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Weakly Nearly Prime Submodules

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Abstract

In this article, unless otherwise established, all rings are commutative with identity and all modules are unitary left R-module. We offer this concept of WN-prime as new generalization of weakly prime submodules. Some basic properties of weakly nearly prime submodules are given. Many characterizations, examples of this concept are stablished.

Keywords: Weakly prime submodules, weakly nearly prime submodules, multiplication modules, finitely generated modules, Jacobson of a modules.

1.Introduction

The concept of weakly prime submodule was first Introduced and studied by Behoodi and Koohi in [1] as a generalization of weakly prime submodule , where a proper submodule H of an R-module U is weakly prime submodule, if whenever $0 \neq ru \in H$, for $r \in R, u \in U$, implies that either $u \in H$ or $r\,U \subseteq H$. Recently, weakly prime submodules have been studied by many authors such as [2-5]. Many generalizations of weakly prime submodule are introduced such as weakly primary submodules, weakly quasi- prime submodules and weakly semi- prime submodules see [6- 8]. In 2018 the concepts WE-prime submodules and WE-semi- prime submodules as a strange from of weakly prime submodules are given; see [9]. In this article, we introduce a new generalization of weakly prime submodule called WN-prime submodule , where a proper submodule H of an R-module U is called WN-prime of U if whenever $0 \neq ru \in H$, for $r \in R$, $u \in U$, implies that either $u \in H + \mathrm{J}(U)$ or $r\,U \subseteq H + \mathrm{J}(U)$, where $\mathrm{J}(U)$ is the Jacobson radical of U. An R-module U is multiplication if each submodule H of U from $H = I\,U$ for some ideal I of R , that is $H = [H:_R U]\,U$ [10]. Several characterizations, examples and basic properties of WN-prime submodules were given in this research.

2. Basic Properties of Weakly Nearly Prime Submodules

In this stage, we offer the definition of weakly nearly prime submodule and establish some of its basic properties and characterizations.

Definition (2.1)

A proper submodule H of R-module U is said to be weakly nearly prime submodule of U (for short WN-prime submodule), if whenever $0 \neq au \in H$, where $a \in R$, $u \in U$, implies that either $u \in H + \mathrm{J}(U)$ or $r\,U \subseteq H + \mathrm{J}(U)$.An ideal A of ring R is WN-prime ideal of R if and only if A is a WN-prime submodule of an R-module R.

For example : consider the Z-module Z_{24} and the submodule $H = \langle\bar{8}\rangle$ of Z_{24} which is a WN-prime submodule of Z_{24} since $\mathrm{J}(Z_{24}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. Thus if $0 \neq rm \in H$ with $r \in Z$, $m \in Z_{24}$, implies that either $m \in H + \mathrm{J}(Z_{24}) = \langle\bar{8}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $r \in [H + \mathrm{J}(Z_{24}): Z_{24}] = [\langle\bar{2}\rangle: Z_{24}] = 2Z$.

Remark (2.2)

1. It is clear that every weakly prime submodule of an R-module U is WN-prime , but not conversely.

For example the submodule $N = Z$ of the Z-module Q is not weakly prime, but N is WN-prime since $\mathrm{J}(Q) = Q$ and for each $a \in Z, u \in Q$ with $0 \neq au \in N$, implies that either $u \in N + \mathrm{J}(Q)$ or $aQ \subseteq Z + \mathrm{J}(Q) = Q$.

2. It is clear that every prime submodule of an R-module U is WN-prime, but not conversely .

For example : consider that the Z-module Z_{12} , and the submodule $H = \langle\bar{4}\rangle$ of Z_{12} is not prime , but $H = \langle\bar{4}\rangle$ is WN-prime submodule of Z_{12} since $\mathrm{J}(Z_{12}) = \langle\bar{2}\rangle \cap \langle\bar{3}\rangle = \langle\bar{6}\rangle$. Thus if $0 \neq ru \in H$ with $r \in Z$, $u \in Z_{12}$, implies that either $u \in H + \mathrm{J}(Z_{12}) = \langle\bar{4}\rangle + \langle\bar{6}\rangle = \langle\bar{2}\rangle$ or $r \in [H + \mathrm{J}(Z_{12}): Z_{12}] = [\langle\bar{2}\rangle: Z_{12}] = 2Z$.

3. If H is proper submodule of an R-module U with $\mathrm{J}(U) \subseteq H$. Then H is a WN-prime if and only if H is weakly prime submodule .

4.If U is a semi-simple R-module and H is a proper submodule of U,then H is a weakly prime if and only if H is WN-prime submodule of U.

Proof

It is well-known if U is a semi-simple, then $\mathrm{J}(U) = (0)$. [14, Theo. (9.2.1) (a)]. So the proof follows direct.

The following propositions give characterizations of WN-prime submodules.

Proposition (2.3)

Let U be an R-module, H be a submodule of U, then H is a WN-prime submodule of U if and only if for every submodule L of U and $r \in R$ with $0 \neq \langle r\rangle L \subseteq H$, implies that either $L \subseteq H + \mathrm{J}(U)$ or $\langle r\rangle\, U \subseteq H + \mathrm{J}(U)$.

Proof

($\Longrightarrow$) Suppose that $0 \neq \langle r\rangle L \subseteq H$, for $r \in R$, and L is a submodule of U, with $L \nsubseteq H + \mathrm{J}(U)$, then $l \notin H + \mathrm{J}(U)$ for some non-zero element $l \in L$. Now $0 \neq rl \in H$, then since H is WN-prime submodule of U, and $l \notin H + \mathrm{J}(U)$, then we have $r \in [H + \mathrm{J}(U): U]$, it follows that $\langle r\rangle \subseteq [H + \mathrm{J}(U): U]$. That is $\langle r\rangle U \subseteq H + \mathrm{J}(U)$

($\Longleftarrow$) Let $0 \neq ru \in H$, for $r \in R$, $u \in U$, it follows that $0 \neq \langle r\rangle\langle u\rangle \subseteq H$, so by hypothesis either $\langle u\rangle \subseteq H + \mathrm{J}(U)$ or $\langle r\rangle U \subseteq H + \mathrm{J}(U)$. That is either $u \in H + \mathrm{J}(U)$ or $rU \subseteq H + \mathrm{J}(U)$. Hence H is a WN-prime submodule of U.

As direct result of Proposition (2.3) we get the following corollary.

Corollary (2.4)

A proper submodule H of an R-module U is WN-prime if and only if for every submodule K of U and every $r \in R$ such that $0 \neq rK \subseteq H$, implies that either $K \subseteq H + \mathrm{J}(U)$ or $r \in [H + \mathrm{J}(U) : U]$.

Proposition (2.5)

Let H be proper submodule of R-module U, then H is WN-prime submodule of U if and, only if $[H:_R x] \subseteq [H + \mathrm{J}(U):_R U] \cup [0:_R x]$ for all $x \in U$ and $x \notin H + \mathrm{J}(U)$.

Proof

($\Longrightarrow$) Let $r \in [H:_R x]$ and $x \notin H + \mathrm{J}(U)$, then $rx \in H$ .If $rx \neq 0$, and H is a WN-prime submodule of U and $x \notin H + \mathrm{J}(U)$, hence $r \in [H + \mathrm{J}(U):_R U]$. If $rx = 0$, then $r \in [0:_R x]$.Thus $r \in [H + \mathrm{J}(U):_R U] \cup [0:_R x]$.Hence $[H:_R x] \subseteq [H + \mathrm{J}(U):_R U] \cup [0:_R x]$.

($\Longleftarrow$) Let $0 \neq rx \in H$ for $r \in R$, $u \in U$, with $x \notin H + \mathrm{J}(U)$, then $r \in [H:_R x]$, by hypothesis $r \in [H + \mathrm{J}(U):_R U] \cup [0:_R x]$, but $rx \neq 0$. Thus, $r \in [H + \mathrm{J}(U):_R U]$ and hence H is a WN-prime submodule of U .

Proposition (2.6)

Let H be a proper submodule of an R-module U with $[H + \mathrm{J}(U):_R U]$ is a maximal ideal of R, then H is a WN-prime submodule of U.

Proof

Suppose that $0 \neq ru \in H$, with $r \in R$, $u \in U$ and $rU \nsubseteq H + \mathrm{J}(U)$. That is, $r \notin [H + \mathrm{J}(U): U]$,but $[H + \mathrm{J}(U): U]$ is maximal, then by [11,Th. 5.1] $R = \langle r\rangle + [H + \mathrm{J}(U):_R U]$. It follows that $1 = ar + b$, for some $a \in R$, $b \in [H + \mathrm{J}(U):_R U]$. Hence, $u = aru + bu \in H + \mathrm{J}(U)$. Hence, H is a WN-prime submodule of U.

Proposition (2.7)

Let H be a proper submodule of an R-module U with $[L:_R U] \nsubseteq [H + \mathrm{J}(U):_R U]$ and $H + \mathrm{J}(U)$ is a proper submodule of L for each submodule L of U .If $[H + \mathrm{J}(U):_R U]$ is a prime ideal of R, then H is a WN-prime submodule of U.

Proof

Assume that $0 \neq ru \in H$, for $r \in R$, $u \in U$ and $u \notin H + \mathrm{J}(U)$.We have $H + \mathrm{J}(U) \nsubseteq H + \mathrm{J}(U) + \langle u \rangle$, put $L = H + \mathrm{J}(U) + \langle u \rangle = L$, then $[L:_R U] \nsubseteq [H + \mathrm{J}(U):_R U]$. That is there exist $a \in [L:_R U]$ and $a \notin [H + \mathrm{J}(U):_R U]$. It follows that $aU \subseteq L$ but $aU \nsubseteq H + \mathrm{J}U$. $aU \subseteq L$, implies that $raU \subseteq rL = r(H + \mathrm{J}(U) + \langle u \rangle) \subseteq H + \mathrm{J}(U)$, that is $ra \in [H + \mathrm{J}(U): U]$. But $[H + \mathrm{J}(U):_R U]$ is a prime ideal of R and $a \notin [H + \mathrm{J}(U):_R U]$ then $r \in [H + \mathrm{J}(U) : U]$. Thus H is a WN-prime submodule of U.

It is well-known that if U is a multiplication R-module and H is a proper submodule of U, then $[L:_R U] \nsubseteq [H :_R U]$ for each submodule L of U with $H \nsubseteq L$ [12, Rem. (2.15)].

Corollary (2.8)

Let H be a proper submodule of a multiplication R-module U,then H is a WN-prime submodule of U**,** if $[H + \mathrm{J}(U):_R U]$ is a prime ideal of R and $H + \mathrm{J}(U)$ is a proper submodule of L for each submodule L of U.

If H is a submodule of an R-module U, then $H(S) = \{u \in U : \exists t \in S \text{ such that } tu \in H\}$ [13].

Proposition (2.9)

Let H be a proper submodule of an R-module U, with $[H + \mathrm{J}(U):_R U]$ is a prime ideal of R, then H is WN-prime if and only if $H(S) \subseteq H + \mathrm{J}(U)$ for each multiplicatively closed subset S of R with $S \cap [H + \mathrm{J}(U):_R U] = \varphi$.

Proof

($\Longrightarrow$) Suppose that H is a WN-prime submodule of U with $S \cap [H + \mathrm{J}(U):_R U] = \varphi$. Let $u \in H(S)$, then $\exists r \in S$ such that $ru \in H$, implies that $r \in [H:_R u] \subseteq [H + \mathrm{J}(U):_R U] \cup [0:_R u]$ by Proposition (2.5) .It follows that $0 \neq ru \in H$ (since H is a WN-prime), implies that either $u \in H + \mathrm{J}(U)$ or $r \in [H + \mathrm{J}(U):_R U]$. If $r \in [H + \mathrm{J}(U):_R U]$, implies that $r \in S \cap [H + J(U):_R U] = \varphi$ which is a contradiction. Thus $u \in H + \mathrm{J}(U)$ and hence $H(S) \subseteq H + \mathrm{J}(U)$.

($\Longleftarrow$) Suppose that $0 \neq ru \in H$ where $r \in R$, $u \in U$ such that $u \notin H + \mathrm{J}(U)$ and $r \notin [H + \mathrm{J}(U):_R U]$. Since $r \in S$, then $S = \{1, r, r^2, r^3, \ldots\}$ is multiplicatively closed subset of R and $S \cap [H + \mathrm{J}(U):_R U] = \varphi$ (since$[H + \mathrm{J}(U):_R U]$ is prime ideal of R). But $u \notin H + \mathrm{J}(U)$ implies that $u \notin H(S)$ and then $0 \neq ru \notin H$ which is a contradiction. Thus $u \in H + \mathrm{J}(U)$ or $r \in [H + \mathrm{J}(U):_R U]$. That is, H is a WN-prime submodule of U.

The following corollary a direct consequence of Proposition (2.9).

Corollary (2.10)

Let U be an R-module, H be a proper submodule of U, with $[H + \mathrm{J}(U):_R U]$ is prime ideal in R, then H is WN-prime if and only if $H(R - ([H + \mathrm{J}(U):_R U]) \subseteq H + \mathrm{J}(U)$.

Proposition (2.11)

Let U be an R-module, and A be a maximal ideal of R, with $A\,U + \mathrm{J}(U) \neq U$. Then $A\,U$ is a WN-prime submodule of U.

Proof:

Since $A\,U \subseteq A\,U + \mathrm{J}(\,U)$, then $A \subseteq$[A U+J(U) :$_R$U] .That is, there exists $r \in [A\,U + \mathrm{J}(\,U): U]$ and $r \notin A$. But A is a maximal ideal of R, then $R = A + \langle r \rangle$, then $1 = a + sr$ for some $s \in R$, it follows that $u = au + sru$ for each $u \in U$. Thus $u \in A\,U + \mathrm{J}(\,U)$ for each $u \in U$, so $A\,U + \mathrm{J}(\,U) = U$ which is a contradiction. Hence, $r \in A$ and it follows that $[A\,U + \mathrm{J}(\,U): U] \subseteq A$.Thus $[A\,U + \mathrm{J}(\,U): U] = A$. That is, $[A\,U + \mathrm{J}(\,U): U]$ is a maximal ideal of R, hence by Proposition(2.6), $A\,U$ is a WN-prime submodule of U.

Proposition (2.12)

Let H be a proper submodule of an R-module U with $[H + \mathrm{J}(\,U):_R U] = [H + \mathrm{J}(\,U):_R K]$ for each submodule K of U such that $H + \mathrm{J}(\,U)$ is a proper submodule of L, then H is a WN-prime submodule of U.

Proof

Suppose that $0 \neq ru \in H$ for each $r \in R$,u ∈ U with $u \notin H + \mathrm{J}(\,U)$. Assume that $K = H + \mathrm{J}(\,U) + \langle u \rangle$, it is clear that $H + \mathrm{J}(\,U) \subseteq K$, then $u \in K$ and so $r \in [H:_R K]$. Since $H \subseteq H + \mathrm{J}(\,U)$, then $[H:_R K] = [H + \mathrm{J}(\,U):_R K] = [H + \mathrm{J}(\,U):_R U]$ by hypothesis. Thus $r \in [H + \mathrm{J}(\,U):_R U]$, it follow that H is a WN-prime submodule of U.

Recall that submodule H of an R-module U is to said to be small, if for any submodule K of U with $U = H + K$ then $K = U$ [14].

Proposition (2.13)

Let H be a small proper submodule of an R-module U and $\mathrm{J}(U)$ is a weakly prime submodule of U, then H is a WN-prime submodule of U.

Proof

Suppose that $0 \neq ru \in H$, where $r \in R, u \in U$. Since H is a small submodule of U, then $0 \neq ru \in H \subseteq \mathrm{J}(U)$. It follows that $0 \neq ru \in \mathrm{J}(U)$, but $\mathrm{J}(U)$ is a weakly prime submodule of U, implies that either $u \in \mathrm{J}(U) \subseteq H + \mathrm{J}(U)$ or $rU \subseteq \mathrm{J}(U) \subseteq H + \mathrm{J}(U)$. Hence H is a WN-prime submodule of U.

Remark (2.14)

If H and L are two submodules of R-module U with H is contained in L, L is a WN-prime submodule of U. Then H not necessary to be WN-prime submodule of U. The following example explains that. Consider the Z-module Z_{24} and the submodule $H = \{\bar{0}, \overline{12}\}$, $L = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}, \overline{12}, \overline{14}, \overline{16}, \overline{18}, \overline{20}, \overline{22}\}$ we have L is a WN-prime (since L is a weakly prime) submodule of the Z-module Z_{24}, but H is not WN-prime because if $3 \in Z, \bar{4} \in Z_{24}$ such that $\bar{0} \neq 3\,\bar{4} \in H$, but $\bar{4} \notin H + \mathrm{J}(Z_{24}) = \{\bar{0}, \bar{6}, \overline{12}, \overline{18}\}$ and $3 \notin [\,H + \mathrm{J}(Z_{24}) : Z_{24}] = 6Z$.

Proposition (2.15)

Let U be an R-module , and H, L are submodules of U with H contained in L, and $\mathrm{J}(\,U) \subseteq \mathrm{J}(L)$. If H is WN-prime submodule of U, then H is WN-prime submodule of L.

Proof

Assume that $0 \neq rx \in H$ with $r \in R, x \in L$. Since L is a WN-prime submodule of U, then $x \in H + \mathrm{J}(\,U)$ or $r \in [H + \mathrm{J}(\,U):_R U]$. But $\mathrm{J}(\,U) \subseteq \mathrm{J}(L)$ so $x \in H + \mathrm{J}(L)$ or $r \in [H + \mathrm{J}(L):_R U] \subseteq [H + \mathrm{J}(L):_R L]$. Hence H is a WN-prime submodule of L.

Remark (2.16)

The resudule of WN-prime submodule of an R-module U need not to be WN-prime ideal of R. The following example shows that:

Let $U = Z_{12}$, $R = Z$ and $H = \{\bar{0}, \bar{4}, \bar{8}\}$, H is a WN-prime submodule of Z_{12} by Remark(2.2)(2). But $[H:_Z Z_{12}] = 4Z$ is not WN-prime ideal of R because $0 \neq 2\,2 \in 4Z, 2 \in Z$ but $2 \notin 4Z + \mathrm{J}(Z) = 4Z$ and $2 \notin [4Z + \mathrm{J}(Z):_Z Z] = 4Z$.

The following propositions show that the resudule of a WN-prime submodule is a WN-prime ideal in the class of multiplication R-module over a good ring, Artinian ring respectively.

Remember that A ring R is called good if $\mathrm{J}(U) = \mathrm{J}(R).U$ where U is an R-module [14].

Proposition (2.17)

Let U be a multiplication module over a good ring R , and H is a WN-prime submodule of U then $[H:_R U]$ is a WN-prime ideal of R.

Proof

suppose that $0 \neq rs \in [H:_R U]$ where $r, s \in R$, implies that $0 \neq r(sU) \subseteq H$. But H is a WN-prime submodule of U, then by Corollary (2.4) either $sU \subseteq H + \mathrm{J}(U)$ or $rU \subseteq H + \mathrm{J}(U)$. For U a multiplication module over good ring, then $\mathrm{J}(U) = \mathrm{J}(R).U$ and $H = [H:_R U].U$. Thus either $sU \subseteq [H:_R U].U + \mathrm{J}(R).U$ or $rU \subseteq [H:_R U]\,U + \mathrm{J}(R)U$. Hence either $s \in [H:_R U] + \mathrm{J}(R)$ or $r \in [H:_R U] + \mathrm{J}(R) = \big[[H:_R U] + \mathrm{J}(R):_R U\big]$. Therefore $[H:_R U]$ is a WN-prime ideal of R.

It is well known if U is a module over Artinian ring R then $\mathrm{J}(U) = \mathrm{J}(R)U$. [14, Co. 9.3.10(c)].

Proposition (2.18)

Let U is a multiplication module over Artinian ring R, and H is a WN-prime submodule of U then $[H:_R U]$ is a WN-prime ideal of R.

Proof

Let $0 \neq rI \in [H:_R U]$ where $r \in R$ and I is an ideal of R, then $0 \neq rI \subseteq H$. Since H is a WN-prime submodule of U, then by Corollary (2.4) either $IU \subseteq H + \mathrm{J}(U)$ or $rU \subseteq H + \mathrm{J}(U)$.But U is a multiplication module over good ring R, then $\mathrm{J}(U) = \mathrm{J}(R)U$ and $H = [H:_R U]U$. It follows that either $IU \subseteq [H:_R U]U + \mathrm{J}(R)U$ or $rU \subseteq [H:_R U]U + \mathrm{J}(R).U$. Hence either $I \subseteq [H:_R U] + \mathrm{J}(R)$ or $r \in [H:_R U] + \mathrm{J}(R) = \big[[H:_R U] + \mathrm{J}(R):_R U\big]$.Therefore $[H:_R U]$ is a WN-prime ideal of R.

It is well known that if U is a projective R-module then $\mathrm{J}(U) = \mathrm{J}(R).U$ [14, Th. 9.2.1(g)].

Proposition (2.19)

Let U be a projective multiplication R-module , and H is a WN-prime submodule of U then $[H:_R U]$ is a WN-prime ideal of R.

Proof

Follows in the same way of Proposition (2.17) and Proposition (2.18).

It is well known if U is a multiplication finitely generated R-module , and A, B are ideals of R, then $A\,U \subseteq B\,U$ if and only if $A \subseteq B + ann(\,U)$ [15, Cor. of th. 9].

Proposition (20)

Let U be a multiplication finitely generated faithful module over good ring R , A is a WN-prime ideal of R.Then $A\,U$ is a WN-prime submodule of U.

Proof

Suppose that $0 \neq aH \subseteq A\,U$ where $a \in R, H$ is a submodule of U,implies that $0 \neq aI\,U \subseteq A\,U$ for U is a multiplication, it follows that $0 \neq aI \subseteq A + ann(\,U)$.But U is faithful, then $ann(\,U) = (0)$. Thus $0 \neq aI \subseteq A$. But A is a WN-prime ideal of R, then either $I \subseteq A + \mathrm{J}(R)$ or $r \in [A + \mathrm{J}(R): R] = A + \mathrm{J}(R)$. Hence $I\,U \subseteq A\,U + \mathrm{J}(R)\,U$ or $r\,U \subseteq A\,U + \mathrm{J}(R)\,U$. That is either $I\,U \subseteq A\,U + \mathrm{J}(\,U)$ or $r\,U \subseteq A\,U + \mathrm{J}(\,U)$. Thus either $H \subseteq A\,U + \mathrm{J}(\,U)$ or $r \in [A\,U + \mathrm{J}(\,U):_R\,U]$. Therefore $A\,U$ is a WN-prime submodule of U.

Proposition (2.21)

Let U be a finitely generated multiplication faithful module over Artinian ring R, and A be a WN-prime ideal of R, then $A\,U$ is a WN-prime submodule of U.

Proof

Similar as in Proposition (2.20).

Proposition (2.22)

Let U be a finitely generated projective multiplication R-module , and A is a WN-prime ideal of R with $ann(U) \subseteq A$ then AU is a WN-prime submodule of U.

Proof

Suppose that $0 \neq au \in AU$ for $a \in R, u \in U$ so, $0 \neq a(u) \subseteq AU$. Since U is a multiplication, then $(u) = \mathrm{J}U$ for some ideal J of R, hence $0 \neq a\mathrm{J}U \subseteq AU$, since U is finitely generated multiplication, then $0 \neq a\mathrm{J} \subseteq A + ann(U)$. But $ann(U) \subseteq A$, then $0 \neq a\mathrm{J} \subseteq A$, since A is a WN-prime ideal of R then by Corollary (2.4) either $\mathrm{J} \subseteq A + \mathrm{J}(R)$ or $a \in [A + \mathrm{J}(R):_R R] = A + \mathrm{J}(R)$. That is either $\mathrm{J}U \subseteq AU + \mathrm{J}(R)U$ or $aU \subseteq AU + \mathrm{J}(R)U$. But U is a projective, then $\mathrm{J}(R)\,U = \mathrm{J}(\,U)$. Thus either $(u) \subseteq A\,U + \mathrm{J}(\,U)$ or $a \in [A\,U + \mathrm{J}(\,U):_R\,U]$. That is either $u \in A\,U + \mathrm{J}(U)$ or $a \in [AU + \mathrm{J}(U):_R U]$. Thus AU is a WN-prime submodule of U.

Proposition (2.23)

Let H be a WN-prime submodule of an R-module U, then $S^{-1}H$ is a WN-prime submodule of $S^{-1}R$-module $S^{-1}\,U$, where S is a multiplicatively closed subset of R.

Proof

Suppose that $(0) \neq \frac{r_1}{s_1}\frac{u}{s_2} \in S^{-1}H$ for $\frac{r_1}{s_1} \in S^{-1}R$ and $\frac{u}{s_2} \in S^{-1}\,U$ and $r_1 \in R,\ s_{1,}s_2 \in S, u \in\ U$. Then $\frac{r_1 u}{t} \in S^{-1}H$, where $t = s_1 s_2 \in S$, that is there exists non-zero element $t_1 \in S$ such that $0 \neq t_1 r_1 u \in H$. But H is a WN-prime submodule of U, then either $t_1 u \in H + \mathrm{J}(\,U)$ or $r_1 \in [H + \mathrm{J}(\,U):_R\,U]$, it follows that either $\frac{t_1 u}{t_1 s_2} \in S^{-1}\big(H + \mathrm{J}(\,U)\big) \subseteq S^{-1}H + \mathrm{J}(S^{-1}\,U)$or $\frac{r_1}{s_1} \in$

$S^{-1}[H+\mathrm{J}(U):_R U] \subseteq [S^{-1}H+\mathrm{J}(S^{-1}U):_R S^{-1}U]$. Hence either $\frac{u}{s_2} \in S^{-1}H+\mathrm{J}(S^{-1}U)$ or $\frac{r_1}{s_1} \in [S^{-1}H+\mathrm{J}(S^{-1}U):_R S^{-1}U]$. Thus $S^{-1}H$ is a WN-prime submodule of $S^{-1}R$-module $S^{-1}U$.

It is well known that if $\varphi : U \longrightarrow Y$ is an R-epimorphism and $Ker\varphi$ small submodule of R-module U, then $\varphi(\mathrm{J}(U)) = \mathrm{J}(Y), \varphi^{-1}(\mathrm{J}(Y)) = \mathrm{J}(U)$ [14, Cor. 9.1.5(a)].

Proposition (2.24)

Let $\varphi : U \longrightarrow U'$ be an R-epimorphism with $Ker\varphi$ is small submodule of U, and K be a WN-prime submodule of U', then $\varphi^{-1}(K)$ is a WN-prime submodule of U.

Proof

Let $0 \neq rx \in \varphi^{-1}(K)$ where $r \in R, x \in U$ with $x \notin \varphi^{-1}(K)+\mathrm{J}(U)$, it follows that $\varphi(x) \notin K+\varphi(\mathrm{J}(U)) = K+\mathrm{J}(U')$. Since $0 \neq rx \in \varphi^{-1}(K)$, implies that $0 \neq r\,\varphi(x) \in K$. But K be a WN-prime submodule of $U'$and $\varphi(x) \notin K+\mathrm{J}(U')$, it follows that $r \in [K+\mathrm{J}(U'):_R U']$, that is $r\,U' \subseteq K+\mathrm{J}(U')$, hence $r\,\varphi(U) = \varphi(r\,U) \subseteq K+\mathrm{J}(U')$. Implies that $r\,U \subseteq \varphi^{-1}(K)+\mathrm{J}(U)$. Therefore $\varphi^{-1}(K)$ is a WN-prime submodule of U.

Proposition (2.25)

Let $f : U \longrightarrow U'$ be an R-epimorphism with $Kerf$ is small submodule of U, and H be a WN-prime submodule of U with $Kerf \subseteq H$. Then $f(H)$ is a WN-prime submodule of U'.

Proof

Since $Kerf \subseteq H$, that's clearly$f(H)$ is a proper submodule of U'. Now, suppose that $0 \neq rx' \in f(H)$, where $r \in R$, $x' \in U'$. Since f is an epimorphism then $f(x) = x'$ for some $x \in U$, thus $0 \neq rx' = rf(x) = f(rx) \in f(H)$,it follows that there exists non-zero $y \in H$ such that $f(rx) = f(y)$, implies that $f(rx-y) = 0$, hence $rx-y \in Ker\,f \subseteq H \Rightarrow 0 \neq rx \in H$. but H is a WN-prime submodule of U, then either $x \in H+\mathrm{J}(U)$ or $rU \subseteq H+\mathrm{J}(U)$, it follows that either $x' = f(x) \in f(H)+\mathrm{J}(U')$ or $r\,U' = rf(U) \subseteq f(H)+\mathrm{J}(U')$. That is $f(H)$ is a WN-prime submodule of U'.

3. Conclusion

In this article the concept WN-prime submodule was introduced and studied as generalization of a weakly prime submodule. The results that we set in this research are the following:

1. Every weakly prime submodule of R-module U is WN-prime, but not conversely .
2. A proper submodule H of an R-module U is a WN-prime if and only if whenever $0 \neq \langle r\rangle L \subseteq H$ where $r \in R$, L is a submodule of U implies that either $L \subseteq H+\mathrm{J}(U)$ or $\langle r\rangle\, U \subseteq H+\mathrm{J}(U)$.
3. A proper submodule H of an R-module U is WN-prime if and only if $[H:_R x] \subseteq [H+\mathrm{J}(U):_R U] \cup [0:_R x]$ for all $x \in U$ and $x \notin H+\mathrm{J}(U)$.
4. Let H be a proper submodule of an R-module U, with $[H+\mathrm{J}(U):_R U]$ is a prime ideal of R, then H is a WN-prime if and only if $H(S) \subseteq H+\mathrm{J}(U)$ for each multiplicatively closed subset S of R with $S \cap [H+\mathrm{J}(U):_R U] = \varphi$.
5. If a submodule H of an R-module U is small and $\mathrm{J}(U)$ is a weakly prime submodule of U, then H is WN-prime submodule of U.

6. Let U be a multiplication module over Artinian ring R, and H is a WN-prime submodule of U then $[H:_R U]$ is a WN-prime ideal of R.
7. If U is a projective multiplication R-module , and H is a WN-prime submodule of U then $[H:_R U]$ is a WN-prime ideal of R.
8. If U is finitely generated faithful multiplication module over good ring R, and A be WN-prime ideal of R, then $A\,U$ is WN-prime submodule of U.
9. If U is finitely generated projective multiplication R-module then $A\,U$ is a WN-prime submodule of U for all WN-prime ideal A of R with $ann(U) \subseteq A$.
10. If H is a WN-prime submodule of an R-module U, then $S^{-1}H$ is a WN-prime submodule of $S^{-1}R$-module $S^{-1}\,U$, where S is a multiplicatively closed subset of R.

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