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Estimate for Survival and Related Functions of Weighted Rayleigh Distribution.

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Abstract

In this paper, we introduce a new class of Weighted Rayleigh Distribution based on two parameters, one is the scale parameter and the other is the shape parameter introduced in Rayleigh distribution. The main properties of this class are derived and investigated[13]. The moment method and least square method are used to obtain estimators of parameters of this distribution. The probability density function, survival function, cumulative distribution and hazard function are derived and found. Real data sets are collected to investigate two methods that depend on in this study. A comparison is made between two methods of estimation and clarifies that MLE method is better than the OLS method by using the mean squares error.

Keyword: weighted Rayleigh distribution, Maximum likelihood method and lest square method.

1. Introduction

The Rayleigh distribution is one of the important continuous distributions; it encounters much attention in the literature in benefit from any other distribution in lifetime sample modeling and data analysis. Many researchers developed various generalizations of Rayleigh probability density function to increase the flexibility in lifetime sample modeling [1]. We

Introduced a new class of density function depending on the shape parameter in the normal distribution, which is known as weighted normal distribution or (skew-normal distribution) [2]. We used the idea of Azzalini to find the shape parameter to an exponential distribution which is known as weighted exponential distribution, as well as putthe general mathematical formula to treat the weighted statistical distributions which are as follow:

$$f_w(X) = \frac{1}{p[x_2 < \alpha x_1]} f(x_1)F(\alpha x_1) \quad (1)$$

Where:

$f_w(X)$ Weighted probability density function.

$f(x_1)$ Standard probability density function for $r.v(x_1)$

$F(\alpha x_1)$ Cumulative distribution functions with respect to weighted parameter α for standard distribution.

$P_r(x_2 < \alpha x_1)$ Probability for $r.v(x_2)$ with respect to the $r.v(x_1)$ and weighted function (α).

[3] Studied weighed Weibull distribution by using the idea of Azzalini and introduced the basic properties for this model [4]. Studied the skew-ness parameter of a gamma distribution by using the idea of Azzalini that resulted a new class of weighted gamma distribution.[5] proposed the extension of the weighted Weibull distribution and the main properties of this class are investigated and derived [6]. Estimated Linley's approximation method for weighted exponential distribution by using a Monte Carlo simulation study [7]. Introduced two shape parameters to the existing weighted exponential distribution to develop the weighted Beta exponential distribution using the log it of beta function[8] proposed a model named exponentiated weighted exponential distribution and some of the basic statistical properties of the proposed model are studied and provided[9]. Studied the Nonparametric method such as the empirical method, kernel method, and modified shrinkage provided in weighted Weibull distribution [10]. Focused on Bayes estimation of weighted exponential distribution with fuzzy data[11]. Derived two parameters inverted weighted Exponential distribution and its various statistical properties thatwere established [12]. presented a new generalization weighted Weibull distribution using topple one family of distribution [13]. Proposed a new class of weighted Rayleigh distribution by using the idea of Azzalini and introduce the main characteristics of this distribution.This paper aimsto introduce a new weighted Rayleigh distribution with its properties which discussed maintained in[13]. Applying this new distribution on real data to estimate the parameters by using two methods and calculate the death density function, survival function, hazard function for these two methods. The rest of this article is as follows: in section two, the new weighted Rayleigh distribution and its properties is presented, section three xplains estimation methods, in section four is devoted to the real data application and section five gives the conclusion.

2- Weighted Rayleigh Distribution

In this section, we will provide the probability density function of new weighted Rayleigh distribution which is as follows:

$$f(x;\alpha,\theta)=\frac{\alpha^2+1}{\alpha^2}\theta x\, e^{-\frac{\theta}{2}x^2}\left[1-e^{-\frac{\theta}{2}x^2\alpha^2}\right]\;,\;\; x>0[13] \tag{2}$$

The cumulative distribution of new distribution is as follow:

$$F(x;\ \alpha,\theta)=1-\left[\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x^2}-e^{-\frac{\theta}{2}x^2(\alpha^2+1)}}{\alpha^2}\right] \tag{3}$$

The survival function and hazard function is as follows:

$$S(x)=\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x^2}-e^{-\frac{\theta}{2}x^2(\alpha^2+1)}}{\alpha^2} \tag{4}$$

$$h(x)=\frac{\theta x(\alpha^2+1)\left[1-e^{-\frac{\theta}{2}x^2d^2}\right]}{\left[(\alpha^2+1)-e^{-\frac{\theta}{2}x^2\alpha^2}\right]} \tag{5}$$

The r^{th} moment of new distribution is as follows:

$$E(x^r)=\frac{\alpha^2+1}{\alpha^2}\left(\frac{2}{\theta}\right)^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{\frac{r}{2}+1}}\right] \tag{6}$$

The mean and the variance of the new distribution is:

$$\mathrm{M}=\mathrm{E}\,(X)=\frac{\alpha^2+1}{\alpha^2}\left[\frac{\pi}{2\theta}\right]^{\frac{1}{2}}\left[1-\frac{1}{(\alpha^2+1)^{3/2}}\right] \tag{7}$$

The variance:

$$\sigma^2=var(X)=\frac{4\alpha^4(\alpha^2+2)-\pi\left[(\alpha^2+1)^{\frac{3}{2}}-1\right]^2}{2\theta\alpha^4(\alpha^2+1)} \tag{8}$$

The *Moment generated function of this distribution is as* follows:

$$m.g.f=\frac{\alpha^2+1}{\alpha^2}\sum_{r=0}^{\infty}\frac{t^r}{r!}\left(\frac{2}{\theta}\right)^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{r/2+1}}\right] \tag{9}$$

The Factorial Moment Generating function is:

$$M_x(t)=\frac{\alpha^2+1}{\alpha^2}\sum_{r=0}^{\infty}\frac{(\ell nt)^r}{r!}\left(\frac{2}{\theta}\right)^{\frac{r}{2}}\Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{\frac{r}{2}+1}}\right] \tag{10}$$

The skewness and the kurtosis of this distribution is :

$$C.S = \frac{1}{\alpha^2}\,\Gamma\left(\frac{5}{2}\right)\left[\frac{(\alpha^2+1)^{\frac{5}{2}}-1}{(\alpha^2+1)^{\frac{3}{2}}}\right] \tag{11}$$

$$C.k = \frac{2[(\alpha^2+1)^3-1]}{\alpha^2(\alpha^2+1)^2} - 3 \tag{12}$$

The characteristic function of this distribution is as follows :

$$Qx\,(x) = \frac{\alpha^2+1}{\alpha^2}\sum_{r=0}^{\infty}\frac{(it)^r}{r!}\left(\frac{2}{\theta}\right)^r \Gamma\left(\frac{r}{2}+1\right)\left[1-\frac{1}{(\alpha^2+1)^{\frac{r}{2}+1}}\right] \tag{13}$$

3. Estimation Methods

In this section, estimate the parameters of weighted Rayleigh distribution by employing two methods (Maximum likelihood estimator method and Ordinary least square estimator method).

3.1: Maximum Likelihood Method

MLE is a famous and classical method used to find the estimators of parameters that maximize the likelihood function. The probability density function is:

$$f(x;\alpha,\theta) = \frac{\alpha^2+1}{\alpha^2}\,\theta x\, e^{-\frac{\theta}{2}x^2}\left[1-e^{-\frac{\theta}{2}x^2\alpha^2}\right] \tag{14}$$

$$L\,(\alpha,\theta;x) = \frac{(\alpha^2+1)^n}{\alpha^{2n}}\theta^n \prod_{i=1}^{n} x_i\; e^{-\frac{\theta}{2}x^2}\prod_{i=1}^{n}\left[1-e^{-\frac{\theta}{2}x^2\alpha^2}\right] \tag{15}$$

$$LnL = nln\,(\alpha^2+1) - 2n\,ln(\alpha) + nln\theta + \sum_{i=1}^{n} Ln\,x_i \;-\frac{\theta}{2}\sum_{i=1}^{n} x_i^2$$

$$+\sum_{i=1}^{n} Ln\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right] \tag{16}$$

$$\frac{dlnl}{d\theta} = \frac{n}{\theta} - \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n}\frac{\frac{1}{2}x_i^2\,\alpha^2 e^{-\frac{\theta}{2}x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]} = g_1(\theta) \tag{17}$$

$$\frac{dlnl}{d\alpha} = \frac{2n\alpha}{(\alpha^2+1)} - \frac{2n}{\alpha} + \sum_{i=1}^{n}\frac{\theta\alpha\, x_i^2\; e^{-\frac{\theta}{2}x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]} = g_2\,(\alpha) \tag{18}$$

$$\frac{dg_1(\theta)}{d\theta} = -\frac{n}{\theta^2} - \sum_{i=1}^{n}\frac{\frac{1}{4}x_i^4\alpha^4 e^{-\theta x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]^2} \tag{19}$$

$$\frac{dg_1(\theta)}{d\alpha}=\sum_{i=1}^{n}\frac{-\frac{1}{2}\theta\alpha^3x_i^4e^{-\frac{\theta}{2}x_i^2\alpha^2}+\alpha x_i^2e^{-\frac{\theta}{2}x_i^2\alpha^2}-\alpha x_i^2e^{-\theta x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]^2} \qquad (20)$$

$$\frac{dg_2(\alpha)}{d\theta}=\sum_{i=1}^{n}\frac{-\frac{1}{2}\theta\alpha^3x_i^4\,e^{-\frac{\theta}{2}x_i^2\alpha^2}+\alpha x_i^2e^{-\frac{\theta}{2}x_i^2\alpha^2}-\alpha x_i^2e^{-\theta x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]^2} \qquad (21)$$

$$\frac{dg_2(\alpha)}{d\alpha}=\frac{2n(1-\alpha^2)}{(\alpha^2+1)^2}+\frac{2n}{\alpha^2}+\sum_{i=1}^{n}\frac{\theta x_i^2e^{-\frac{\theta}{2}x_i^2\alpha^2}-\theta x_i^2e^{-\theta x_i^2\alpha^2}-\theta^2\alpha^2x_i^4e^{-\frac{\theta}{2}x_i^2\alpha^2}}{\left[1-e^{-\frac{\theta}{2}x_i^2\alpha^2}\right]^2} \qquad (22)$$

The above equations are nonlinear and must use multivariate Newton-Raphson method which is as follow:

$$\begin{bmatrix}\theta_{k+1}\\ \alpha_{k+1}\end{bmatrix}=\begin{bmatrix}\theta_k\\ \alpha_k\end{bmatrix}-j^{-1}\begin{bmatrix}g_1(\theta)\\ g_2(\alpha)\end{bmatrix} \qquad (23)$$

Where the Jacobean matrix is as follows:

$$J=\begin{bmatrix}\frac{dg_1(\theta)}{d\theta} & \frac{dg_1(\theta)}{d\alpha}\\ \frac{dg_2(\alpha)}{d\theta} & \frac{dg_2(\alpha)}{d\alpha}\end{bmatrix} \qquad (24)$$

Where J is Jacobean is a symmetric and square matrix.

Then we find the stopping Rule for getting convergence which is as follows:

$$\left\|\begin{bmatrix}\theta_{k+1}\\ \alpha_{k+1}\end{bmatrix}-\begin{bmatrix}\theta_k\\ \alpha_k\end{bmatrix}\right\|\le\begin{vmatrix}\epsilon_\theta\\ \epsilon_\alpha\end{vmatrix} \qquad (25)$$

3.2. The Ordinary Least Square Method

The least squares method is a statistical procedure to find the best fit for a set of data points by minimizing the sum of the offsets or residuals of points from the plotted curve. Least squares regression is used to predict the behavior of dependent variables.

$$Y_i=B_0+B_1x+\varepsilon \qquad (26)$$

Where: B_0 represents the intercept term

B_1 represents the slop term

ε represents the error term

The idea of this method is to minimize the sum of the squared difference between observed sample values and the estimate expected values by linear approximation:

$$\varepsilon = Y_i - \hat{B}_0 - \hat{B}_1 x \tag{27}$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} [y_i - E(\hat{y})^2]^2$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left[Y_i - \hat{B}_0 - \hat{B}_1 x\right]^2$$

Now, we apply this method to minimize the square difference between the estimate cumulative distribution function and the empirical cumulative distribution function, where the empirical (CDF) is:

$$F(x) = \frac{i-0.5}{n} \tag{28}$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left[\hat{F}(x_i) - F(x_i)\right]^2$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left[1 - \left[\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right] - \frac{i-0.5}{n}\right]^2$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left[1 - \frac{(i-0.5)}{n} - \left[\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]\right]^2$$

$$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n} \left[\frac{n-i+0.5}{n} - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]$$

Let $\quad y_i = \dfrac{n-i+0.5}{n}$ (29)

$$\frac{d\sum_{i=1}^{n} \varepsilon_i^2}{d\theta} = 2\sum_{i=1}^{n} \left[y_i - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]$$

*

$$* \left[0 - \left(\frac{1}{\alpha^2}\left[-\frac{1}{2}x_i^2(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} + \frac{1}{2}x_i^2(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]\right)\right]$$

$$\frac{d\sum_{i=1}^{n}\varepsilon_i^2}{d\theta} = 2\sum_{i=1}^{n}\left[y_i - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right] * \left(-\left[\frac{x_i^2(\alpha^2+1)}{2\alpha^2}\left(-e^{-\frac{\theta}{2}x_i^2} + e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right)\right]\right)$$

$$\frac{d\sum_{i=1}^{n}\varepsilon_i^2}{d\theta} = \sum_{i=1}^{n}\left[y_i - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right] * \frac{x_i^2(\alpha^2+1)}{x^2} * \left[e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]$$

$$\frac{d\sum_{i=1}^{n}\varepsilon_i^2}{d\theta} = \sum_{i=1}^{n}\frac{x_i^2(\alpha^2+1)}{\alpha^2} * \left[y_i - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right] * \left[e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]$$

$$g_1(\theta) = \sum_{i=1}^{n}\frac{x_i^2(\alpha^2+1)}{\alpha^2}\left[y_i - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right] * \left[e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right] \quad (30)$$

$$\frac{d\sum\varepsilon_i^2}{d\alpha} = 2\sum_{i=1}^{n}\left[y_i - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right] * \left(-\left[\left[\frac{1}{\alpha^2}\left[2\alpha e^{-\frac{\theta}{2}x_i^2} + \theta\alpha x^2 e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right] + \left[(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]\right]\left(-\frac{2}{\alpha^3}\right)\right)$$

$$\frac{d\sum\varepsilon_i^2}{d\alpha} = 2\sum_{i=1}^{n}\left[y_i - \frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right] * \left(\frac{2}{\alpha^3}\left[(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2} - e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right] - \frac{1}{\alpha^2}\left[2\alpha e^{-\frac{\theta}{2}x_i^2} + \theta\alpha x_i^2 e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]\right)$$

$$\frac{d\sum\varepsilon_i^2}{d\alpha}=2\sum_{i=1}^{n}\left[y_i-\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]\left[\frac{2}{\alpha^3}\alpha^2e^{-\frac{\theta}{2}x_i^2}+\frac{2}{\alpha^3}e^{-\frac{\theta}{2}x_i^2}-\frac{2}{\alpha^3}e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-\frac{2\alpha^2}{\alpha^3}e^{-\frac{\theta}{2}x_i^2}\right.$$
$$\left.-\frac{\theta\alpha^2x_i^2}{\alpha^3}e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]$$

$$g_2(\alpha)=\sum_{i=1}^{n}\left[y_i-\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]\left[\frac{4e^{-\frac{\theta}{2}x_i^2}-4e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-2\theta\alpha^2x_i^2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^3}\right] \quad (31)$$

$$\frac{dg_1}{d\theta}=\sum_{i=1}^{n}\frac{x_i^4(\alpha^2+1)}{2\alpha^2}\left(\left[y_i-\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]\left[e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-e^{-\frac{\theta}{2}x_i^2}\right]\right.$$
$$\left.+\frac{(\alpha^2+1)}{\alpha^2}\left[e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]^2\right) \quad (32)$$

$$\frac{dg_1}{d\alpha}=\sum_{i=1}^{n}\left[y_i-\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]\left(\frac{\theta\alpha^2(\alpha^2+1)x_i^4e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-2x^2\left[e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]}{\alpha^3}\right)$$
$$+\frac{x_i^2(\alpha^2+1)}{\alpha^2}\left[e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]\left[\frac{2e^{-\frac{\theta}{2}x_i^2}-2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-\theta\alpha^2x^2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^3}\right] \quad (33)$$

$$\frac{dg_2}{d\theta}$$
$$=\sum_{i=1}^{n}\left[y_i-\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]\left(\frac{\theta\alpha^2(\alpha^2+1)x_i^4e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-2x_i^2\left[e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]}{\alpha^3}\right)$$
$$+\frac{x_i^2(\alpha^2+1)}{\alpha^2}\left[\frac{2e^{-\frac{\theta}{2}x_i^2}-2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-\theta\alpha^2x_i^2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^3}\right]\left[e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right] \quad (34)$$

$$\frac{dg_2}{d\alpha}=\sum_{i=1}^{n}\left[y_i-\frac{(\alpha^2+1)e^{-\frac{\theta}{2}x_i^2}-e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^2}\right]$$
$$\left(\frac{2\theta^2\alpha^2x_i^4e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-3\left[4e^{-\frac{\theta}{2}x_i^2}-4e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-2\theta^2\alpha^2x_i^2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}\right]}{\alpha^4}\right) \quad (35)$$
$$+\left[\frac{4e^{-\frac{\theta}{2}x_i^2}-4e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}-2\theta\alpha^2x_i^2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^3}\right]\left[\frac{2e^{-\frac{\theta}{2}x_i^2}-2e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}+\alpha e^{-\frac{\theta}{2}x_i^2(\alpha^2+1)}}{\alpha^3}\right]$$

$$\begin{bmatrix}\theta_{k+1}\\ \alpha_{k+1}\end{bmatrix} = \begin{bmatrix}\theta_k\\ \alpha_k\end{bmatrix} - j^{-1}\begin{bmatrix}g_1(\theta)\\ g_2(\alpha)\end{bmatrix} \tag{36}$$

$$J^{-1} = \begin{bmatrix}\frac{dg_1}{d\theta} & \frac{dg_1}{d\alpha}\\ \frac{dg_2}{d\theta} & \frac{dg_2}{d\alpha}\end{bmatrix}^{-1} \tag{37}$$

The Jacobean matrix is Symmetric and square, where:

$$\frac{dg_1}{d\alpha} = \frac{dg_2}{d\theta}$$

Then we can terminate and stop when:

$$\left\|\begin{bmatrix}\theta_{k+1}\\ \alpha_{k+1}\end{bmatrix} - \begin{bmatrix}\theta_k\\ \alpha_k\end{bmatrix}\right\| \leq \begin{vmatrix}\varepsilon_\theta\\ \varepsilon_\alpha\end{vmatrix} \tag{38}$$

4. Real Data Application

In this section, real data for brain cancer disease is analyzed because it is widespread and deadly in Iraq; the time from infection to death was registered for period from 1/1/2018 to 31/12/2018 at Republic of Iraq / Ministry of Health/Medical City. The data of size (111) patients are considered as complete data set.

X= { 23 10 14 4 14 11 20 15 20 9 7 15 10 16 12 7 11 15 16 13 14 15 12 9 14 14 21 28 16 16 10 5 5 13 17 9 9 9 9 22 10 13 18 22 8 19 20 10 6 12 10 20 5 12 10 26 8 9 21 12 16 12 14 14 19 17 28 6 10 5 20 6 8 11 14 17 9 18 24 9 10 9 10 14 14 8 16 8 8 7 13 11 5 14 24 7 11 15 2 18 10 11 15 20 28 14 19 9 15 7 9 }.

After applying the chi square goodness of fit to test the hypothesis:

$H_0 = \textit{the data is distributed as weighted Rayleigh distribution.}$

$H_1 = \textit{the data not distributed as weighted Rayleigh distribution.}$

Where the chi square goodness of fit statistic depends on the differences of the theoretical frequencies under the assumed distribution and observed differences from data and we applying the formula:

$$X^2 = \sum_{i=1}^{N} \frac{(O_i - E_i)^2}{E_i} \tag{39}$$

We apply the chi square goodness of fit to the data to get:

The calculate chi square test = 4.2963

The calculate chi square test = 35.17

Calculate chi- square < tabulate chi-square.

We accept H_0 which means the data are distributed as weighted Rayleigh distribution.

We apply the equation (23) and (25) in Maximum likelihood method by using Math Lab program version (2014) to find and estimate the values of $\hat{\theta}$ and $\hat{\alpha}$.

Finally, we apply the equation (36) and (38) in Ordinary least square method by using MATHLAB program version (2014) to find and estimate the values of $\hat{\theta}$ and $\widehat{\alpha}$, then we get:

$\hat{\alpha}._{MLE}$ =1.9881, $\hat{\theta}_{MLE}$= 0.0118

$\hat{\alpha}._{OLS}$ =5.9299, $\hat{\theta}_{OLS}$= 0.0218

4.1. Maximum Likelihood Method

The value of estimate parameters $(\hat{\alpha}, \hat{\theta})$ will be (1.9881, 0.0118) respectively. The estimated Probability Density Function, Survival Function, Cumulative Distribution and Hazard Functions will be as illustrated in **Table 1.**

Table 1. Estimate the values of $\hat{f}$(t), $\hat{F}$(t), $\hat{S}$(t), $\hat{h}$(t) for MLE method.

X	$\hat{S}$(t)	$\hat{f}$(t)	$\hat{F}$(t)	$\hat{h}$(t)
2	0.998678	0.002583	0.001322	0.002586
4	0.981538	0.016821	0.018462	0.017137
5	0.959154	0.028277	0.040846	0.029481
5	0.959154	0.028277	0.040846	0.029481
5	0.959154	0.028277	0.040846	0.029481
5	0.959154	0.028277	0.040846	0.029481
5	0.959154	0.028277	0.040846	0.029481
6	0.924601	0.040877	0.075399	0.04421
6	0.924601	0.040877	0.075399	0.04421
6	0.924601	0.040877	0.075399	0.04421
7	0.877585	0.052931	0.122415	0.060314
7	0.877585	0.052931	0.122415	0.060314
7	0.877585	0.052931	0.122415	0.060314
7	0.877585	0.052931	0.122415	0.060314
7	0.877585	0.052931	0.122415	0.060314
8	0.819409	0.062997	0.180591	0.076881
8	0.819409	0.062997	0.180591	0.076881

8	0.819409	0.062997	0.180591	0.076881
8	0.819409	0.062997	0.180591	0.076881
8	0.819409	0.062997	0.180591	0.076881
8	0.819409	0.062997	0.180591	0.076881
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
9	0.752565	0.070164	0.247435	0.093233
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
10	0.680162	0.074106	0.319838	0.108953
11	0.605378	0.074984	0.394622	0.123864
11	0.605378	0.074984	0.394622	0.123864
11	0.605378	0.074984	0.394622	0.123864
11	0.605378	0.074984	0.394622	0.123864
11	0.605378	0.074984	0.394622	0.123864
11	0.605378	0.074984	0.394622	0.123864
12	0.53106	0.073266	0.46894	0.137963
12	0.53106	0.073266	0.46894	0.137963
12	0.53106	0.073266	0.46894	0.137963
12	0.53106	0.073266	0.46894	0.137963
12	0.53106	0.073266	0.46894	0.137963
12	0.53106	0.073266	0.46894	0.137963
13	0.459511	0.06955	0.540489	0.151356
13	0.459511	0.06955	0.540489	0.151356
13	0.459511	0.06955	0.540489	0.151356
13	0.459511	0.06955	0.540489	0.151356
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197
14	0.392426	0.064435	0.607574	0.164197

15	0.330929	0.058456	0.669071	0.176641
15	0.330929	0.058456	0.669071	0.176641
15	0.330929	0.058456	0.669071	0.176641
15	0.330929	0.058456	0.669071	0.176641
15	0.330929	0.058456	0.669071	0.176641
15	0.330929	0.058456	0.669071	0.176641
15	0.330929	0.058456	0.669071	0.176641
16	0.275656	0.05205	0.724344	0.188823
16	0.275656	0.05205	0.724344	0.188823
16	0.275656	0.05205	0.724344	0.188823
16	0.275656	0.05205	0.724344	0.188823
16	0.275656	0.05205	0.724344	0.188823
16	0.275656	0.05205	0.724344	0.188823
17	0.226855	0.045562	0.773145	0.200842
17	0.226855	0.045562	0.773145	0.200842
17	0.226855	0.045562	0.773145	0.200842
18	0.184472	0.03925	0.815528	0.212767
18	0.184472	0.03925	0.815528	0.212767
18	0.184472	0.03925	0.815528	0.212767
19	0.148234	0.033299	0.851766	0.224641
19	0.148234	0.033299	0.851766	0.224641
19	0.148234	0.033299	0.851766	0.224641
20	0.11771	0.027837	0.88229	0.236489
20	0.11771	0.027837	0.88229	0.236489
20	0.11771	0.027837	0.88229	0.236489
20	0.11771	0.027837	0.88229	0.236489
20	0.11771	0.027837	0.88229	0.236489
20	0.11771	0.027837	0.88229	0.236489
21	0.092372	0.022938	0.907628	0.248324
21	0.092372	0.022938	0.907628	0.248324
22	0.071635	0.018636	0.928365	0.260153
22	0.071635	0.018636	0.928365	0.260153
23	0.0549	0.014932	0.9451	0.27198
24	0.04158	0.011801	0.95842	0.283806
24	0.04158	0.011801	0.95842	0.283806
26	0.02302	0.007078	0.97698	0.307456
28	0.012156	0.004025	0.987844	0.331107
28	0.012156	0.004025	0.987844	0.331107
28	0.012156	0.004025	0.987844	0.331107

From **Table 2.** We find that the death density function is increasing with the failure times until (0.074984) when t=11, then the values decreasing with failure times from (0.073266) when t=12 until the end of failure times .We find that the survival function is decreasing with the increasing of failure times .We find that the hazard function is increasing with increasing failure times.

4.2: Ordinary Least square method

The value of estimated $(\hat{\alpha}, \hat{\theta})$ will be (5.9299, 0.0218) respectively and the estimated Probability Density, Survival, Distribution and Hazard Functions will be as illustrated in **Table**

Table 2. Estimate the values of $\hat{f}$(t), $\hat{F}$(t), $\hat{S}$(t), $\hat{h}$(t) for OLS method

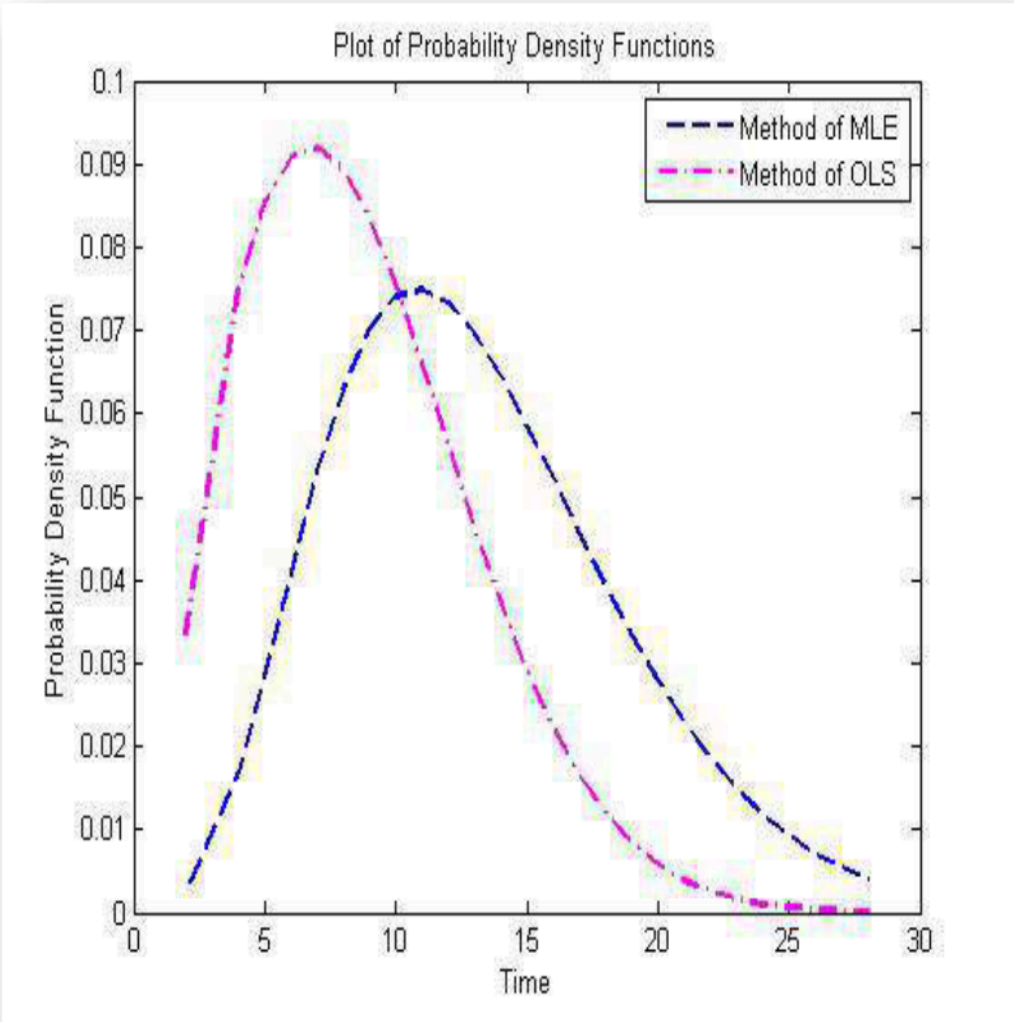
X	$\hat{S}$(t)	$\hat{f}$(t)	$\hat{F}$(t)	$\hat{h}$(t)
2	0.97873	0.033599	0.02127	0.034329
4	0.863998	0.075079	0.136002	0.086898
5	0.783413	0.085272	0.216587	0.108846
5	0.783413	0.085272	0.216587	0.108846
5	0.783413	0.085272	0.216587	0.108846
5	0.783413	0.085272	0.216587	0.108846
5	0.783413	0.085272	0.216587	0.108846
6	0.695008	0.090785	0.304992	0.130624
6	0.695008	0.090785	0.304992	0.130624
6	0.695008	0.090785	0.304992	0.130624
7	0.6033	0.09194	0.3967	0.152395
7	0.6033	0.09194	0.3967	0.152395
7	0.6033	0.09194	0.3967	0.152395
7	0.6033	0.09194	0.3967	0.152395
7	0.6033	0.09194	0.3967	0.152395
8	0.512414	0.089245	0.487586	0.174166
8	0.512414	0.089245	0.487586	0.174166
8	0.512414	0.089245	0.487586	0.174166
8	0.512414	0.089245	0.487586	0.174166
8	0.512414	0.089245	0.487586	0.174166
8	0.512414	0.089245	0.487586	0.174166
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
9	0.425848	0.083439	0.574152	0.195937
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
10	0.346284	0.075389	0.653716	0.217707
11	0.275522	0.065981	0.724478	0.239478
11	0.275522	0.065981	0.724478	0.239478
11	0.275522	0.065981	0.724478	0.239478
11	0.275522	0.065981	0.724478	0.239478
11	0.275522	0.065981	0.724478	0.239478
11	0.275522	0.065981	0.724478	0.239478
12	0.214499	0.056038	0.785501	0.261249

12	0.214499	0.056038	0.785501	0.261249
12	0.214499	0.056038	0.785501	0.261249
12	0.214499	0.056038	0.785501	0.261249
12	0.214499	0.056038	0.785501	0.261249
12	0.214499	0.056038	0.785501	0.261249
13	0.163395	0.046244	0.836605	0.28302
13	0.163395	0.046244	0.836605	0.28302
13	0.163395	0.046244	0.836605	0.28302
13	0.163395	0.046244	0.836605	0.28302
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
14	0.121786	0.037119	0.878214	0.30479
15	0.088818	0.029004	0.911182	0.326561
15	0.088818	0.029004	0.911182	0.326561
15	0.088818	0.029004	0.911182	0.326561
15	0.088818	0.029004	0.911182	0.326561
15	0.088818	0.029004	0.911182	0.326561
15	0.088818	0.029004	0.911182	0.326561
15	0.088818	0.029004	0.911182	0.326561
16	0.06338	0.022077	0.93662	0.348332
16	0.06338	0.022077	0.93662	0.348332
16	0.06338	0.022077	0.93662	0.348332
16	0.06338	0.022077	0.93662	0.348332
16	0.06338	0.022077	0.93662	0.348332
16	0.06338	0.022077	0.93662	0.348332
17	0.044253	0.016378	0.955747	0.370102
17	0.044253	0.016378	0.955747	0.370102
17	0.044253	0.016378	0.955747	0.370102
18	0.030233	0.011848	0.969767	0.391873
18	0.030233	0.011848	0.969767	0.391873
18	0.030233	0.011848	0.969767	0.391873
19	0.02021	0.00836	0.97979	0.413644
19	0.02021	0.00836	0.97979	0.413644
19	0.02021	0.00836	0.97979	0.413644
20	0.013219	0.005756	0.986781	0.435415
20	0.013219	0.005756	0.986781	0.435415
20	0.013219	0.005756	0.986781	0.435415
20	0.013219	0.005756	0.986781	0.435415
20	0.013219	0.005756	0.986781	0.435415
20	0.013219	0.005756	0.986781	0.435415
21	0.00846	0.003868	0.99154	0.457185
21	0.00846	0.003868	0.99154	0.457185
22	0.005298	0.002537	0.994702	0.478956
22	0.005298	0.002537	0.994702	0.478956
23	0.003246	0.001625	0.996754	0.500727
24	0.001946	0.001017	0.998054	0.522498
24	0.001946	0.001017	0.998054	0.522498
26	0.000655	0.000371	0.999345	0.566039

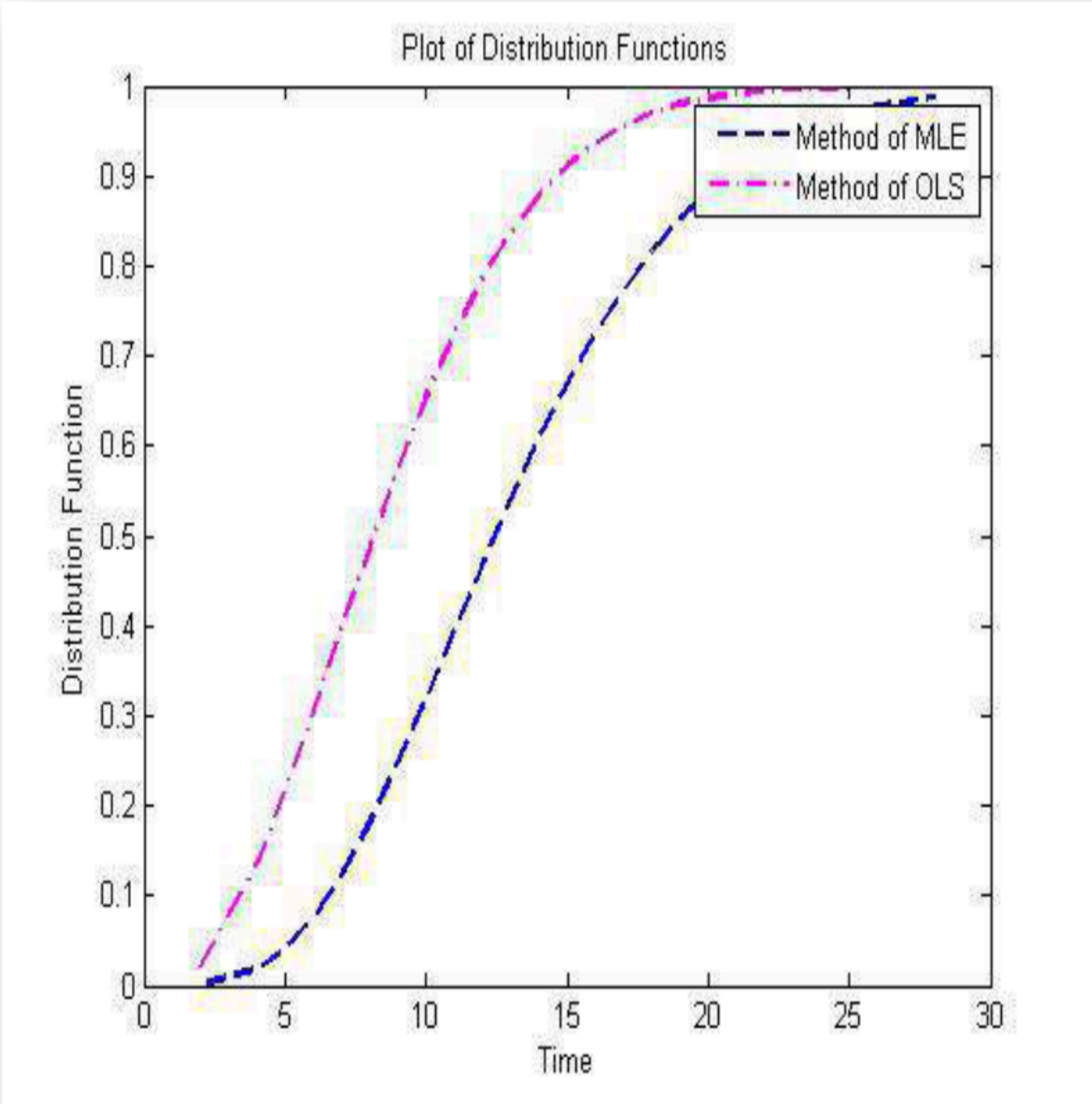
28	0.000202	0.000123	0.999798	0.609581
28	0.000202	0.000123	0.999798	0.609581
28	0.000202	0.000123	0.999798	0.609581

From table (2), we find that the death density function is increasing with the failure times until (0.09194) when t=7, then decreasing with failure times from (0.089245) when t=8 until the end of failure times .We find that the survival function is decreasing with the increasing of failure times. We find that the hazard function is increasing with the increasing failure times.

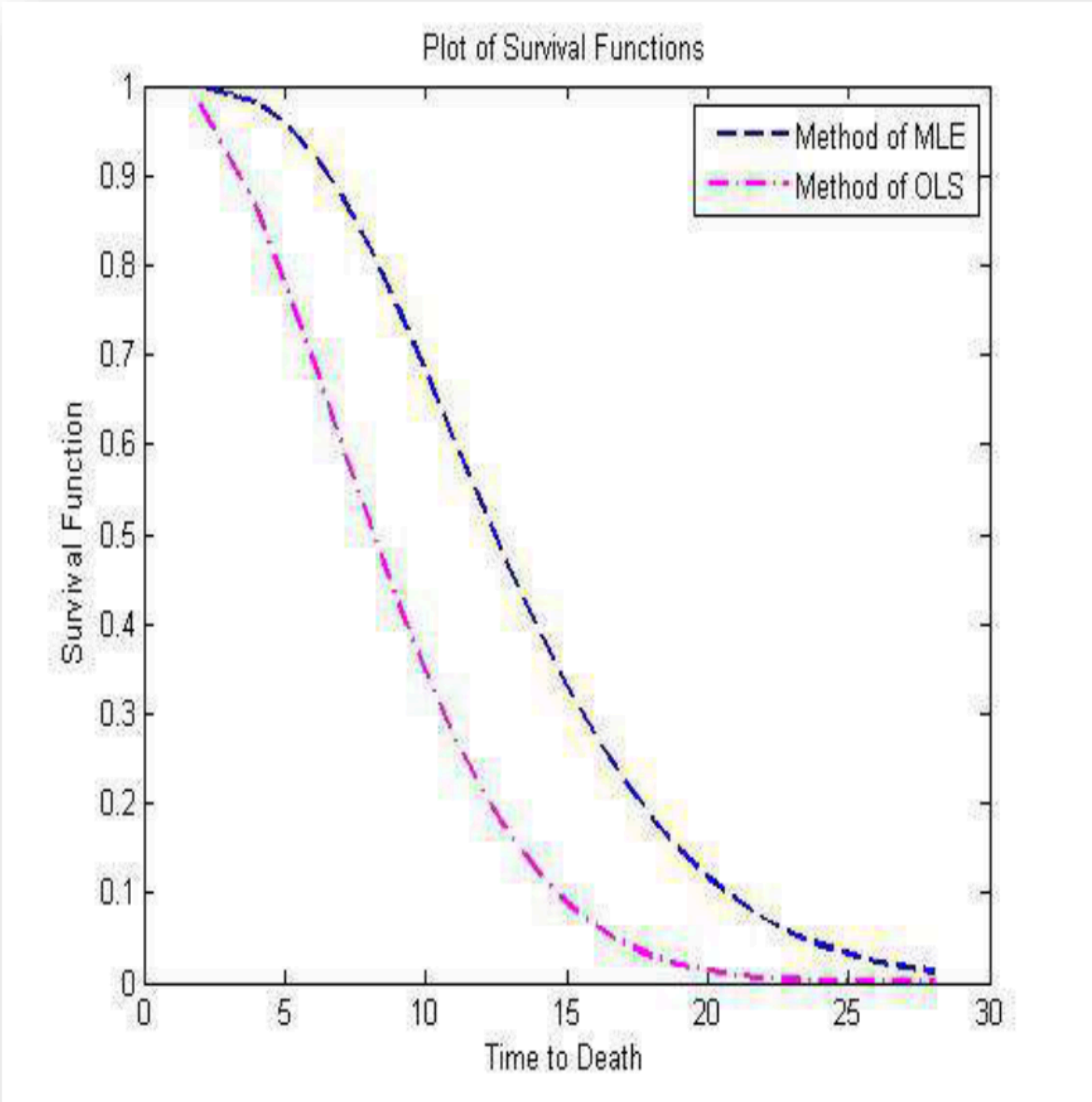
$\hat{\alpha}_{MLE} = 1.9881, \hat{\theta}_{MLE} = 0.0118, \hat{\alpha}_{OLS} = 5.9299, \hat{\theta}_{OLS} = 0.0218$



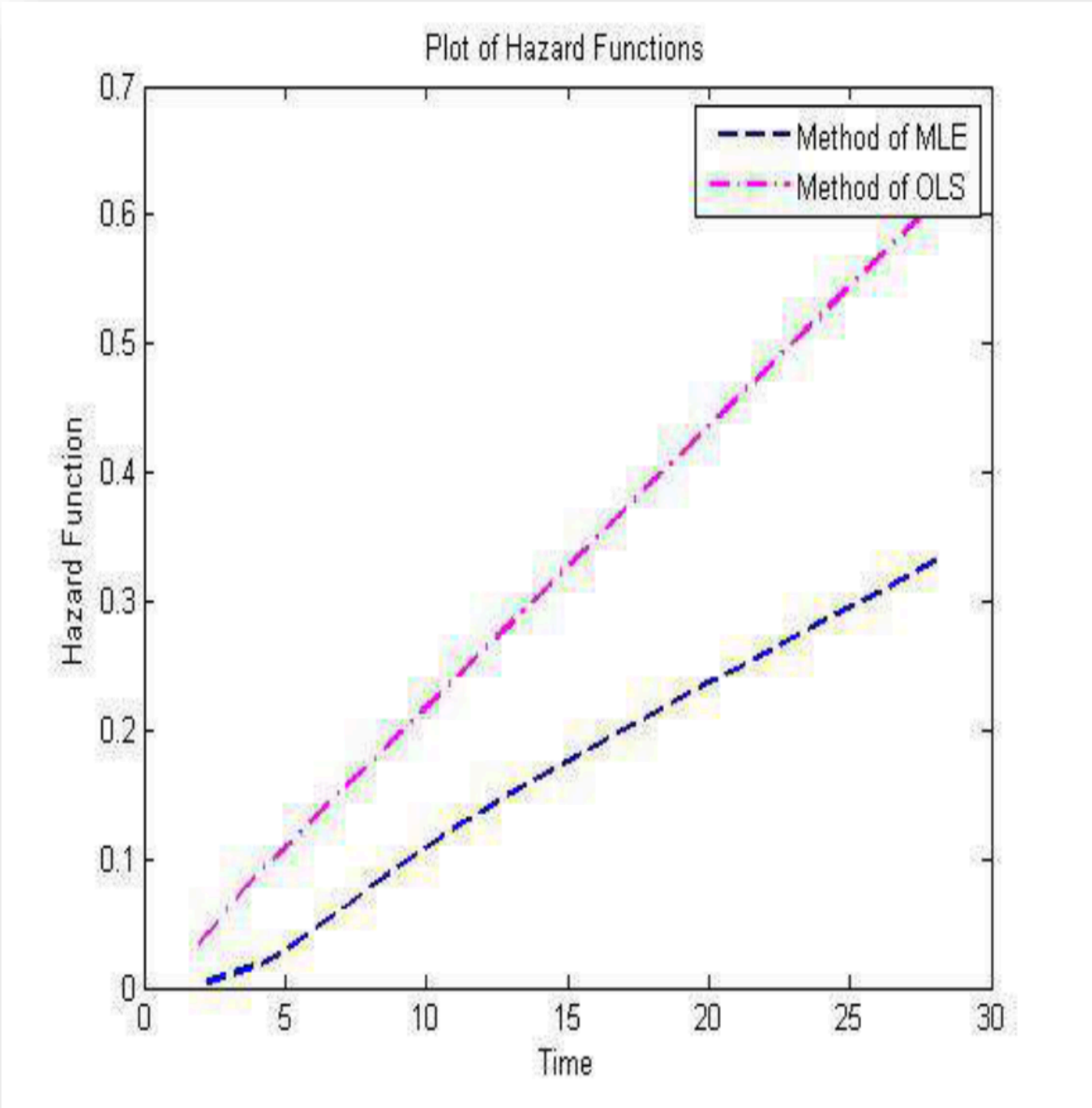
$\hat{\alpha}_{MLE} = 1.9881, \hat{\theta}_{MLE} = 0.0118, \hat{\alpha}_{OLS} = 5.9299, \hat{\theta}_{OLS} = 0.0218$



$\hat{\alpha}_{MLE} = 1.9881, \hat{\theta}_{MLE} = 0.0118, \hat{\alpha}_{OLS} = 5.9299, \hat{\theta}_{OLS} = 0.0218$



$\hat{\alpha}_{MLE} = 1.9881, \hat{\theta}_{MLE} = 0.0118,\ \hat{\alpha}_{OLS} = 5.9299, \hat{\theta}_{OLS} = 0.0218$



5 .Conclusion

1. In the two estimation methods above, the probability estimation values of the survival function are decreasing with the increase in failure time values. This shows that there is an inverse relationship between the probability survival function and the failure times.

2. In the two estimation methods, we can watch the estimation value of the probability hazard function is increasing with the increase in failure times, and then it is very clear that this relationship is direct between the times of failure and the probability hazard function.

3. From our reading of the values of the estimated probability density function of the two methods used are increasing values until the time of failure ($x = 11$) in MLE and ($x=7$)in OLS, then the probability density function decreases with increasing failure times. This shows us that the relationship between the probability density function and the failure times is not constant i.e. according to the values of failure times.

4. That clear the values of the cumulative distribution function are increasing with increasing the values of failure times.

5. Observing that the estimation values of the parameters which are the shape parameter and the scale parameter in the weighted Rayleigh distribution. It is clear that for the $\hat{\alpha}$ values is one close to the other in the MLE, OLS method but on the other hand, the value of $\hat{\theta}$ is close to one unit of the other.
In the table below, the estimate values of the two parameters for weighted Rayleigh distribution by MLE and OLS can be show in following **Table:**

The methods	$\hat{\alpha}$	$\hat{\theta}$
MLE	1.9881	0.0118
OLS	5.9299	0.0218

Table 3. Estimate the values of $\hat{\alpha}, \hat{\theta}$ by MLE and OLS methods

6. Note that, as is known, the values of the probability hazard function depend on the value of the shape parameter, thus the probability hazard function is an increasing such that $(\alpha > 1)$ for the two estimation methods.
9. The mean squares error values for the MLE, OLS methods by using the equation is:

$$MSE[\hat{s}(\mathrm{t})] = \frac{\sum_{i=1}^{n}[\hat{s}(\mathrm{t})-\mathrm{s}(\mathrm{t})]^2}{n} \quad (40)$$

We find the MSE for MLE and OLS and there is a clear difference and eventually. The MLE method is better than OLS as below:

MSE	
OLS	**MLE**
0.0626	**0.0006**

References

1. Azzalini, A. A Class of Distributions which includes. The normal ones, *Board of the foundation of the Scandinavian journal of statistics*.**1985**, *12*, *2*, 171-178.
2. Gupta, R.D.; Kundu, D. A new class of weighted exponential distributions, *A journal of the article and Applied statistics*.**2009**, 43, *6*, 621-634.
3. Shahbaz, S.; Shah76ؼbaz, M.Q.; Butt, N.S. A class of weighted Weibull Distribution, *Scorn Electronic Journal*.**2010**, *6*, *1*, 53-59.
4. Ramadan, M. M A class of weighed Gamma Distributions its properties, *Economic Quality Control*.**2010**, *26*, *2*, 133-144.
5. Essam, A.A.; Mohamed, A.H. A weighted three –parameters weibull distribution, *Journal of Applied Sciences Research*.**2014**, *9*, *13*, 6627-6635.
6. Farahani, Z.S.M.; Khorram, E. Bayesian statistical inference for weighted Exponential Distribution, *Taylor and Francis group. LLc*.**2014**, *43*, *6*, 1384-1462.
7. Tawfiq, LNM.; Jabber, AK. Mathematical Modeling of Groundwater Flow. *Global Journal of Engineering Science and Researches*. **2016**, *3, 10*, 15-22.

8. Oguntunde, p. E. On the Exponentiated Weighted Exponential Distribution and its Basic statistical properties. *PSCI Publication*.**2015**, *10*, *3*, 160-167.
9. Zahrani, B. on the estimation of Reliability of Weighted weibull Distri- bution: A comparative study, *international journal of statistics and probability*.**2016**, *5*, *4*, 1427-7040.
10. Al-Noor, N.H.; Hussein, L.K. Weighted Exponential distribution: Approximate Bayes Estimations with Fuzzy Data, *scientific international conference college of science*, *Al-Nahrain Unevercity.2017*, *0*, *1, 174*-185.
11. Oguntunde, P.E.; Ilori, K.A.; Okagbue, H. I. The inverted weighted exponential distribution with applications, *interracial journal of Advanced and Applied Sciences*.**2018**, *5*, *11*, 46-50.
12. Tawfiq, LNM; Oraibi. YA. Fast Training Algorithms for Feed Forward Neural Networks. *Ibn Al-Haitham Journal for Pure and Applied Science*. **2017**, *26, 1*: 1275-280.
13. Hussein. Iden Hasan and Zain. Saad Adnan**.** Some Aspects of Weighted Rayleigh Distribution. *Ibn Al- Haitham Journal for Pure and Applied Science*. Date of acceptance of publication in the above journal. **2019**, ***33***, that will out in 2020.