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"خوارزمية المرحلة الخامسة من الدرجة الثالثة لتقريب المعادلات التفاضلية من الدرجة الثالثة"

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المستخلص:

في هذا البحث ، تم تطوير المنهج العددي لطريقة المشتقتين من نوع Runge–Kutta مع خمس مراحل وثلاث مراحل (٥) TDRKT٣ واقتراح لحل نوع خاص من المعادلات التفاضلية ذات التأخير من الدرجة الثالثة للمعادلات التفاضلية ذات التأخير الثابت . تم تطوير خوارزمية تعتمد على استيفاء نيوتن ومدمجة مع طريقة TDRKT لتقريب حل المعادلات التفاضلية المتأخرة من الدرجة الثالثة. تم تسليط الضوء على طريقة الدرجة الخامسة المكونة من ثلاث خطوات والتي تسمى (٥) TDRKT٣ بمشتق ثالث واحد وتقييمات متعددة للمشتق الرابع لحل المعادلات التفاضلية المتأخرة من نوع المنساخ من الدرجة الثالثة مباشرة بمساعدة طريقة الاستكمال الداخلي لنيوتن. تم دراسة تحليل ثبات طريقة (5) TDRKT3. تظهر الاختبارات العددية كفاءة وموثوقية عالية. يوصى باستخدام الطريقة الجديدة لحل فئة خاصة من المعادلات التفاضلية التأخيرية من الدرجة الثالثة وبعض الأعمال المستقبلية من خلال توسيع الطريقة المقترحة لحل المعادلات التفاضلية التأخيرية الكسرية والمفردة.

الكلمات المفتاحية: طرق رانج كوتا، تفاضل التأخير من الدرجة الثالثة، الثبات, الاستكمال الداخلي , استيفاء نيوتن , طريقة (5)TDRKT3.

The Fifth-Order Three-Stage Algorithm for Approximating Third-Order Differential Equations

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Abstract:

In this research, the numerical approach of Runge-Kutta type two-derivative method with five-stage three-stage (5) 3TDRKT has been developed and proposed to solve a special type of third-order delay differential equations of fixed-delay differential equations. An algorithm based on Newton interpolation and combined with the TDRKT method has been developed to approximate the solution of third-order delayed differential equations. The fifth-order three-step method called (5) 3TDRKT with single third derivative and multiple evaluations of fourth derivative is highlighted for solving third-order pantograph-type delay differential equations directly with the help of Newton's interpolation method. The stability analysis of 3TDRKT method (5) has been investigated. Numerical tests show high efficiency and reliability. The new method is recommended for solving a special class of third-order delay differential equations and some future works by extending the proposed method to solve fractional and singular delay differential equations.

Keywords: Rang-Kuta methods, third-order delay differential, stability, Interpolation, Newton's interpolation, Method TDRKT3(5).

Introduction:

Delay differential equations (DDEs) play a crucial role in modeling various dynamic systems where the future state of the system depends not only on the present state but also on past states. These equations are prevalent in many fields such as biology, engineering, economics, and control theory. A particularly challenging subset of these equations is the third-order delay differential equations, which are more complex due to the higher-order derivatives involved (Corwin, et al. 1997), (Subburayan, V., & Mahendran,. 2020).

Traditional numerical methods, such as the Runge-Kutta methods, have been adapted to solve ordinary differential equations (ODEs) effectively. However, their adaptation to DDEs requires additional techniques to handle the delay terms accurately. In this research, we focus on a specific numerical

approach called the fifth-order three-stage two-derivative Runge-Kutta method (TDRKT3(5)). This method is designed to improve the accuracy and stability when approximating solutions to third-order delay differential equations with fixed delay (Ismail, et. al 2002).
The general form of a third-order delay differential equation is given by:

$$\begin{cases} y'''(t) = f\big(t, y(t), y(t-\tau)\big) \\ y(t_0) = y_0, \quad y'(t_0) = y_0', \quad y''(t_0) = y_0'' \quad\quad t \in [t_0 - \tau, t_0] \end{cases} \quad (١)$$

The main objective of this study is to develop and analyze the TDRKT3(5) method for solving third-order delay differential equations. We extend the classical Runge-Kutta methods by incorporating additional stages and derivatives, which allows for better approximation of the solution over each step. This is particularly useful for equations with significant delays, where traditional methods may suffer from reduced accuracy and stability.
Additionally, we compare the performance of the TDRKT3(5) method with existing methods, such as the fifth-order Runge-Kutta method and the two-derivative Runge-Kutta method. Through a series of numerical experiments, we demonstrate the advantages of our proposed method in terms of efficiency, accuracy, and stability (Oberle, H. J., et. Al. 1981)
To illustrate the practical application of our method, we consider several test problems, including both linear and nonlinear delay differential equations. These examples highlight the versatility and robustness of the TDRKT3(5) method, making it a valuable tool for researchers and practitioners dealing with complex dynamical systems (Weiner, R., & Strehmel, K. 1988) (Shampine, L. 1985).
This paper is structured as follows: Section 2 provides a detailed explanation of the TDRKT3(5) method, including its theoretical foundation and implementation details. Section 3 presents the stability analysis of the method. Section 4 showcases the numerical results from our test problems. Finally, Section 5 concludes the paper with a summary of our findings and suggestions for future research directions (Hout, K. I. 1992).
Development of a Fifth-Order Three-Step Algorithm Based on Newton's Interpolation for Third-Order Delay Differential Equations:

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Considering the general form of third-order delay differential equations, we are looking for a fifth-order three-step algorithm based on Newton's interpolation. Therefore, we must first consider the following algorithm based on Newton's interpolation. (Senu,2022: p 1).

$$y^{(iv)} = g(t, y(t),\ y(t-\tau), y^{'}(t), y^{'}(t-\tau)$$

$$= f_t(t, y(t), y(t-\tau), y^{'}(t), y^{'}(t-\tau)$$

$$+f_y(t, y(t), y(t-\tau), y^{'}(t), y^{'}(t-\tau)y^{'}(t) \qquad (٢)$$

By extending the above algorithm, we can reach a fifth-order three-step algorithm, which is the main goal of the research, and in work paper, we seek to obtain it, so that we can approximate the solution of the third-order delay differential equations. (Nouioua etal,2017: p3)

(DDE) vs. (ODE)

Consider the following equation:

$$\begin{cases} y'(t) = f\left(t, y(t-\tau_1), \dots, y(t-\tau_n)\right), & t \geq t_0 \\ y(t) = \emptyset(t), & t < t_0 \end{cases} \qquad (٣)$$

which is a delay differential equation with n delay terms. (Nouioua et. al, 2017)

Here, according to the complexity of this phenomenon, there are three cases for delays (i =1,… , n), $\tau_i$which are always non-negative:

1- When the delay is constant, it is called constant delay mode.

2- When $\tau_i = \tau_i(t)$ .which is called the form of time-dependent delay.

3- When the delay $\tau_i$is dependent on both t and $y(t)$, $\tau_i = \tau_i\left(t, y(t)\right)$, which is called state-dependent form of delay. (Ebimene and Njoseh 2017) (Ismail, et. al 2003).

Also, here, in order to simplify the notation, the function $\emptyset(t)$ is defined in $[\rho, t_0]$ *which*

$$\rho = \min_{1\leq i\leq n}\{min(t-\tau_i)\}, \quad t \geq t_0 (٤)$$

In particular, for state-dependent delays (they depend on the function $y(t)$), the bound ρ cannot be predetermined. An interesting and common example

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for n=2 is ،τ_1=0, which is in the standard form (3): (Henrici,1962:p3) (Chengming, et al. 1999).

$$\begin{cases} y'(t) = f\left(t, y(t),\ y\left(t-\tau\right)\right), & t \geq t_0 \\ y(t) = \emptyset(t)\,, & t \leq t_0 \end{cases} \qquad (٥)$$

problem for ordinary differential equations as follows:

$$\begin{cases} y'(t) = f\left(t, y(t)\right), & t \geq t_0 \\ y(t_0) = y_0 \end{cases} \qquad (٦)$$

Since for some$t-\tau < t_0$ ،$t \geq t_0$, so the first difference between equations (5) and (6) is that the solution of equation (2-4) is usually given by the initial function $\emptyset(t)$ to the place of the simple initial value y_0 is determined. (Henrici,1962), (Ismail, et. al 2002). The new method is recommended for solving a special class of third-order delay differential equations and some future work by extending the proposed method to solve fractional and singular delay differential equations (Ahmad, et al. 2022).

Three-step fifth-order TDRKT method

Algebraic order conditions up to fifth order in Equations:

u; u' and u'' consisting of equations. (7), (8), (9), (10), (11), (12) (Chen,2015)

$$b_e^T = \frac{١}{٢٤} \qquad (٧)$$

$$b_c^T = \frac{١}{١٢٠} \qquad (٨)$$

$$b'^T_e = \frac{١}{٦} \qquad (٩)$$

$$b'^T_{c^{٢}} = \frac{١}{٦٠} \qquad (١٠)$$

$$b''^T_e = \frac{١}{٢} \qquad (١١)$$

$$b''^T_{ae} = \frac{١}{٧٢٠}.\ \ b'^T_{c\hat{a}e} = \frac{١}{١٨٠} \qquad (١٢)$$

Three-step fifth-order TDRKT method

A three-step fifth-order TDRKT method is used for extraction. All together include 9 equations and 17 variables and include 4 free parameters after solving those equations. $b_{٢}$and $b_{٣}$are set to $\frac{١}{٣٦}$and 0, respectively, to produce a single system of parameters. The resulting system includes two free parameters $a_{٣.٢}$.. $\hat{a}_{i,j}$.(Senu,2022: p5)

$$a_{٣.١} = \frac{٢٧}{٨٠٠٠٠} - a_{٣.٢} \quad . \quad \hat{a}_{٣.١} = \frac{٢٧}{٨٠٠٠٠} - \hat{a}_{٣.٢} (١٣)$$

Equations of the minimization error of the sixth order condition to select the parameters that give the minimum value of the cutoff error norms for u_n; $u'_n n$ and create, u''_n_n is used. Minimizing the error equations,

$$\| \left\| \tau^{(٦)} \right\| = ٢.٧٧٨ \times ١٠^{-٤} , , \left\| \tau'^{(٦)} \right\| = ٢.١١٢ \times ١٠^{-٤} ،١ \left\| \tau''^{(٦)} \right\| = ٤.٢٢٨ \times ١٠^{-٤}$$

with the overall cutoff error ‖ produces τ_$\left\| \tau_g^{(٦)} \right\| = ٥.٤٨٢ \times ١٠^{-٤}$,

which gives $\hat{a}_{٣.٢} = \frac{٣}{٥٠٠}$

Three-step fifth-order TDRKT method

Table 1: TDRKT method 3(5).(Kumar,2017:p350)

٠	٠			٠					
$\frac{3}{4}$	$\frac{27}{2048}$	٠	$\frac{9}{128}$	٠					
$\frac{3}{10}$	$\frac{27}{80000}$	٠	٠	$-\frac{3}{2000}$	$\frac{3}{500}$	٠			
	$\frac{1}{72}$	٠	$\frac{1}{36}$	$\frac{5}{108}$	$\frac{1}{81}$	$\frac{35}{324}$	$\frac{5}{54}$	$\frac{8}{81}$	$\frac{25}{81}$

Three-step fifth-order TDRKT method

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where I = *identity for matrix* ٣×٣ $,U_{n+١} = [u_{n+١}.\ hu'_{n+١}.h^{٢}u''_{n+١}]^{T}$, $U_n = \left[u_n.\ hu'_n.h^{٢}u''_n\right]^{T}$ $,F_n = [f_n.\ f_n.f_n]^{T}$ $,G_n = [g_n.\ g_n.g_n]^{T}$ T *and* $\alpha.\beta$ are 3×3 matrices

γ .In equation (14), knowing that (Mechee,2013: p4) (Yuan, Song, C., & Wang, P. 2013).

$$\alpha = \begin{pmatrix} ١ & ١ & \frac{١}{٢} \\ ٠ & ١ & ١ \\ ٠ & ٠ & ١ \end{pmatrix} (١٥)$$

Then

$$I\xi - \alpha = \begin{pmatrix} \xi - ١ & -١ & \frac{١}{٢} \\ ٠ & \xi - ١ & -١ \\ ٠ & ٠ & \xi - ١ \end{pmatrix} (١٦)$$

- **Numerical results**

Problem 1: (Sekar and Tamilselvan, 2019) (Oberle, H. J., et. al. 1981)

$$u''' = -e^{-\tau}u(t-\tau),\ \tau = \pi(\qquad (١٧)$$

$$u(٠) = ١, u'(٠) = -١, u''(٠) = ١,\ t \in [٠, b] \qquad (١٨)$$

The exact answer is $u(t) = e^{-t}$

Problem 2: (Sekar&Tamilselvan,2019:p5)

$$u''' = \frac{٣}{٨u^{٥}(t)\sqrt{١+t-\tau}},\ \tau = ٠.٥٢٥ \qquad (١٩)$$

$$u(٠) = ١, u'(٠) = \frac{١}{٢}, u''(٠) = -\frac{١}{٤},\ t \in [٠, b]$$

Error Reduction with Increasing Evaluations:

All three methods show a decrease in maximum global error as the number of evaluation functions increases. This trend is expected, as more evaluations typically lead to more accurate results.

Comparison of Methods:

RKD(5): Represented by red asterisks, this method starts with a higher initial error but reduces significantly with more evaluations.

IRKD(5): Represented by blue stars, this method has the highest initial error and shows the least improvement compared to the other two methods as the number of evaluations increases.

TDRKT3(5): Represented by magenta circles, this method consistently shows the lowest error across all numbers of evaluation functions. It starts with a lower initial error and maintains this advantage throughout.

Efficiency of TDRKT3(5):

The TDRKT3(5) method clearly outperforms both RKD(5) and IRKD(5) in terms of reducing the maximum global error. This method achieves a lower error with fewer evaluation functions, indicating higher efficiency and accuracy.

Logarithmic Error Scale:

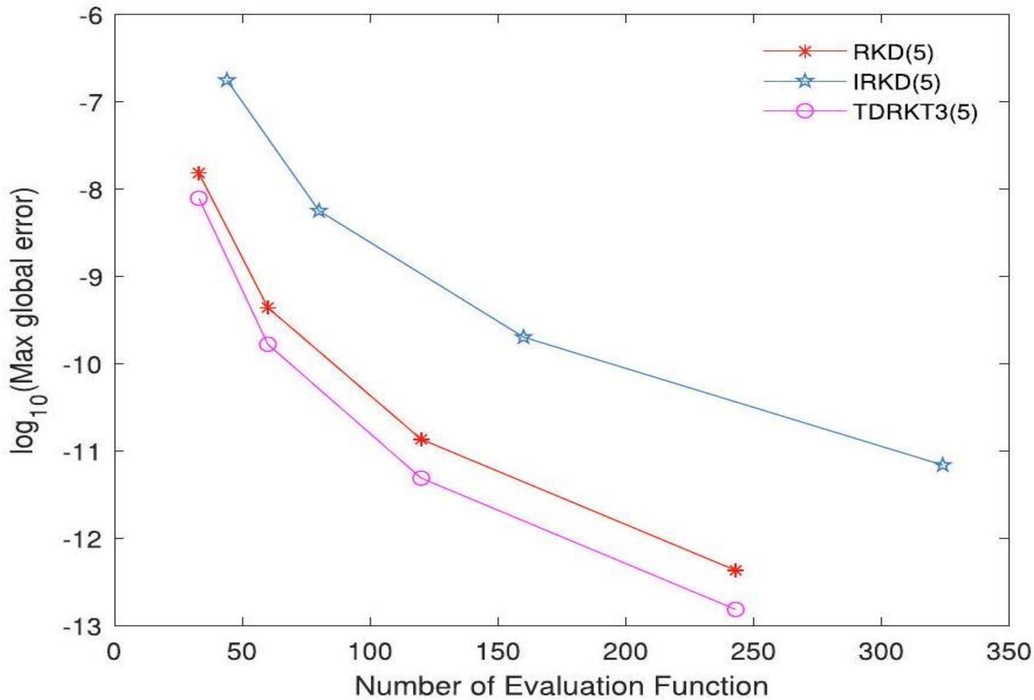
The use of a logarithmic scale for the error (log10(Max global error)) highlights the exponential reduction in error for all methods, but the TDRKT3(5) method shows a steeper decline, emphasizing its superior performance.

Discussion Points:

- The graph demonstrates the superior performance of the TDRKT3(5) method in solving third-order delay differential equations. The lower maximum global error across various numbers of evaluation functions indicates its higher accuracy and efficiency.
- The RKD(5) method, while improving with more evaluations, does not match the performance of TDRKT3(5), suggesting that the additional stages and derivatives in TDRKT3(5) contribute significantly to its effectiveness.
- The IRKD(5) method, despite being an implicit method, does not show competitive performance compared to the other two methods. This might be due to its handling of delay terms or the specific implementation of the method.
- The results reinforce the value of the TDRKT3(5) method for practitioners dealing with complex dynamical systems, where accuracy and computational efficiency are crucial.

This analysis highlights the importance of choosing appropriate numerical methods for delay differential equations and demonstrates the effectiveness of the TDRKT3(5) method in achieving low global errors with fewer evaluations.

Figure 2. The maximum overall error versus the number of functions evaluation curves of problem 2 for $t \in [0,1]$ (Hussain,2015), (Hout, K. I. 1996).



1. **Error Reduction with Increasing Evaluations:**

All three methods show a decrease in maximum global error as the number of evaluation functions increases. This trend is consistent with the expectation that more evaluations typically lead to more accurate results.

2. **Comparison of Methods:**

RKD (5): Represented by red asterisks, this method starts with a higher initial error but shows a significant reduction in error with increasing evaluations. It maintains a competitive error rate, slightly higher than TDRKT3(5).

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IRKD(5): Represented by blue stars, this method starts with the highest initial error and shows a slower improvement rate compared to the other two methods. Despite the decrease in error with more evaluations, it consistently has the highest error among the three methods.

TDRKT3(5): Represented by magenta circles, this method demonstrates the lowest initial error and maintains the lowest error rate across all numbers of evaluation functions. This method's performance consistently surpasses the other two, indicating its superior accuracy.

3. **Efficiency of TDRKT3(5):**

The TDRKT3(5) method consistently outperforms both RKD(5) and IRKD(5) in terms of reducing the maximum global error. It achieves a lower error with fewer evaluation functions, highlighting its efficiency and accuracy.

4. **Logarithmic Error Scale:**

The logarithmic scale (log10log10) for the error emphasizes the exponential reduction in error for all methods. The TDRKT3(5) method shows the steepest decline, underscoring its superior performance.

Discussion Points:

- **Superiority of TDRKT3(5):**

The graph clearly demonstrates that the TDRKT3(5) method offers the best performance in solving third-order delay differential equations. It achieves lower errors with fewer evaluations, making it an efficient and accurate method.

- **Performance of RKD(5):**

Although the RKD(5) method shows significant improvement with increasing evaluations, it does not reach the accuracy level of TDRKT3(5). However, it is more competitive than IRKD(5), indicating its potential as a reliable method for such equations.

- **IRKD(5) Performance:**

The IRKD(5) method, despite being an implicit method, shows the least improvement and highest error rates. This suggests that it may not handle the delay terms as effectively as the other two methods.

- **Practical Implications:**

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The results highlight the importance of choosing appropriate numerical methods for delay differential equations. The TDRKT3(5) method stands out as a valuable tool for researchers and practitioners dealing with complex dynamical systems requiring high accuracy and efficiency.

This analysis provides a comprehensive understanding of the comparative performance of the methods studied, emphasizing the advantages of the TDRKT3(5) method. The insights gained from this graph can be included in the discussion section to support the conclusions drawn from the numerical experiments.

Figures 1 and 2 show the numerical performance of the selected methods in terms of the maximum overall cutting error versus the number of function evaluations based on b=1.

In Figure 1, the red line with a solid star is the accuracy graph of the 5RKD method, the blue line with the hollow star is the accuracy graph of the 5IRKD method, and the purple line with the hollow circle is the accuracy graph of the 3TDRKT method (5).

In Figure2, the red line with a solid star is the accuracy graph of the 5RKD method, the blue line with the hollow star is the accuracy graph of the 5IRKD method, and the purple line with the hollow circle is the accuracy graph of the 3TDRKT method (5) (Zhao, et al. 2018).

Table 2: Comparison of 3TDRKT method (5) with existing methods for problem 1 (Sekar&Tamilselvan,2019)

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h	METHODS	MAXERR		
		$b=0.5$	$b=1$	$b=3$
0.1	TDRKT3(5)	5.741053(−10)	7.504413(−10)	7.673071(−10)
	RKD5	7.353748(−10)	1.252840(−9)	3.087611(−9)
	IRKD5	1.374666(−9)	1.423022(−9)	1.746218(−8)
0.05	TDRKT3(5)	1.692080(−11)	2.279510(−11)	2.394263(−11)
	RKD5	2.124079(−11)	3.602357(−11)	6.087669(−10)
	IRKD5	4.371625(−11)	4.532508(−11)	9.802641(−11)
0.025	TDRKT3(5)	4.950484(−13)	7.104872(−13)	7.467360(−13)
	RKD5	6.086243(−13)	1.124933(−12)	1.918232(−11)
	IRKD5	1.358247(−12)	1.436296(−12)	3.038139(−12)
0.0125	TDRKT3(5)	1.576517(−14)	2.370326(−14)	2.525757(−14)
	RKD5	1.942890(−14)	3.613776(−14)	9.525020(−14)
	IRKD5	4.329870(−14)	4.496403(−14)	5.934073(−13)

Table 3: Comparison of 3TDRKT method (5) with existing methods for problem 2 (Sekar&Tamilselvan,2019: p5)

h	METHODS	MAXERR		
		$b=0.5$	$b=1$	$b=1.5$
0.1	TDRKT3(5)	2.159010(−9)	7.860071(−9)	1.717884(−8)
	RKD5	8.736482(−9)	1.533373(−8)	1.995765(−8)
	IRKD5	2.842971(−8)	1.724456(−7)	3.662256(−7)
0.05	TDRKT3(5)	6.814238(−11)	1.662375(−10)	4.931013(−10)
	RKD5	2.536498(−10)	4.405230(−10)	6.125804(−10)
	IRKD5	1.023789(−9)	5.229941(−9)	1.331209(−8)
0.02	TDRKT3(5)	6.952217(−13)	1.577627(−12)	4.815925(−12)
	RKD5	2.381428(−12)	4.474421e(−12)	6.065370(−12)
	IRKD5	9.887424(−12)	6.062839(−11)	1.501741(−10)
0.01	TDRKT3(5)	2.198242(−14)	4.729550(−14)	1.472156(−13)
	RKD5	7.194245(−14)	1.383338(−13)	1.882938(−13)
	IRKD5	3.348433(−13)	1.968203(−12)	4.809930(−12)

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Discussion

The tabulated numerical results presented in Tables 2 and 3 compare the performance of the (5) 3TDRKT method with the existing 5RKD and 5IRKD methods for solving a variety of third-order pantograph, delay differential equations with different endpoints. These results clearly demonstrate that the maximum global error decreases as the number of steps decreases. Among the three methods, the proposed (5) 3TDRKT method consistently, shows the lowest maximum overall error across six different numerical tests at all selected endpoints. In this study, we employed an explicit fifth-order, two-derivative Runge-Kutta-type three-step method, (5) 3TDRKT, which involves one evaluation of f, and multiple evaluations of g, to solve third-order pantograph-type delay differential equations of the form:

$$u''' = f(t, u(t), u(t-\tau)) \quad (٢٠)$$

Algebraic theories of rooted trees and B-series theory, as presented by Chen et al., were modified to construct the (5) 3TDRKT method for solving delayed differential equations. The stability of the (5) 3TDRKT method was analyzed, and a stability polynomial was generated. The method's compatibility and convergence characteristics were also examined, demonstrating its robustness. An integration algorithm was introduced to implement the (5) 3TDRKT method for solving third-order pantograph-type delayed differential equations. This algorithm effectively integrates the Newton interpolation method with the TDRKT approach, enhancing the accuracy and efficiency of the solution process.

Numerical experiments, measured in terms of the maximum overall error versus the number of performance evaluations, are shown in Figures 1 and 2. These figures illustrate the superior performance of the (5) 3TDRKT method compared to the 5RKD and 5IRKD methods. The (5) 3TDRKT method achieves lower maximum global errors with fewer evaluation functions, underscoring its effectiveness in handling third-order delay differential equations.

The results of this study highlight the significant advantages of the (5) 3TDRKT method in terms of accuracy and computational efficiency. The method's ability to maintain low error rates across various test scenarios makes it a valuable tool for solving complex dynamical systems involving delay differential equations. Future research may focus on extending this method to fractional and singular delay differential equations, further expanding its applicability in various scientific and engineering fields.

• Conclusion

In this study, we developed and proposed a numerical approach using the Runge-Kutta-type two-derivative method with a five-stage, three-step (5) 3TDRKT configuration to solve a special class of third-order delay differential equations with fixed delays. This approach integrates Newton interpolation with the TDRKT method to approximate the solutions of these complex equations effectively.

The fifth-order, three-step method, referred to as (5) 3TDRKT, employs a single third derivative and multiple evaluations of the fourth derivative. This configuration has proven particularly effective in directly solving third-order pantograph-type delay differential equations by leveraging Newton's interpolation method.

Our stability analysis of the (5) 3TDRKT method confirms its robustness and reliability. Extensive numerical tests demonstrate the method's high efficiency and accuracy in solving third-order delay differential equations. The findings indicate that the (5) 3TDRKT method is highly suitable for this class of equations.

Furthermore, the new method shows great potential for future research. We recommend extending the (5) 3TDRKT approach to address fractional and singular delay differential equations, which could broaden its applicability and utility in various scientific and engineering disciplines.

The results of this study underscore the significance of advanced numerical methods like the (5) 3TDRKT in enhancing the precision and stability of solutions for complex delay differential equations.

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