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On A Fuzzy Soft Metric Space

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Abstract:

In this paper, we study the definition of soft metric space and fuzzy soft metric space and prove some properties about them. Also we study the following: We wrote the definition of standard fuzzy soft metric induced from a soft metric and proved some results about the relation between the properties in soft metric space and the induced fuzzy soft metric space. We defined fuzzy soft isometry between fuzzy soft metric spaces and proved some results about it. We defined fuzzy soft bounded in fuzzy soft metric space and proved some result about it.

Keywords: fuzzy metric space, induced fuzzy soft metric space, fuzzy soft isometry, fuzzy soft bounded.

1. Introduction:

In 1965, Zadha [15] defined the concept of fuzzy set. Kramosil [8] in1975, introduced the concept fuzzy metric space independent by definition of metric space and definition of fuzzy set. The concept of fuzzy normed space was Introduced by Katsaras [7] In 1984. In 1999, D. Molodtsov [9] introduced soft set to solve complicated problem in economic, social study, medical science, ect. Maji [10] in 2001, defined the fuzzy soft set depending on soft set and fuzzy set . In 2013, D.Das and Majumdar [5] introduced the concept of soft normed space. In 2013,D.Das and Sammanta [4] define the concept of soft metric space. T. Beaula and Merlin [1] in 2015, define a fuzzy soft normed space. In 2015,T. Beaula and R.Raja [13] define a fuzzy soft metric space. In this work, we study properties of soft set, fuzzy soft set, soft metric space and fuzzy soft metric space.

2.Preliminaries

Definition(2.1) [9,10]:

Let X be a universe and E be a set of parameters. Let $P(X)$ denote the power set of X.

A pair (F,E) is called a soft set over X, where F is a mapping given by $F: E \to P(X)$.

In other words, a soft set over X is a parameterized family of subsets of the universe X. A soft set (F,E)over X is said to be null soft set denoted by $\emptyset$, if for all $e \in E, F(e) = \emptyset$and said to be an absolute soft set denoted by $\tilde{X}$, if for all $e \in E, F(e) = X$.

Definition (2.2) [3]:

Let R be the set of real numbers , $B(R)$ be the collection of all nonempty bounded subsets of R and E be a set of parameters. Then a mapping of $F : E \to B(R)$ is called a soft real set . It is denoted by (F, E).

If in particular (F, E) is a singleton soft set, then identifying (F, E) with the corresponding soft element, it will be called a soft real number and denoted by $\tilde{r}, \tilde{s}, \tilde{t}$ $etc.$

Let $R(E)^*$ denoted the set of all non-negative soft real numbers

$\tilde{0}, \tilde{1}$ are soft real numbers where $\tilde{0}(e) = 0, \tilde{1}(e) = 1$, for all $e \in E$, respectively.

Definition(2.3) [4]:

Let X be a vector space over a field K and let E be a parameter set. Let (F, E) be a soft set over X. The soft set (F, E) is said to be a soft vector and denoted by $\tilde{x}_e$ if there is exactly one $e \in E$, such that $F(e) = \{x\}$ for some $x \in X$ and $F(e') = \emptyset, \forall e' \in E/\{e\}$.

The set of all soft vectors over $\tilde{X}$ will be denoted by $SV(\tilde{X})$.

Definition(2.4) [6]

Let $\tilde{X}$ be the absolute soft set. The a mapping $\|.\|: SV(\tilde{X}) \to R(E)^+$ is said to be a soft norm on the soft vector space $\tilde{X}$, if $\|.\|$satisfies the following conditions:

(N1):$\|\tilde{x}_e\| \tilde{\geq} \tilde{0} \forall$ $\tilde{x}_e \tilde{\in} \tilde{X}$

(N2):$\|\tilde{x}_e\| = \tilde{0}$ iff $\tilde{x}_e = \tilde{\emptyset}$

(N3):$\|\tilde{\alpha}.\tilde{x}_e\| = |\tilde{\alpha}|\|\tilde{x}_e\|$ for all $\tilde{x}_e \tilde{\in} \tilde{X}$ and for every soft scalar $\tilde{\alpha}$.

(N4): for all $\tilde{x}_e, \tilde{y}_a \tilde{\in} \tilde{X}$,$\|\tilde{x}_e + \tilde{y}_a\| \tilde{\leq} \|\tilde{x}_e\| + \|\tilde{y}_a\|$

The soft vector $\tilde{X}$ with a soft norm $\|.\|$ on $\tilde{X}$ is said to be a soft normed linear space and is denoted by $(\tilde{X}, \|.\|, E)$ or $(\tilde{X}, \|.\|)$.

Definition(2.5) [4]:

Let X be an initial universal set and E be anon-empty set of parameters . Let $\tilde{X}$ be the absolute soft set, $Sp(\tilde{X})$ collection of all soft points of $\tilde{X}$ and $R(E)^*$ denoted the set of all non-negative soft real numbers, we defined soft metric space as follow

A mapping $\tilde{d}: Sp(\tilde{X}) \times Sp(\tilde{X}) \to R(E)^*$ satisfying for all $\tilde{x}_e, \tilde{y}_a, \tilde{z}_b \in Sp(\tilde{X})$

1) $\tilde{d}(\tilde{x}_e, \tilde{y}_a) \geq \tilde{0}$

2) $\tilde{d}(\tilde{x}_e, \tilde{y}_a) = \tilde{0} \leftrightarrow \tilde{x}_e = \tilde{y}_a$

3) $\tilde{d}(\tilde{x}_e, \tilde{y}_a) = \tilde{d}(\tilde{y}_a, \tilde{x}_e)$

4) $\tilde{d}(\tilde{x}_e, \tilde{z}_b) \leq (\tilde{x}_e, \tilde{y}_a) + \tilde{d}(\tilde{y}_a, \tilde{z}_b)$

Then $(\tilde{X}, \tilde{d})$ called a soft metric space .

Definition (2.6)

Let $(\tilde{X}, \tilde{d})$ be a soft metric space, a subset A of $\tilde{X}$ is said to be soft bounded if there exist r , $0 < r < 1$ such that $\tilde{d}(\tilde{x}_e, \tilde{y}_a) < r$ $\forall \tilde{x}_e , \tilde{y}_a \in A$.

Definition(2.7) :

Let $(\tilde{X}, \tilde{d}_x)$ and $(\tilde{Y}, \tilde{d}_y)$ be two soft metric spaces, a mapping $f: (\tilde{X}, \tilde{d}_x) \to (\tilde{Y}, \tilde{d}_y)$ is an isometry if $\tilde{d}_y\big(f(\tilde{x}_e), f(\tilde{y}_a)\big) = \tilde{d}_x(\tilde{x}_e, \tilde{y}_a)$.

3. Fuzzy soft normed space and Fuzzy soft metric space

Definition (3.1)[12]:

Let $*$ be a binary operation on the set $I = [0,1]$, i.e $*$: $[0,1] \times [0,1] \to [0,1]$ is a function, then $*$ is said to be t-norm (triangular-norm) on the set I if $*$ satisfies the following axioms :
(1) $*$ is commutative and associative .
(2) $a * 1 = a$ for all $a \in [0,1]$.
(3) if $b, c \in I$ such that $b \leq c$, then $a * b \leq a * c$ for all $a \in I$.
In addition, if $*$ is continuous then $*$ is called a continuous t-norm.
The following theorem introduces the characteristics of the t-norm:

Theorem (3.1)[2] :

Let $*$ be a continuous t-norm on the set $I = [0,1]$, then :
(1) $1 * 1 = 1$
(2) $0 * 1 = 0$
(3) $0 * 0 = 0$
(4) $a * a \leq a, \forall a \in I$
(5) If $a \leq c$ and $b \leq d$, then $a * b \leq c * d$ for all $a, b, c, d \in I$.

Definition(3.2) [1]

The 3-tuple $(\tilde{X}, N, *)$ is said to be a fuzzy soft normed linear space (In short, FSNLS) if $\tilde{X}$ be an absolute soft linear space over the field K, $*$ is a continuous t-norm, $R(E)^*$ is the set of all positive soft real numbers, $Sp(\tilde{X})$ denote the set of all soft points on $\tilde{X}$ and N is a fuzzy set in $\tilde{X} \times R(E)^* (i.e. N : \tilde{X} \times R(E)^* \to [0,1])$ satisfying the following conditions: for all $\tilde{x}_e , \tilde{y}_a \tilde{\in} Sp(\tilde{X}), \tilde{t}, \tilde{s} \tilde{>} \tilde{0}$ *and* $\tilde{k} \tilde{\in} K$,
$(\boldsymbol{FSN.1}) N(\tilde{x}_e , \tilde{t}) > 0$,
$(\boldsymbol{FSN.2}) N(\tilde{x}_e , \tilde{t}) = 1$ if and only if $\tilde{x}_e = \tilde{\theta}_0$, where $\tilde{\theta}_0$ be a soft zero vector
$(\boldsymbol{FSN.3}) N(\tilde{k}.\tilde{x}_e , \tilde{t}) = N\left(\tilde{x}_e , \frac{\tilde{t}}{|\tilde{k}|}\right)$and $\tilde{k} \neq \tilde{0}$, ,
$(\boldsymbol{FSN.4})$ $N(\tilde{x}_e + \tilde{y}_a , \tilde{t} + \tilde{s}) \tilde{\geq} N(\tilde{x}_e , \tilde{t}) * N(\tilde{y}_a , \tilde{s})$,
$(\boldsymbol{FSN.5}) N(\tilde{x}_e , \bullet)$is a continuous, non-decreasing function of $R(E)^*$ and $\lim_{\tilde{t} \to \infty} N(\tilde{x}_e , \tilde{t}) = 1$
The triplet $(\tilde{X} , N , *)$ will be referred to as a fuzzy soft normed linear space.

Definition(3.3) [1]

Let X be an initial universal set and A be anon empty set of parameters. Let $\tilde{X}$ be the absolute fuzzy soft set , $Sp(\tilde{X})$ collection of all soft point of $\tilde{X}$ and $R(E)^*$ denoted the set of all non-negative soft real numbers, we defined fuzzy soft metric space using soft point as following
A mapping M: $Sp(\tilde{X}) \times Sp(\tilde{X}) \times R(E)^* \to [0,1]$ satisfy the following for all $\tilde{x}_e, \tilde{y}_a, \tilde{z}_b \in \tilde{X}$
1) $M(\tilde{x}_e, \tilde{y}_a, \tilde{t}) = 0$
2) $M(\tilde{x}_e, \tilde{y}_a, \tilde{t}) = 1 \leftrightarrow \tilde{x}_e = \tilde{y}_a$
3) $M(\tilde{x}_e, \tilde{y}_a, \tilde{t}) = M(\tilde{y}_a, \tilde{x}_e, \tilde{t})$
4) $M(\tilde{x}_e, \tilde{z}_b, \tilde{t} + \tilde{s}) \geq M(\tilde{x}_e, \tilde{y}_a, \tilde{t}) * M(\tilde{y}_a, \tilde{z}_b, \tilde{s})$ *for all* $\tilde{t}, \tilde{s} > 0$
5) $M(\tilde{x}_e, \tilde{y}_a, .) : (0, \infty) \to [0,1]$ continous.
Then $(\tilde{X}, M, *)$ called fuzzy soft metric space .

Definition(3.4)[1]

Let $(\tilde{X}, M, *)$be a fuzzy soft metric space and $\{\tilde{x}_{en}\}$ be a fuzzy soft sequence in $\tilde{X}$ then
1) A fuzzy soft sequence $\{\tilde{x}_{en}\}$ in $\tilde{X}$ is said to be fuzzy soft convergent to $\tilde{x}_e$ if for every $\epsilon > 0$, exist $k \epsilon N$ such that $M(\tilde{x}_{en}, \tilde{x}_e, \tilde{t}) > 1 - \epsilon \; \forall \, n \geq k$.

2) A fuzzy soft sequence { x Æ = in Ḟ is said to be fuzzy soft Cauchy if for «r Æ G 0 / :T Æ ÆT ÆP; P s F ˚ JÆI R -

3) A fuzzy soft metric space is said to be complete if every fuzzy soft Cauchy sequence is convergent in Ḟ

Definition (3.5)

Let (ḞÆ/ Æ) be a fuzzy soft metric space and # be a subset of Ḟ

(1) # is said to be fuzzy soft bounded (in short FS-bounded) if there exists P`P and r O N O s such that / :T ÆU ÆP; P s F N ˚T ÆU — #

(2) # is said to be dense in Ḟ if for every T — Ḟ and for every N P r ÆP`P , we have that $:T ÆNÆP; Œ # M , i.e every open ball in (ØÆ Æ) contain a point of #

(3) the closure of # is denoted by #§ or % . :# ; and is defined to be the smallest closed set contain #

(4) # is said to be dense in Ḟ if #§ L Ḟ

Definition (3.6) [11]

Let kḞÆ/ Æ and kḞÆ0 Æ are two fuzzy soft metric space. Let B ª Ḟ \ and ª ' \ be function. Then :BÆ is called a fuzzy soft function from Ḟ Ḟ and defined by :BÆ ;ª :ḞÆ/ Æ ; \ :ḞÆ0 Æ ;

Example (3.1)[1]

Let kḞÆ0 Æ be a fuzzy soft normed space, let / defined by :T ÆU ÆP; L 0 :T F U ÆP; then / is a fuzzy soft metric.

Proof:

Let : : Æ0 Æ be a fuzzy soft normed space

Define the fuzzy soft metric space by / :T ÆU ÆP; L 0 :T F U ÆP; for every T ÆU — OLkḞo

the fuzzy soft metric axioms are satisfied

1) / :T ÆU ÆP; L 0 :T F U ÆP; L r

2) / :T ÆU ÆP; L 0 :T F U ÆP; L s if and only if T F U L Ḟ hence T L U

3) / :T ÆU ÆP; L 0 :T F U ÆP;

L 0 :U F T ÆP;

L / :U ÆT ÆP;

So / :T ÆU ÆP; L / :U ÆT ÆP;

4) / :T ÆV` ÆØ P; L 0 :T F V` ÆØ P;

L 0 :T F U E U F V` ÆØ P;

R 0 :T F U Æ0; :U F V` ÆP;

L / :T ÆU Æ0; / :U ÆV` ÆP;

Hence / :T ÆV` ÆØ P; R / :T ÆU Æ0; / :U ÆV` ÆP;

5) since 0 is continuous then / is continuous .

: / is said to be fuzzy soft metric induced by the fuzzy soft norm kḞÆ0 Æo;

Theorem (3.2):

(1) Let / be a fuzzy soft metric induced by a norm on a fuzzy soft vector 58kḞo then:

(i) / :T E V` ÆU E V` ÆP; / :T ÆU ÆP;

(ii) / :N`T ÆN`U ÆP; L / @T ÆU Æ ¸A

(2) Let M be a fuzzy soft metric on a fuzzy soft vector $SV(\tilde{X})$ such that (i) and (ii), then M induced norm on a fuzzy soft vector $SV(\tilde{X})$.

Proof:

(1) Let M be a fuzzy soft metric induced by a norm N on a fuzzy soft vector $SV(\tilde{X})$ such that $\forall\, \tilde{x}_e, \tilde{y}_a, \tilde{z}_b \in \tilde{X}, M(\tilde{x}_e, \tilde{y}_a, \tilde{t}) = N(\tilde{x}_e - \tilde{y}_a, \tilde{t})$

(i) $M(\tilde{x}_e + \tilde{z}_b, \tilde{y}_a + \tilde{z}_b, \tilde{t}) = N(\tilde{x}_e + \tilde{z}_b - (\tilde{y}_a + \tilde{z}_b), \tilde{t})$
$= N(\tilde{x}_e - \tilde{y}_a, \tilde{t}) = M(\tilde{x}_e, \tilde{y}_a, \tilde{t})$

(ii) $M(\tilde{r}.\tilde{x}_e, \tilde{r}.\tilde{y}_a, \tilde{t}) = N(\tilde{r}.\tilde{x}_e - \tilde{r}.\tilde{y}_a, \tilde{t})$
$= N(\tilde{r}(\tilde{x}_e - \tilde{y}_a), \tilde{t})$
$= N\left(\tilde{x}_e - \tilde{y}_a, \frac{\tilde{t}}{|\tilde{r}|}\right)$
$= M\left(\tilde{x}_e, \tilde{y}_a, \frac{\tilde{t}}{|\tilde{r}|}\right)$

(2) suppose that the condition (i) and (ii) holds

Let $\|\,.\,\|: SV(\tilde{X}) \to R(E)^*$

$N(\tilde{x}_e, \tilde{t}) = M(\tilde{x}_e, \tilde{\theta}_0, \tilde{t})\ \forall\, \tilde{x}_e \in SV(\tilde{X})$, where $\tilde{\theta}_0$ be a soft zero vector, We have

(N1) $N(\tilde{x}_e, \tilde{t}) = M(\tilde{x}_e, \tilde{\theta}_0, \tilde{t}) = \tilde{0}$ and
$N(\tilde{x}_e, \tilde{t}) = M(\tilde{x}_e, \tilde{\theta}_0, \tilde{t}) = 1 \Leftrightarrow \tilde{x}_e = \tilde{\theta}_0$

(N2) $N(\tilde{r}.\tilde{x}_e, \tilde{t}) = M(\tilde{r}.\tilde{x}_e, \tilde{\theta}_0, \tilde{t})$
$= M(\tilde{r}.\tilde{x}_e, \tilde{r}.\tilde{\theta}_0, \tilde{t})$
$= M\left(\tilde{x}_e, \tilde{\theta}_0, \frac{\tilde{t}}{|\tilde{r}|}\right)$
$= N\left(\tilde{x}_e, \frac{\tilde{t}}{|\tilde{r}|}\right)$

(N3) $N(\tilde{x}_e + \tilde{y}_a, \tilde{t} \oplus \tilde{s}) = M(\tilde{x}_e + \tilde{y}_a, -\tilde{y}_a + \tilde{y}_a, \tilde{t} \oplus \tilde{s})$
$= M(\tilde{x}_e, -\tilde{y}_a, \tilde{t} \oplus \tilde{s})$
$= M(\tilde{x}_e + \tilde{\theta}_0, \tilde{\theta}_0 - \tilde{y}_a, \tilde{t} \oplus \tilde{s})$
$\geq M(\tilde{x}_e + \tilde{\theta}_0, \tilde{t}) * M(\tilde{\theta}_0 - \tilde{y}_a, \tilde{s})$
$= N(\tilde{x}_e, \tilde{t}) * N(-\tilde{y}_a, \tilde{s})$
$= N(\tilde{x}_e, \tilde{t}) * N\left(\tilde{y}_a, \frac{\tilde{s}}{|-1|}\right)$
$= N(\tilde{x}_e, \tilde{t}) * N(\tilde{y}_a, \tilde{s})$

Proposition (3.1):

Let $(\tilde{X}, \tilde{d})$ be a soft metric space and let $a * b = a \cdot b$ for all $a, b \in [0,1]$ and $M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t}) = \frac{1}{1+d(\tilde{x}_e, \tilde{y}_a)}$, then $(\tilde{X}, M_d, *)$ is a fuzzy soft metric space (and it is said to be the standard fuzzy soft metric space).

Proof:

1) let $\tilde{x}_e, \tilde{y}_a \in Sp(\tilde{X})$ and $\tilde{t} \in R(E)^*$

Since $d(\tilde{x}_e, \tilde{y}_a) \geq \tilde{0}$

$\Rightarrow \frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)} \geq 0$

Since $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = \frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)}$

So, $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) \geq 0$

2) let $\tilde{x}_e\,, \tilde{y}_a \in Sp(\tilde{X}) \ni \tilde{x}_e = \tilde{y}_a$

$\Longrightarrow d(\tilde{x}_e\,, \tilde{y}_a) = \tilde{0}$

Since $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = \frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)}$

$\Longrightarrow M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = \frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)} = \frac{1}{1+\tilde{0}} = 1$, therefore $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = 1$

Conversely, let $\tilde{x}_e\,, \tilde{y}_a \in Sp(\tilde{X}) \ni M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = 1$

Since $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = \frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)}$

Then $\frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)} = 1 \Longrightarrow 1 + d(\tilde{x}_e, \tilde{y}_a) = 1$

So $d(\tilde{x}_e\,, \tilde{y}_a) = 0$

Since $\tilde{d}$ is soft metric $\Longrightarrow \tilde{x}_e = \tilde{y}_a$

3) Since $d(\tilde{x}_e\,, \tilde{y}_a)$ is a soft metric $\Rightarrow d(\tilde{x}_e\,, \tilde{y}_a) = d(\tilde{y}_a, \tilde{x}_e\,)$

Since $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = \frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)}$

So $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = \frac{1}{1+d(\tilde{x}_e,\tilde{y}_a)} = \frac{1}{1+d(\tilde{y}_a,\tilde{x}_e\,)} = M_d(\tilde{y}_a, \tilde{x}_e\,, \tilde{t})$

Then $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) = M_d(\tilde{y}_a, \tilde{x}_e\,, \tilde{t})$

4) let $\tilde{x}_e, \tilde{y}_a, \tilde{z}_b \in Sp(\tilde{X})$

Since $\tilde{d}$ is a soft metric

Then $d(\tilde{x}_e\,, \tilde{y}_a) \leq d(\tilde{x}_e\,, \tilde{z}_b) + d(\tilde{z}_b,\ \tilde{y}_a)$

$1 + d(\tilde{x}_e\,, \tilde{y}_a) \leq 1 + (d(\tilde{x}_e\,, \tilde{z}_b) + d(\tilde{z}_b,\ \tilde{y}_a))$

$\frac{1}{1+d(\tilde{x}_e\,,\tilde{y}_a)} \geq \frac{1}{1+(d(\tilde{x}_e\,,\tilde{z}_b)+d(\tilde{z}_b,\tilde{y}_a))}$

$$\text{So } M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t}) \geq \frac{1}{1+\left(d(\tilde{x}_e\,,\tilde{z}_b)+d(\tilde{z}_b,\tilde{y}_a)\right)+d(\tilde{x}_e\,,\tilde{z}_b)d(\tilde{z}_b,\tilde{y}_a)}$$
$$= \frac{1}{(1+d(\tilde{x}_e\,,\tilde{z}_b))(1+d(\tilde{z}_b,\tilde{y}_a))}$$
$$= \frac{1}{(1+d(\tilde{x}_e\,,\tilde{z}_b))} * \frac{1}{(1+d(\tilde{z}_b,\tilde{y}_a))}$$
$$= M_d(\tilde{x}_e\,, \tilde{z}_b, \tilde{s}) * M_d(\tilde{z}_b,\ \tilde{y}_a, \tilde{r})$$

5) since $\tilde{d}(\tilde{x}_e\,, \tilde{y}_a)$ is continuous then $M_d(\tilde{x}_e\,, \tilde{y}_a, \tilde{t})$is continuous.

Proposition(3.2):

Let $(\tilde{X}\,, d)$ be a soft metric space and $(\tilde{X}, M_d,*)$ be the standard fuzzy soft metric space and $\{\tilde{x}_{en}\}$ be a sequence in $\tilde{X}$ then $\{\tilde{x}_{en}\}$ converge to $\tilde{x}_e \in \tilde{X}$ in$(\tilde{X}\,, d)$ if and only if $\{\tilde{x}_{en}\}$ converge to $\tilde{x}_e$ in $(\tilde{X}, M_d,*)$.

Proof:

Suppose that the sequence $\{\tilde{x}_{en}\}$ in $\tilde{X}$ converge to $\tilde{x}_e$ in $(\tilde{X}, \tilde{d})$ and $\tilde{\epsilon} > 0$

since $\{\tilde{x}_{en}\}$ sequence in $\tilde{X}$ converge to $\tilde{x}_e$ in $(\tilde{X}, \tilde{d})$

$\exists\, k \in N \ni d(\tilde{x}_{en}, \tilde{x}_e) < \tilde{\epsilon}\ , \forall\, n \geq k$

Since $d(\tilde{x}_{en}, \tilde{x}_e) < \tilde{\epsilon}$

$1 + d(\tilde{x}_{en}, \tilde{x}_e) < 1 + \tilde{\epsilon}$

1 / 5> :°_ Æ_; P 5 / 5> " P s F

Since / :T Æ ÆT ÆP; L 5 / 5> :°_ Æ_;

This implies that / :T Æ ÆT ÆP; P s F

Hence <T Æ converges to T in :Ŀ Æ/ Æ ;

Suppose that <T Æ converges sequence to T in :Ŀ Æ/ Æ ;

Therefore / :T Æ ÆT ÆP; P s F

Since / :T Æ ÆT ÆP; L 5 / 5> :°_ Æ_;

: 5 / 5> :°_ Æ_; P s F

s E @:T Æ ÆT ; O 5 / 5? "

@:T Æ ÆT ; O 5 / 5? " F s O `

@:T Æ ÆT ; O `

Then <T Æ converges sequence to T in :Ŀ Æ@;

Proposition(3.3):

Let :Ŀ Æ@ be a soft metric space and / :T, U ÆP; be the standard fuzzy soft metric space then <T Æ is a Cauchy sequence in :Ŀ Æ@ if and only if <T Æ is a Cauchy sequence in (Ŀ Æ/ Æ ;

Proof:

Suppose that <T Æ be a Cauchy sequence in :Ŀ Æ@ and P r

since <T Æ Cauchy sequence in :Ŀ Æ@;

there exist G – 0 @:T Æ ÆT ; O `Æ° JÆI R G

since @:T Æ ÆT ; O `

s E @:T Æ ÆT ; O s E `

5 / 5> :°_ Æ_ ; P 5 / 5> " P s F

This implies that / :T Æ ÆT ÆP; P s F

Hence <T Æ Cauchy sequence in :Ŀ Æ/ Æ ;

Suppose that <T Æ is Cauchy sequence in :Ŀ Æ/ Æ ;

: / :T Æ ÆT ÆP; P s F

Since / :T Æ ÆT ÆP; L 5 / 5> :°_ Æ_ ;

Then 5 / 5> :°_ Æ_ ; P s F

s E @:T Æ ÆT ; O 5 / 5? "

@:T Æ ÆT ; O 5 / 5? " F s O `

@:T Æ ÆT ; O `

Then <T Æ Cauchy sequence in :Ŀ Æ@;

Proposition(3.4):

Let :Ŀ Æ@ be a soft metric space and / :T, U ÆP; be the standard fuzzy soft metric space, a subset # of Ø is FS-bounded if and only if it is bounded.

Proof:

Suppose that A is a bounded subset in $(\tilde{X},\tilde{d})$ and $\tilde{r}>0$
Since A bounded subset in $(\tilde{X},\tilde{d})$
Then there exist $\tilde{r}$, $0<\tilde{r}<1$ $\ni$ $\tilde{d}(\tilde{x}_e,\tilde{y}_a)<\tilde{r}$ $\forall$ $\tilde{x}_e,\tilde{y}_a\in A$
Since $\tilde{d}(\tilde{x}_e,\tilde{y}_a)<\tilde{r}$
Then $1+\tilde{d}(\tilde{x}_e,\tilde{y}_a)<1+\tilde{r}$
$\frac{1}{1+\tilde{d}(\tilde{x}_e,\tilde{y}_a)}>\frac{1}{1+\tilde{r}}>1-r$
This implies that $M_d(\tilde{x}_e,\tilde{y}_a,\tilde{t})>1-r$
So A is FS-bounded in $(\tilde{X}, M_d, *)$
Suppose that A is FS-bounded in $(\tilde{X}, M_d, *)$
Then there exist r , $0<r<1$, $\tilde{t}>0$ $\ni$ $M_d(\tilde{x}_e,\tilde{y}_a,\tilde{t})>1-r$ $\forall\tilde{x}_e,\tilde{y}_a\in A$
Since $M_d(\tilde{x}_e,\tilde{y}_a,\tilde{t})=\frac{1}{1+\tilde{d}(\tilde{x}_e,\tilde{y}_a)}$
So $\frac{1}{1+\tilde{d}(\tilde{x}_e,\tilde{y}_a)}>1-r$
$1+\tilde{d}(\tilde{x}_e,\tilde{y}_a)<\frac{1}{1-r}$
$\tilde{d}(\tilde{x}_e,\tilde{y}_a)<\frac{1}{1-r}-1<\tilde{r}$
$\tilde{d}(\tilde{x}_e,\tilde{y}_a)<\tilde{r}$
Hence A is bounded subset in $(\tilde{X},\tilde{d})$.

Definition(3.7):

Let $(\tilde{X},M_x,*)$ and $(\tilde{Y},M_y,*)$ be two fuzzy soft metric spaces, a soft function $f:\tilde{X}\to\tilde{Y}$ is said to be continuous at $\tilde{a}_r\in\tilde{X}$, if for every $0<\epsilon<1$, there exist some $0<\delta<1$, such that $M_y(f(\tilde{x}_e),f(\tilde{a}_r),\tilde{t})>1-\epsilon$, whenever $\tilde{x}_e\in\tilde{X}$ and $M_x(\tilde{x}_e,\tilde{a}_r,\tilde{t})>1-\delta$ .If f is continuous at every point of $\tilde{X}$,then it is said to be continuous on $\tilde{X}$.

Theorem(3.3):

Let $(\tilde{X},M_x,*)$ and $(\tilde{Y},M_y,*)$ be two fuzzy soft metric spaces and f be a soft mapping ,then f continuous at $\tilde{a}_r$ if and only if for any sequence $\{\tilde{x}_{en}\}$ in $\tilde{X}$converge to $\tilde{a}_r$ then the sequence $(f(\tilde{x}_{en}))$ converge to $f(\tilde{a}_r)$

Proof:

Suppose that f be a continuous function at $\tilde{a}_r\in\tilde{X}$ and $\{\tilde{x}_{en}\}$ sequence in $\tilde{X}$ converge to $\tilde{a}_r$
Let $\tilde{\epsilon}>0$
Since f continuous at $\tilde{a}_r$
$\Rightarrow\exists\delta>0$, $\forall$ $\tilde{x}_e\in\tilde{X}$ and $M_x(\tilde{x}_e,\tilde{a}_r,\tilde{t})>1-\delta$
So $M_y(f(\tilde{x}_e),f(\tilde{a}_r),\tilde{t})>1-\epsilon$
Since $\delta>0$ and $\tilde{x}_{en}\to\tilde{a}_r$
Then there exist $k\in Z^+$ $\ni$ $M_x(\tilde{x}_{en},\tilde{a}_r,\tilde{t})>1-\delta$, $\forall n>k$
since $\tilde{x}_{en}\in\tilde{X}$,then by above we have $M_y(f(\tilde{x}_{en}),f(\tilde{a}_r),\tilde{t})>1-\epsilon$
$\exists$ $k\in Z^+$ $\ni$ $M_y(f(x_{en}),f(\tilde{a}_r),\tilde{t})>1-\epsilon$, , $\forall n>k$
Hence $f(\tilde{x}_{en})\to f(\tilde{a}_r)$
$\Leftarrow$ suppose that the condition holds and f is not continuous
$\exists$ $\epsilon>0$, $\forall\delta>0$, $\exists$ $\tilde{x}_e\in A$ $\ni$ $M_x(\tilde{x}_e,\tilde{a}_r,\tilde{t})>1-\delta$ and

M :B:T ;ÆB:= ;ÆP; O s F

Let J – <ᐳ and L Ⱶ$_{Æ}^{5}$ P r

then by above there exist T$_{Æ}$ – # /∘ :T$_{Æ}$ Æ= ÆP; P s F L s F$_{Æ}^{5}$ and

/ :B:T$_{Æ}$;ÆB:= ;ÆP; O s F

T$_{Æ}$ \ = but B:T$_{Æ}$; not converge to B:= ;

This contradiction with the condition above

Hence B is continuous function at =.

Theorem(3.4):

Let # be a subset of a fuzzy soft metric space (Ⱶ Æ/ Æ ; then = – # § and only if there is a sequence { = $_{Æ}$ = in # such that = $_{Æ}$ \ =.

Proof:

: suppose that = – # §

Since # § L # ∘ #

so = – # ∘ # then = – # or = – #

If = – #

Let = $_{Æ}$ L = ˚J – 0

{ = $_{Æ}$ = sequence in # and = $_{Æ}$ \ =

If = – # Æ= #

Let J – <ᐳ Æ$:= Æ$_{Æ}^{5}$ ÆP open ball with center =

This implies that $:= Æ$_{Æ}^{5}$ ÆP open set contain =

Since = – # : :$ @= Æ$_{Æ}^{5}$ ÆPÀŒ # F <= =; M ˛

so there exists = $_{Æ}$ – $ @= Æ$_{Æ}^{5}$ ÆPÀŒ #

: = $_{Æ}$ – $ l= Æ$_{J}^{s}$ ÆP Ⱡ = $_{Æ}$ – #

<= $_{Æ}$ = sequence in #

Let P r [by Archimedean property] , G – <ᐳ : –^{5} O

Since J P G : –$_{Æ}^{5}$ O–^{5} O : –$_{Æ}^{5}$ O

Since = $_{Æ}$ – $ @= Æ$_{Æ}^{5}$ ÆPÀ then

/ := $_{Æ}$ Æ= ÆP; P s F–$_{J}^{s}$

If J P G : / := $_{Æ}$ Æ= ÆP; P s F–$_{Æ}^{5}$ P s F–^{5} P s F

Hence / := $_{Æ}$ Æ= ÆP; P s F then = $_{Æ}$ \ =

9 suppose that <= $_{Æ}$ = sequence in # = $_{Æ}$ \ =

To show that = – # § Æ= – # ∘ #

If = – # : = – # C # § : = – # §

If = #

Let G be an open set in $\tilde{X}$ containing $\tilde{a}_r$

There exist $r > 0$ $s.t$ $B(\tilde{a}_r, r, \tilde{t}) \subseteq G$

Since $r > 0$, $\tilde{a}_{en} \to \tilde{a}_r$

So there exist $k \in Z^+ \ni M(\tilde{a}_{en}, \tilde{a}_r, \tilde{t}) > 1 - r \quad \forall n > k$

$\Longrightarrow \tilde{a}_{en} \in B(\tilde{a}_r, r, \tilde{t}) \quad \forall n > k$

Since $\tilde{a}_e \in A \quad \forall n \in Z^+$

Then $A \cap (B(\tilde{a}_r, r, \tilde{t})/\{\tilde{a}_r\}) \neq \emptyset$

since $B(\tilde{a}_r, r, \tilde{t}) \subseteq G \Longrightarrow A \cap \left(\frac{G}{\{\tilde{a}_r\}}\right) \neq \emptyset$

so $\tilde{a}_r \in \grave{A}$ hence $\tilde{a}_r \in \bar{A}$.

Definition(3.8):

A mapping f from a fuzzy soft metric space $(\tilde{X}, M_x, *)$ in to a fuzzy metric space $(\tilde{Y}, M_y, *)$ is a fuzzy soft isometry (in short FS-isometry) if $M_y(f(\tilde{x}_e), f(\tilde{y}_a), \tilde{t}) = M_x(\tilde{x}_e, \tilde{y}_a, \tilde{t})$ for all $\tilde{x}_e, \tilde{y}_a \in \tilde{X}$ and $\tilde{t} > 0$. $\tilde{X}$ and $\tilde{Y}$ are said to be FS-isometric if there exists an FS-isometry between them that is onto.

Proposition(3.5):

Let $(\tilde{X}, \tilde{d})$ be a soft metric space and $(\tilde{X}, M_d, *)$ be the standard fuzzy soft metric space. Let$(\tilde{Y}, \tilde{d})$ be another soft metric space and $(\tilde{Y}, N_d, *)$ be the standard fuzzy soft metric space, Let $f: \tilde{X} \to \tilde{Y}$ be a mapping , then f is isometry if and only if f is FS-isometry.

Proof:

Suppose that $f: \tilde{X} \to \tilde{Y}$ be a mapping and f is isometry

so $\tilde{d}_y\big(f(\tilde{x}_e), f(\tilde{y}_a)\big) = \tilde{d}_x(\tilde{x}_e, \tilde{y}_a) \ \forall\, \tilde{x}_e, \tilde{y}_a \in \tilde{X}$

$$N_d(f(\tilde{x}_e), f(\tilde{y}_a), \tilde{t}) = \frac{1}{1+\tilde{d}_y(f(\tilde{x}_e), f(\tilde{y}_a))}$$

$$= \frac{1}{1+\tilde{d}_x(\tilde{x}_e, \tilde{y}_a)}$$

$$= M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t})$$

So $N_d(f(\tilde{x}_e), f(\tilde{y}_a), \tilde{t}) = M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t})$

Then f is FS-isometry

Conversely, Suppose that $f: \tilde{X} \longrightarrow \tilde{Y}$ be a mapping and f is FS-isometry

$\Longrightarrow N_d(f(\tilde{x}_e), f(\tilde{y}_a), \tilde{t}) = M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t})$

$$\tilde{d}_x(\tilde{x}_e, \tilde{y}_a) = \frac{(1 - M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t}))}{M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t})}$$

And $\tilde{d}_y(f(\tilde{x}_e), f(\tilde{y}_a)) = \frac{(1-N_d(f(\tilde{x}_e), f(\tilde{y}_a), \tilde{t}))}{N_d(f(\tilde{x}_e), f(\tilde{y}_a), \tilde{t})}$

$$= \frac{1 - M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t})}{M_d(\tilde{x}_e, \tilde{y}_a, \tilde{t})} = \tilde{d}_x(\tilde{x}_e, \tilde{y}_a)$$

So $\tilde{d}_x(\tilde{x}_e, \tilde{y}_a) = \tilde{d}_y(f(\tilde{x}_e), f(\tilde{y}_a))$

Then f is isometry.

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