M}|_{\mathcal{B}}$.

Thus, $\mathcal{M}|_{\mathcal{B}}$is a $\mathcal{P}^*$− field on $\mathcal{B}$.

Proposition 3.3

If $\mathcal{M}$ is $\mathcal{P}^*$− field of $\mathcal{U}$ and $C \subseteq \mathcal{B} \subseteq \mathcal{U}$ such that $C \in \mathcal{M}$, then $C \in \mathcal{M}|_{\mathcal{B}}$.

Proof.

Clearly.

The following examples explain that if $\mathcal{M}$ is a $\mathcal{P}^*$− field of a set $\mathcal{U}$, then it is not necessarily that :

1- $\mathcal{M}|_{\mathcal{B}} \subseteq \mathcal{M}$.
2- $\mathcal{M} \subseteq \mathcal{M}|_{\mathcal{B}}$

Example 3.4

Let $\mathcal{U} = \{1,2,3,4\}$and $\mathcal{M} = \{ \Phi, \{1,3\}, \{1,2,3\}, \{1,3,4\}, \mathcal{U} \}$. Then, $\mathcal{M}$ is a $\mathcal{P}^*$− field of $\mathcal{U}$. If $\mathcal{B} = \{2,3,4\}$, then $\mathcal{M}|_{\mathcal{B}} = \{ \Phi, \{3\}, \{2,3\}, \{3,4\}, \mathcal{B}\}$. It is clear that $\mathcal{M}|_{\mathcal{B}} \nsubseteq \mathcal{M}$, since $\{3\} \in \mathcal{M}|_{\mathcal{B}}$ but $\{3\} \notin \mathcal{M}$.

Example 3.5

Let $\mathcal{U} = \{1,2,3,4\}$and $\mathcal{M} = \{ \Phi, \{1,2\}, \{1,2,3\}, \{1,2,4\}, \mathcal{U} \}$. Then, $\mathcal{M}$ is a $\mathcal{P}^*$− field of $\mathcal{U}$. If $\mathcal{B} = \{2,3,4\}$, then $\mathcal{M}|_{\mathcal{B}} = \{ \Phi, \{2\}, \{2,3\}, \{2,4\}, \mathcal{B}\}$. It is clear that $\mathcal{M} \nsubseteq \mathcal{M}|_{\mathcal{B}}$, since $\{1, 2\} \in \mathcal{M}$ but $\{1,2\} \notin \mathcal{M}|_{\mathcal{B}}$.

Proposition 3.6

If $\mathcal{M}$ is $\mathcal{P}^*-$field on $\mathcal{U}$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{B} \in \mathcal{M}$.
Then $\mathcal{M}|_{\mathcal{B}} = \{C \subseteq \mathcal{B}: C \in \mathcal{M}\}$.

Proof.

Assume that $N \in \mathcal{M}|_{\mathcal{B}}$, then $N = M \cap \mathcal{B}$, for some $M \in \mathcal{M}$ and thus $N \in \mathcal{M}$.
Hence, $N \in \{C \subseteq \mathcal{B}: C \in \mathcal{M}\}$. Therefore, $\mathcal{M}|_{\mathcal{B}} \subseteq \{C \subseteq \mathcal{B}: C \in \mathcal{M}\}$. Let $D \in \{C \subseteq \mathcal{B} : C \in \mathcal{M}\}$. Then $D \subseteq \mathcal{B}$ and $D \in \mathcal{M}$, hence $D = D \cap \mathcal{B}$, but $D \in \mathcal{M}$, then $D \in \mathcal{M}|_{\mathcal{B}}$. So, we get $\{C \subseteq \mathcal{B} : C \in \mathcal{M}\} \subseteq \mathcal{M}|_{\mathcal{B}}$. Consequentially, $\mathcal{M}|_{\mathcal{B}} = \{C \subseteq \mathcal{B} : C \in \mathcal{M}\}$.

Corollary 3.7

If $\mathcal{M}$ is $\mathcal{P}^*-$field on $\mathcal{U}$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{B} \in \mathcal{M}$.
Then, $\mathcal{M}|_{\mathcal{B}} \subseteq \mathcal{M}$.

Proof.

The proof follows **Proposition 3.6**.

Definition 3.8

If $\mathcal{U}$ is a universal set and $\mathcal{J} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, then a restriction of $\mathcal{J}$ on $\mathcal{B}$ is defined as:
$\mathcal{J}|_{\mathcal{B}} = \{N: N = M \cap \mathcal{B}, \text{ for some } M \in \mathcal{J}\}$.

Proposition 3.9

If $\mathcal{J} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$. Assume $\mathcal{M}$ is a $\mathcal{P}^*-$field of $\mathcal{U}$ that contains $\mathcal{J}$ and $\mathcal{B} \in \mathcal{M}$, then $\mathcal{P}^*(\mathcal{J})|_{\mathcal{B}}$ is a $\mathcal{P}^*-$field of $\mathcal{B}$.

Proof.

The proof is done by proposition 2.6 and 3.2

Theorem 3.10

Assume $\mathcal{J} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, then $\mathcal{P}^*(\mathcal{J}|_{\mathcal{B}})$ is the smallest $\mathcal{P}^*-$field on $\mathcal{B}$ that contain $\mathcal{J}|_{\mathcal{B}}$, where
$\mathcal{P}^*(\mathcal{J}|_{\mathcal{B}}) = \cap\{\mathcal{M}_i|_{\mathcal{B}}: \mathcal{M}_i|_{\mathcal{B}} \text{ is a } \mathcal{P}^*-\text{field of } \mathcal{B} \text{ and } \mathcal{M}_i|_{\mathcal{B}} \supseteq \mathcal{J}|_{\mathcal{B}}, \forall i \in I\}$.

Proof.

In the same way as in proposition 2.4, we can prove that $\mathcal{P}^*(\mathcal{J}|_{\mathcal{B}})$ is a $\mathcal{P}^*-$field on $\mathcal{B}$. To prove that $\mathcal{P}^*(\mathcal{J}|_{\mathcal{B}}) \supseteq \mathcal{J}|_{\mathcal{B}}$, assume that $\mathcal{M}_i|_{\mathcal{B}}$ is a $\mathcal{P}^*-$field on $\mathcal{B}$ and $\mathcal{M}_i|_{\mathcal{B}} \supseteq \mathcal{J}|_{\mathcal{B}}, \forall i \in I$, then $\mathcal{J}|_{\mathcal{B}} \subseteq \cap_{i\in I} \mathcal{M}_i|_{\mathcal{B}}$; hence $\mathcal{J}|_{\mathcal{B}} \subseteq \mathcal{P}^*(\mathcal{J}|_{\mathcal{B}})$. Now, let $\mathcal{M}^*|_{\mathcal{B}}$ be a $\mathcal{P}^*-$field on $\mathcal{B}$ such that $\mathcal{M}^*|_{\mathcal{B}} \supseteq \mathcal{J}|_{\mathcal{B}}$. Then, $\mathcal{M}^*|_{\mathcal{B}} \supseteq \mathcal{P}^*(\mathcal{J}|_{\mathcal{B}})$.
Therefore, $\mathcal{P}(\mathcal{J}|_{\mathcal{B}})$ is the smallest $\mathcal{P}^*-$field on $\mathcal{B}$ containing $\mathcal{J}|_{\mathcal{B}}$.

Theorem 3.11

If $\mathcal{J} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, define a class $\mathcal{M}$ by:
$\mathcal{M} = \{M \subseteq \mathcal{U} : M \cap \mathcal{B} \in \mathcal{P}^*(\mathcal{J}|_{\mathcal{B}})\}$. Then $\mathcal{M}$ is a $\mathcal{P}^*-$field on a set $\mathcal{U}$.

Proof.

By Theorem 3.10, we have $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$ as a $\mathcal{P}^*-$field on $\mathcal{B}$, so $\Phi \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$.
Since $\Phi = \Phi \cap \mathcal{B}$, then we get $\Phi \in \mathcal{M}$.
Assume that $M_1, M_2 \in \mathcal{M}$. Then $(M_i \cap \mathcal{B}) \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$, for each i=1,2.
Now, $(M_1 \cap M_2) \cap \mathcal{B} = (M_1 \cap \mathcal{B}) \cap (M_2 \cap \mathcal{B})$. Since $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})\}$ is a $\mathcal{P}^*-$field on $\mathcal{B}$, then $(M_1 \cap \mathcal{B}) \cap (M_2 \cap \mathcal{B}) \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$ and hence $(M_1 \cap M_2) \cap \mathcal{B} \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$, thus $M_1 \cap M_2 \in \mathcal{M}$.
Let $M_1, M_2, \ldots \in \mathcal{M}$. Then $(M_i \cap \mathcal{B}) \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$, for i=1,2,…
Since $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})\}$ is $\mathcal{P}^*-$field on $\mathcal{B}$, then $\bigcup_{i=1}^{\infty}(M_i \cap \mathcal{B}) \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$.
Now, $(\bigcup_{i=1}^{\infty} M_i) \cap \mathcal{B} = \bigcup_{i=1}^{\infty}(M_i \cap \mathcal{B}) \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$, thus $\bigcup_{i=1}^{\infty} M_i \in \mathcal{M}$.
Therefore, $\mathcal{M}$ is $\mathcal{P}^*-$field on a universal set $\mathcal{U}$.

Theorem 3.12

If $\mathcal{U}$ is a universal set and $\mathcal{I} \subseteq P(\mathcal{U})$ such that $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$, then $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}}) = \mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$.

Proof.

By proposition 2.6, we have $\mathcal{P}^*(\mathcal{I})$ is $\mathcal{P}^*-$field on $\mathcal{U}$. So, we get $\mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$ is *a* $\mathcal{P}^*-$field on $\mathcal{B}$ by proposition 3.2. Assume*that* $N \in \mathcal{I}|_{\mathcal{B}}$. Then $N = M \cap \mathcal{B}$ for some $M \in \mathcal{I}$.
But $\mathcal{I} \subseteq \mathcal{P}^*(\mathcal{I})$, so we have $M \in \mathcal{P}^*(\mathcal{I})$ and thus $N \in \mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$.
Hence $\mathcal{I}|_{\mathcal{B}} \subseteq \mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$. Therefore, $\mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$ is a $\mathcal{P}^*-$field on $\mathcal{B}$ that containing $\mathcal{I}|_{\mathcal{B}}$.
By Theorem 3.10, we have $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$ is the smallest $\mathcal{P}^*-$field on $\mathcal{B}$ that containing $\mathcal{I}|_{\mathcal{B}}$, which implies that $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}}) \subseteq \mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$.
Now, if we define a class $\mathcal{M}$ by
$\mathcal{M} = \{C \subseteq \mathcal{U} : C \cap \mathcal{B} \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})\}$, then in Theorem 3.11, we have $\mathcal{M}$ as *a* $\mathcal{P}^*-$field on $\mathcal{U}$. Let $C \in \mathcal{I}$, then $(C \cap \mathcal{B}) \in \mathcal{I}|_{\mathcal{B}}$, but $\mathcal{I}|_{\mathcal{B}} \subseteq \mathcal{P}(\mathcal{I}|_{\mathcal{B}})$ implies that $(C \cap \mathcal{B}) \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$, hence $C \in \mathcal{M}$ and $\mathcal{I} \subseteq \mathcal{M}$.
Now, if we assume that $N \in \mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$, then $N = M \cap \mathcal{B}$, for some $M \in \mathcal{P}(\mathcal{I})$. But $\mathcal{P}^*(\mathcal{I}) \subseteq \mathcal{M}$, then $M \in \mathcal{M}$, hence $N \in \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$. Consequentially, $\mathcal{P}^*(\mathcal{I})|_{\mathcal{B}} \subseteq \mathcal{P}^*(\mathcal{I}|_{\mathcal{B}})$.
This completes the proof.

3. Conclusions

We tried to define the concept of measure relative to the $\mathcal{P}^*-$field $\mathcal{M}$ of $\mathcal{U}$ and also define the idea of the restriction of measure on $\mathcal{M}|_{\mathcal{B}}$ of a set $\mathcal{B}$. Also, we discuss many properties of these notions. In this article, the idea of $\mathcal{P}^*-$field is given to refer to the generalization of each σ− field and σ−ring. Furthermore, some properties of the purposed notion are proven as explained below:

1. Let $\mathcal{M}$ be a $\mathcal{P}^*-$field of a set $\mathcal{U}$ and let $\mathcal{B}$ be a nonempty subset of $\mathcal{U}$. Then, $\mathcal{M}|_{\mathcal{B}}$ is a $\mathcal{P}^*-$field of a set $\mathcal{B}$.
2. Assume *that* $\mathcal{M}$ is a $\mathcal{P}^*-$field on $\mathcal{U}$ and $A \subseteq \mathcal{B} \subseteq \mathcal{U}$. If $A \in \mathcal{M}$, then $A \in \mathcal{M}|_{\mathcal{B}}$.
3. If $\mathcal{M}$ is a $\mathcal{P}^*-$field and $\mathcal{B}$ be a nonempty subset of $\mathcal{U}$ such that $\mathcal{B} \in \mathcal{M}$. Then $\mathcal{M}|_{\mathcal{B}} = \{A \subseteq \mathcal{B} : A \in \mathcal{M}\}$.
4. Suppose that $\mathcal{M}$ is a $\mathcal{P}^*-$field and $\mathcal{B} \subseteq \mathcal{U}$ such that $\mathcal{B} \in \mathcal{M}$. Then $\mathcal{M}|_{\mathcal{B}} \subseteq \mathcal{M}$.
5. If $\mathcal{I} \subseteq P(\mathcal{U})$ and $\Phi \neq \mathcal{B} \subseteq \mathcal{U}$ and $\mathcal{P}^*(\mathcal{I})|\mathcal{B}$ is a $\mathcal{P}^*-$field on $\mathcal{B}$. Then, $\mathcal{P}^*(\mathcal{I}|_{\mathcal{B}}) = \mathcal{P}^*(\mathcal{I})|_{\mathcal{B}}$.

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