Extensive set in Topological Transformation Group

المجموعة الموسعة في زمرة التحويل التبولوجي

Enas Yahya Abdullah
Kufa University College Of Education For Girls/Department Of Mathematic
inasy.abdullah@uokufa.edu.iq

Abstract :

In this paper we discussed properties extensive set through studied semi group replete in topological transformation group , and it will given necessary condition for extensive set to be syndetic set ,so image and inverse image of extensive set also extensive set and introduce new define of point dynamics be strongly periodic using concept of the extensive set as well as we introduce that many of new relations and theorem.

المستخلص

في هذا البحث ناقشنا خصائص المجموعة الموسعة من خلال دراسة شبه الزمرة المفعمة في زمرة التحويل التبولوجي وقدمنا الشروط الضرورية لتكون المجموعة الموسعة مجموعة رابطة كذلك أثبتنا الصورة والصورة العكسية للمجموعة الموسعة ايضا موسعة كما قدمنا في البحث تعريفا جديد لنقطة ديناميكية هي الدورية بقوة باستخدام المجموعة الموسعة كما قدم البحث عدد من النظريات والعلاقات الجديدة

Introduction :

Let X invariant topological group and V topological group , The first has been introduced the topological transformation group (X, V, π) with compact hausdorff X was Ellis R. and Gottschalk W.[1][3].

In this article ,we study extensive set on topological group ,where group be extensive this a much stronger from group itself and obtain many of properties dynamics . We show relation extensive set with some topological properties (closure set ,close set , interior set),and show that intersection extensive set also extensive set . also we studied strongly periodic and feebly recurrent in depended on extensive set and invariant neighborhood so show that relation strongly periodic and feebly recurrent with transitive point , and we studied the image of extensive set under homomorphism ,.We use symbol ■ to indication the end.

1.Preliminaries:

In this section we given important concepts that we needed in this work.

Definition (1.1) [4]:
A topological group is a set V with two structures:
1. V is a group
2. V is a topological space

such that the two structures are compatible i.e., the multiplication map $f: V \times V \to V$ and the inversion map $v: V \to V$ are both continuous.

Definition (1.2) [3]:

A right topological transformation group is a triple (X, V, π) where X is a topological space called the phase space , V is a topological group called the phase group $\pi: X \times V \to X, \pi(x, v) \to xv$ is a continuous mapping such that :
1. $xe = x$ $(x \in X)$,where e is the identity of T
2. $(xv)w = x(vw)$ $(x \in X, v, w \in V)$

Definition (1.3) [3]:
A semi group S in V is said to be replete provided that S contains some translate of each compact set in V.

Definition (1.4) [3]:
A subset ρ of V is said to be $\{left\},\{right\}$ syndetic in V if and only if there exists a compact subset k of V such that $\rho k = k\rho = V$.

Definition (1.5) [3]:
1. A set A in V is said to be extensive set provided that A intersects every semi replete group in V non-empty set.
2. Let (X,V) be a topological transformation group A subset $A \subset$X is said to be invariant set if $AV = A$.
3. The transformation group (X,V) is said to be transitive at x and the point x is said to be transitive under (X,V) provided that if U is a neighborhood subset of X ,then there exists $v \in V$ such that $xv \in U$.

Example (1.6):
(Z, +) with topological discrete be a strongly effective topological transformation group and (2Z, +) semi - group such that $2Z = \{\dots,-2,0,2,4,\dots\}$and $\{0\}$ compact subset of Z there exist $v \in Z$ such that left translation of compact set is contained in 2Z ,it follows that 2Z semi replete group .

Definition (1.7):
Let (X,V,π) and (H,G,σ) be a topological transformation group, and $\tau: X \to H$ be continuous ,$\gamma: V \to G$ be continuous homomorphism then $(\tau,\gamma): (X,V,\pi) \to (H,G,\sigma)$ is said to be homomorphism and $((z,t)\pi)\tau = \big((z)\tau,(t)\gamma\big)\sigma$.If for each τ,γ onto then homomorphism is said to be epimorphism .

Definition (1.8):
Let $x \in X$ .The replete period of V at x or the replete period of x under V is defined to be the extensive subset P of V such that $xP = x$.

Definition (1.9):
Let (X,V,π) be a topological transformation group, and $x \in X$
1. x is said to be strongly periodic point under V or V strongly periodic at x if replete period x extensive subset of V.
2. topological transformation group(X,V,π) is said to be strongly periodic if V strongly periodic .

Definition (1.10):
Let $x \in X$, is said to be feebly recurrent point if for each invariant neighborhood U of X there exist extensive subset A of V such that $xA \subset U$.

2 Extensive set :
In this section ,we introduce properties extensive set .

Theorem (2.1):
Let (X,V) be a topological transformation group , the following statements are valid.
1. if K compact subset of V then K^{-1} compact set of V.
2. if A semi replete group of V then A syndetic set of V .
3. V is extensive set .

Proof (3). Let V group ,it easy find semi replete subset P of V such that $V \cap P \neq \emptyset$ Then V is extensive group . ■

Theorem (2.2):
Let (X, V) be a topological transformation group, and V be an abelian.Then the following statements are equivalent:
1. A extensive set of V
2. A syndetic set of V

Proof: Assume (1) .We prove (2). Let A be a extensive subset of V ,there exist semi replete P of V such that $A \cap P \neq \emptyset$, since intersects be abelian then $P \subset A$ from (theorem (2.1) number 2) there exists a compact subset K of V such that $PK = V$ so $V \subset AK$ .It follows that A syndetic set of V.
Assume (2) .We prove (1).Let A be a syndetic subset of V , there exists a compact subset K of V such that $AK = V$, since V be an abelian then $V = KA$ from (theorem (2.1) number 1) for each $v \in V$ there exist $k \in K^{-1}, a \in A$ such that $vk = a$ and $vK^{-1} \subset A$ .Hence A semi replete of V and $A \cap A \neq \emptyset$,therefore A extensive set of V .■

Theorem (2.3):
Let (X, V) be a topological transformation group then :
1. If A extensive set, $A \subset B \subset \boldsymbol{V}$, of V then B extensive set of V.
2. If A, B extensive set of V and $e \in B$ then $A . B$, $B.A$ extensive set of V.

Proof : (1) .Let A be a extensive subset of V ,there exist semi replete P of V such that$A \cap P \neq \emptyset$,so that $A \cap P \subset B \cap P \subset V \cap P$,since $V \cap P \neq \emptyset$ from (theorem (2.1) number 3) .it follows that B extensive set of V. ■

Theorem (2.4):
Let (X, V) be a topological transformation group, and V be an abelian Then the following statements are equivalent:
1. A extensive set of V
2. Ag extensive set of V

Proof: Assume (1) .We prove (2) .Let A extensive set of V ,then there exit semi- replete P of V such that $A \cap P \neq \emptyset$ so $Ag \cap Pg \neq \emptyset$ for some $g \in V$. It is enough to prove that Pg semi-replete of V ,by hypothesis we have that compact set M of V such that $g_1 M \subset P$ for some $g_1 \in V$ .since V group then there exist $g^{-1} \in V$ such that $g_1 M g g^{-1} \subset P$ and $g_1 M g \subset Pg$,since M compact set of V continuous transformation of $X \times V$ into X then Mg compact set of V ,therefore Pg semi- replete and Ag extensive set of V.

Assume (2) .We prove (1), .Let Ag extensive set of V ,then there exit semi- replete P of V such that $Ag \cap P \neq \emptyset$, for each $a \in A$ there $g \in V, p \in P$ such that $ag = p$, so $a = pg^{-1}$ thus $A \subset Pg^{-1}$,. It is enough to prove that Pg^{-1} semi- replete of V , by hypothesis we have that compact set M of V such that $g_1 M \subset P$ for some $g_1 \in V$. Since V then $g_1 M g^{-1} \subset P g^{-1}$ since M compact set of V continuous transformation of $X \times V$ into X then Mg^{-1} compact set of V ,therefore Pg^{-1} semi- replete it follows that A extensive set of V. ■

Corollary (2. 5):
Let (X, V) be a topological transformation group, and V be an abelian Then A extensive set of V if and only if $g_1 A g$ extensive set of V.

Theorem (2. 6):
Let (X, V) be a topological transformation group, and V be an abelian , A extensive set of V then P semi replete syndetic .

Proof: Let A extensive set of V ,then there exit semi replete P of V such that $A \cap P \neq \emptyset$ by hypothesis there exist compact subset K of V such that $AK \cap PK \neq \emptyset$, by theorem (2.2 number (2)) , $V \subset PK$ it follows that P semi replete syndetic ■

Theorem (2.7):
Let (X,V) be a topological transformation group, and A,B extensive set of V then $A\times B$ extensive set in $V\times V$.
Proof: Let A,B extensive set there exist semi replete P_1,P_2 of V such that $A\cap P_1\neq\emptyset$ and $B\cap P_2\neq\emptyset$, It is enough to prove $P_1\times P_2$ semi replete in $V\times V$,since P_1,P_2 semi group then $P_1\times P_2$ semi group in $V\times V$,since each P_1,P_2semi replete in V then for each compact subset K_1 , K_2 of V there exit $g_1,g_2\in V$ such that $g_1K_1\subset P_1$ and $g_2K_2\subset P_2$,let K compact subset $K_1\times K_2$ Thus $(g_1,g_2)K\subset(g_1,g_2)K_1\times K_2=g_1K_1\times g_2K_2\subset P_1\times P_2$therefore $P_1\times P_2$ semi replete in $V\times V$ for some $(g_1,g_2)\in V\times V$, then it obtain $A\times B\subset P_1\times P_2$ and $A\times B$ extensive set in $V\times V$. ■

Theorem (2.8):
Let (X,V,π) and (H,G,σ) be a topological transformation, $\gamma:V\to G$ be continuous homomorphism Then
1. If A extensive set of V then $\gamma(A)$ extensive set of G.
2. If A extensive set of G then $\gamma^{-1}(A)$ extensive set of V.
Proof:
(1) Let A extensive set of V ,then there exit semi- replete P of V such that $A\cap P\neq\emptyset$ and $\gamma(A\cap P)\subset\gamma(A)\cap\gamma(P)$ .It is enough to prove $\gamma(P)$ semi- replete in G, by hypothesis for each compact subset K of V there exist $g\in V$ such that $gK\subset P$ and $\gamma(gK)\subset\gamma(P)$ since γ be homomorphism then $\gamma(g)\gamma(K)\subset\gamma(P)$,since γ be continuous function and K be compact set then $\gamma(K)$ compact subset in G therefore $\gamma(P)$ semi- replete in G then $\gamma(A)\cap\gamma(P)\neq\emptyset$ and $\gamma(A)$ extensive set of G. ■

(2) Let A extensive set of G ,then there exit semi- replete P of G such that $A\cap P\neq\emptyset$ and $\gamma^{-1}(A\cap P)\subset\gamma^{-1}(A)\cap\gamma^{-1}(P)$ .It is enough to prove $\gamma^{-1}(P)$ semi- replete in V, since P semi-group in G so $\gamma^{-1}(P)$ semi –group in V , assume that K compact subset in V ,then $\gamma(K)$ compact subset in G there exist $n\in G$ such that $n\,\gamma(K)\subset P$ and $v\in V$ such that $\gamma(vK)\subset P$ thus $vK\subset\gamma^{-1}(P)$ and $\gamma^{-1}(P)$ semi- replete in V ,it follows that $\gamma^{-1}(A)$ extensive set of V. ■

Theorem (2.9):
Let (X,V) be a topological transformation group, then A extensive set of V if and only if A° is extensive set of V
Proof: : since $A\cap P\neq\emptyset$ and P semi- replete of V ,since aP left translation for each $a\in P$ and $(aP)^{\circ}=aP^{\circ}$ imply that $(PP)^{\circ}=PP^{\circ}$ since $PP\subset P$ then $P^{\circ}P^{\circ}\subset PP^{\circ}=(PP)^{\circ}\subset P^{\circ}$ and P° semi – group .assume that K compact open sub set of V there exist $g\in V$ such that $gK\subset P$ and $(gK)^{\circ}=gK^{\circ}\subset P^{\circ}$ imply that P°semi- replete , $(A\cap P)^{\circ}=A^{\circ}\cap P^{\circ}\neq\emptyset$ therefore A° is extensive set of V.
Conversely .Let A° be extensive set of V and $A^{\circ}\cap P\neq\emptyset$ where P semi- replete of V since $A^{\circ}\subset A$ and $A^{\circ}\cap P\subset A\cap P$ then $A\cap P\neq\emptyset$ and A extensive set of V. ■

Theorem (2.10):
Let (X,V) be a topological transformation group, and A extensive set of V then :
1. $\bar{A}$ is extensive set of V
2. A is closed set of V

(1) .Since A extensive set of V , then there exit semi- replete P of G such that $A\cap P\neq\emptyset$ and $\overline{A\cap P}\subset\bar{A}\cap\bar{P}$. Since $PP\subset P$, $\bar{P}\,\bar{P}\subset\bar{P}$ then $\bar{P}$ semi group
Now prove $\bar{P}$ by hypothesis for each compact subset K of V there exist $g\in V$ such that $gK\subset P$,since P be a subset of a topological space ,then $\bar{P}$ is the smallest closed set containing P imply $gK\subset\bar{P}$ it follows $\bar{A}\cap\bar{P}\neq\emptyset$ and $\bar{A}$ is extensive set of V. ■

(2)since A be a subset of a topological space ,then $\bar{A}$ is the smallest closed set containing A imply that $A \subset \bar{A}$,now it want to prove $\bar{A} \subset A$ since A extensive set of V there exist semi –replete P of V such that $A \subset P$ by theorem ((2.10) number 1) $\bar{A} \subset \bar{P}$ it only proved P closed set by hypothesis for each compact subset K of V there exist $e \in V$ such that $eK \subset P$ so that $e\bar{K} \subset \bar{P}$ since K compact set then $eK \subset \bar{P}$ and $eK \cap P \subset \bar{P} \cap P$ therefore $\bar{P} \subset P$ and P closed set
since V abelian group $\bar{A} \subset A$ it follows that A closed set of V . ■

Theorem (2.11):
Let (X, V) be a topological transformation group, then A extensive set of V if and only if A^{-1} is extensive set of V
Proof: Let V semi-replete and $A \cap V \neq \emptyset$ there exist $a \in A, g^{-1} \in V$ such that $a = g^{-1}$, $a^{-1} = g$, then $a^{-1} \in V$ then $A^{-1} \cap V \neq \emptyset$ .Then A^{-1} is extensive set of V. Conversely .Clearly $A^{-1} \cap V \neq \emptyset$ there exist $a^{-1} \in A^{-1}, g \in V$ such that $a^{-1} = g$, $g^{-1} = a$ then $g^{-1} \in A$ since V be abelian $A \cap V \neq \emptyset$ .Then A is extensive set of V. ■

Theorem (2.12):
Let (X, V) be a topological transformation group, if A and B are extensive sets in V then $A \cap B$ is a extensive set in V
Proof: Let A, B are extensive sets in V then there exist semi- replete P_1, P_2 of V such that $A \cap P_1 \neq \emptyset$ and $B \cap P_2 \neq \emptyset$,since P_1, P_2 semi group then $P_1 \cap P_2$ Semi- group ,it is enough to prove $P_1 \cap P_2$ semi- replete by hypothesis for each compact subset K_1 and K_2 of V there exist $g_1, g_2 \in V$ such that $g_1 K_1 \subset P_1$ and $g_2 K_2 \subset P_2$ .Assume that K compact subset of $K_1 \cap K_2$ then $(g_1, g_2)K \subset (g_1, g_2)(K_1 \cap K_2) = g_1 K_1 \cap g_2 K_2 \subset P_1 \cap P_2$ since $K_1 \cap K_2$ compact set of V

Then $P_1 \cap P_2$ semi- replete of V, Let $P = (P_1 \cap P_2)$

$$(A \cap B) \cap (P_1 \cap P_2) = A \cap (P_1 \cap P_2) \cap B \cap (P_1 \cap P_2)$$
$$= (A \cap P) \cap (B \cap P)$$
$$\neq \emptyset$$

It follows that $A \cap B$ are a extensive sets in V. ■

Theorem (2.13):
Let (X, V) be a topological transformation abelian group, A is extensive sets in V and P group –replete then $V = AP^{-1}$
Proof: Clearly P semi group –replete set then for each compact subset K of V th ere exit $g \in V$ such that $gK \subset P$ and $gKV \subset PV$ since V syndetic set then $gV \subset V \subset PV$, $V \cap A \subset PV \cap A$ since $V \cap A \neq \emptyset$ then for each $v \in V$ there exist $a \in A, p \in P$ such that $pv = a$ so $v = ap^{-1}$ therefore $V \subset AP^{-1}$ foe some P^{-1} ,It follows that $V = AP^{-1}$. ■

3.Strongly periodic point and feebly recurrent point :

In this section study relation strongly periodic with some dynamical points.

Theorem (3.1):
Let (X, V) be a topological transformation group, and x strongly periodic under V then x feebly recurrent under V
Proof: Assume that N invariant neighborhood of x ,since x strongly periodic under V then replete period of x be extensive subset of V such that $xP = x$ since $x \in N$ it follows that $xP \subset N$ and x feebly recurrent under V . ■

Theorem (3.2):
Let (X, V) be a topological transformation group, and x strongly periodic under V Then:
1. If N neighborhood of x then $xV \cap N \neq \emptyset$
2. If N invariant neighborhood of x then x feebly recurrent under V^n

Proof (1) : Assume that N neighborhood of x M be group – replete , since x strongly periodic under V then replete period of x be extensive subset of V such that $xP = x$ and $xPM^{-1} = xM^{-1}$ by theorem (2.13) it obtain $xV = xM^{-1}$ since $x \in N$ then $xV \cap N \neq \emptyset$.

(2) Assume that M be group – replete , since x strongly periodic under V then replete period of x be extensive subset of V such that $xP = x$ and $xPV = xV$ by hypothesis $xPV\mathrm{M}^{-1} = (xV)\mathrm{M}^{-1} \subset xV$ since N neighborhood of x then $(xV)V \subset N$ so $(xV)V^2 \subset NV$ since N invariant then $(x)V^3 \subset N$ also $(x)V^4 \subset NV \subset U$ .Thus $(x)V^n \subset U$ for each integer $n \geq 2$. ■

Theorem (3.3):
Let (X, V) be a topological transformation group, Then the following statements are equivalent:
1. x feebly recurrent under V
2. x transitive under V

Proof : Assume (1) .We prove (2) .Let x feebly recurrent under V then for each invariant neighborhood U of X there exist extensive subset A of V such that $xA \subset U$. There exist $v \in A$ and A subset of V such that $xv \in U$ then x transitive under V.
Assume (2) .We prove (1) .Let x transitive under V then for each neighborhood U of X there exist $v \in V$ such that $xv \in U$ so that $xV \subset U$ .It is enough to prove
U invariant set , $xVT \subset UT$ since V be invariant then $xV \cap U \subset UT \cap U$,it follows that $UT \subset U$ implies that $U \subset UT$ therefore U invariant set .From (theorem 2.1 number (3)) . it follows that x feebly recurrent under V . ■

Remark (3.4) :
Let (X, V) be a topological transformation group **,** x strongly periodic then x transitive under V.

References

1. Sidney A.Morris " An overview of topological group ".Academic Editor :Humber to bustince (2016).
2. Ellis. R ," Lectures on topological dynamics " ,W.A. Benjamin ,New York, 1969.
3. Gottschalk, W.H and Hedlund ,G.A "Topological dynamics " American mathematical society (1955)
4.Higgins ,P,J,An introduction to topological group ,cambridge university (1974)