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Finding All Natural Solutions of the Hyperbola Equation by a new Form

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Abstract:

In this paper a new form has been found that gives all natural solutions to the hyperbola equation $x^2 - dy^2 = c^2$ where $c \epsilon N$, d is a positive square free number. It has been depended of Pell's equation $x^2 - dy^2 = 1$ and a continued fraction method for a real number $\sqrt{d}$.

Keywords: *Diophantine equation, Pell's equation, continued fraction.*

1. Introduction

For

$$u = \frac{m^2 + n^2}{1 + mn} \tag{1}$$

where $m, n \in Z$ such that $1 + mn \neq 0$, the Diophantine equation has infinitely many real solutions [1]. Indeed, when $(m, n) \in Z$ then Eq. (1) has infinitely many rational solutions. The integer numbers (m, n) sometimes give square positive integer number u in Eq. (1). For example, $(m, n) \in \{(-100,0), (7,0), (0,-50), (1,1)\}$.

For $u \in \{2^2, 3^2, 4^2, 5^2, 6^2, \dots\}$, there are infinitely many natural pairs that can be obtained from Eq. (1) resulting in the following hyperbola equation

$$x^2 - dy^2 = c^2 \tag{2}$$

where d is a positive square free integer and $c \in N$. Note that, there are only one natural pair $(m, n) = (1,1)$ for $u = 1^2$.

We denote by (s_r, t_r), $r = 1,2,3, \dots$ to the infinitely many natural solutions of Eq. (2) that depends on the following equation

$$x^2 - dy^2 = 1 \tag{3}$$

Which is known as Pell's equation and it was named after John Pell. In the seventeenth century Pell [2] searched for integer solution of this type. He was not the first to work on this problem, Fermat [2,3] found the smallest solution for d up to 150, John Wallis[2] solved Eq. (3) for d = 151 or 313. Lagrange[2,3] developed the general theory of Pell's equation, based on continued fractions and algebraic manipulations with numbers of the form $x + \sqrt{d}y$ in (1766–1769).

For Eq. (3), we denote by $(x_r, y_r), r = 1,2,3, \dots$ to all natural solutions. The first non-trivial fundamental solution (x_1, y_1) for Eq. (3) can be found using the cyclic method [3], or using the slightly less efficient but more regular English method defined in [3,4]. The rest of solutions $(x_r, y_r), r = 2,3,4, \dots$ are easily computed from (x_1, y_1). There are another methods to find this fundamental solution, in this paper we have used a continued fraction method for a real number $\sqrt{d}$ see remark (2.6), (For further details on Pell equation see [3,4,5,7]). In theorem (2.9), (x_1, y_1) has been used to give the form of finding all the rest natural solutions$(x_r, y_r), r = 2,3,4, \dots$ for Eq.(3).

2. Preliminaries:

In this section, the basic definitions, theorems and remarks which will be used in this work have been introduced.

Definition 2.1 [3]: The Diophantine equation is a polynomial equation, usually in two or more unknowns, such that only the integer solutions are sought or studied (an integer solution is a solution such that all the unknowns take integer values).

Definition 2.2 [1]: The square-free, or quadrate free integer, is an integer which is divisible by no other perfect square than 1. For example, 10 is square-free but 18 is not, as 18 is divisible by 9 $=3^2$.

Definition 2.3 [6]: The quadratic Diophantine equation of the form $x^2 - dy^2 = \pm 1$ is called a Pell's equation where d is a positive square free integer. In this paper the Pell equation of the form $x^2 - dy^2 = 1$ was discussed.

Example 2.4:

i. $x^2 - 8y^2 = 1$

ii. $x^2 - 13y^2 = 1$

iii. $x^2 - 13 = -1$

The solutions for equations (i), (ii) are given by $(x, y) = (3,1)$, $(x, y) = (649,180)$ respectively, while equation (iii) does not have any solution; it is not solvable [6].

Definition 2.5 [6]: The expression of the form

$$\mathrm{a}_0 + \cfrac{1}{\mathrm{a}_1 + \cfrac{1}{\mathrm{a}_2 + \cfrac{1}{\mathrm{a}_3 + \cfrac{1}{.+\cfrac{1}{.+\cfrac{1}{.+\cdots}}}}}}$$

where a_i's are integers, is called the continued fraction expression to any real number denoted by the notation $[a_0;\ a_1, a_2, a_3, a_4, \ldots, \overline{2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4, \ldots, 2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4, \ldots}\ ]$. This expression will be used to find the fundamental solution (x_1, y_1) for $\sqrt{d}$ in Eq. (3).

The following remark has been explained shortly the continued fraction method [2] for finding the non-trivial fundamental natural solution (x_1, y_1) to Eq. (3).

Remark 2.6: For $\sqrt{d}$ in Eq. (3) assume that $\alpha_0 = \sqrt{d}$, $a_0 = \lfloor \alpha_0 \rfloor$. In general,
$\alpha_k = a_k + \frac{1}{\alpha_{k+1}}$, $a_k = \lfloor \alpha_k \rfloor$ For $k = 0,1,2,3, \ldots$

We obtain $\sqrt{d} = [a_0;\ a_1, a_2, a_3, a_4, \ldots, \overline{2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4, \ldots, 2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \mathrm{a}_3, \mathrm{a}_4, \ldots}\ ]$

For finding (x_1, y_1), only the numbers $[a_0;\ a_1, a_2, a_3, a_4, \ldots]$ will be used such that

$$\sqrt{d} = \mathrm{a}_0 + \cfrac{1}{\mathrm{a}_1 + \cfrac{1}{\mathrm{a}_2 + \cfrac{1}{\mathrm{a}_3 + \cfrac{1}{\mathrm{a}_4 + \cfrac{1}{.+\cfrac{1}{.+\cfrac{1}{.+1}}}}}}} = \frac{x_1}{y_1}$$

Example 2.7: The continued fraction expression for

$$\sqrt{7} = 2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{8}{3} = \frac{x_1}{y_1}$$

where, $\sqrt{7} = [a_0;\ a_1, a_2, \ldots, \overline{2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots, 2\mathrm{a}_0, \mathrm{a}_1, \mathrm{a}_2, \ldots}\ ] = [2; 1,1,1,4,2,1,1,1,4, \ldots]$

Remark 2.8: A continued fraction is purely periodic with period m if the initial block of partial quotients $a_0, a_1, \dots, a_{m-1}$ repeats infinitely and no block for length less than m is repeated, and it is periodic with period m if it consists of an initial block of length n followed by a repeating block of length m. Purely periodic continued fraction $\rightarrow [\overline{a_0; a_1, \dots, a_{m-1}}]$. Periodic continued fraction $\rightarrow [a_0; a_1, \dots, a_{t-1}, \overline{a_t; a_{t+1}, \dots, a_{t+m-1}}]$
the length of the period was denoted by t .

Theorem 2.9 [6]: If (a, b) is a solution to $x^2 - dy^2 = 1$ where $a > 1$ and $b \geq 1$, then (x, y) is also a solution such that

$$x + y\sqrt{d} = \left(a + b\sqrt{d}\right)^j$$

for $j = 1, 2, 3, 4, \dots$. Similarly, if (c, k) is a solution for $x^2 - dy^2 = -1$ where $c > 1$ and $k \geq 1$, then (x, y) is also a solution such that

$$x + y\sqrt{d} = \left(c + k\sqrt{d}\right)^j$$

for $j = 1, 3, 5, 7, \dots$.

Theorem 2.10 [6]: The equation $x^2 - \mathrm{dy}^2 = 1$ is always solvable and the fundamental solution is (A_k, B_k) where $k = t$ or $2t$ and A_k/B_K is a convergent to $\sqrt{d}$. The equation $x^2 - \mathrm{dy}^2 = 1$ is solvable if and only if the period length of the continued expansion of $\sqrt{d}$ is odd. The fundamental solution is (A_k, B_k) where $k = t$ or $t + 1$.

3. The main result

3.1 Finding of the new relation

In the following steps, we give a new form that will be used to find all natural solutions (s_r, t_r), $r = 1,2,3, \dots$ for Eq. (2).

Step 1: We are looking for a natural solution $(x, y) = (s_1, t_1)$ to Eq. (2).

Solving the equation $x^2 - dy^2 = c^2 \Rightarrow x^2 = c^2 + dy^2 \Rightarrow x = \mp\sqrt{c^2 + dy^2}$.

The required natural solution is, $x = \sqrt{c^2 + dy^2}$

then $(x, y) = (s_1, t_1)$ such that $\gamma = s_1 + \sqrt{d}t_1$, $\delta = s_1 - \sqrt{d}t_1$

$$\gamma\delta = \left(s_1 + \sqrt{d}t_1\right)\left(s_1 - \sqrt{d}t_1\right) = {s_1}^2 - d{t_1}^2 = c^2$$

$$\gamma + \delta = \left(s_1 + \sqrt{d}t_1\right) + \left(s_1 - \sqrt{d}t_1\right) = 2s_1$$

By this step we have been found one natural solution (s_1, t_1) to Eq. (2) such that

$$\gamma = s_1 + \sqrt{d}t_1$$

Step 2: In general, assume that all natural solutions of Eq. (2) are $(x, y) = (s_r, t_r), r = 1,2,3, \dots$, where

$$\gamma_r = s_r + \sqrt{d}t_r , \delta_r = s_r - \sqrt{d}t_r$$

$$\gamma_r + \delta_r = s_r + \sqrt{d}t_r + s_r - \sqrt{d}t_r = 2s_r$$

$$\gamma_r\delta_r = \left(s_r + \sqrt{d}t_r\right)\left(s_r - \sqrt{d}t_r\right) = {s_r}^2 - d{t_r}^2 = c^2$$

Step 3: Assume that (x_1, y_1) is the fundamental natural solution to Eq. (3), which has been found by continued fraction expression for $\sqrt{d}$ in remark (2.6) such that

$$\alpha = x_1 + \sqrt{d}y_1 \ , \ \beta = x_1 - \sqrt{d}y_1$$

$$\alpha\,\beta = \left(x_1 + \sqrt{d}y_1\right)\left(x_1 - \sqrt{d}y_1\right) = {x_1}^2 - d{y_1}^2 = 1$$

Hereby, we obtain the fundamental natural solution$(x_1\,, y_1)$ for Eq. (3) such that

$$\alpha = x_1 + \sqrt{d}y_1$$

Step 4: All natural solutions $(x_r\,, y_r),\ r = 1,2,3,4,\ldots$ for Eq. (3) will be given, as follows: We have $\alpha = x_1 + \sqrt{d}y_1$, using theorem (2.9) for $i = 1,2,3,\ldots$ then

$$\alpha^i = (x_1 + \sqrt{d}y_1)^i$$

is a natural solution for Eq. (3) .

Step 5: From **Step1** and **Step4,** we have $\gamma = s_1 + \sqrt{d}t_1$, and $\alpha^i = (x_1 + \sqrt{d}y_1)^i$ respectively. All natural solutions $(s_r\,, t_r),\ r = 1,2,3,4,\ldots$ to Eq. (2) can be obtained from the following for

$$s_r + \sqrt{d}t_r = \gamma\alpha^i \quad where \quad i = 0,1,2,3,\ldots \qquad (4)$$

3.2 Applying the new form

In this section we have applied our a new form in some examples as follows :

Example 3.3: For find all natural solutions to $x^2 - 3y^2 = 4$ then

$x^2 - 3y^2 = 4 \ \Rightarrow\ x = \sqrt{c^2 + dy^2}$. Here $c^2 = 4,\ d = 3,$ therefore, let $y = 2 \ \Rightarrow\ x = 4$, resulting $(s_1\,, t_1) = (4,2)$ such that $\gamma = 4 + 2\sqrt{3}$.

We have $x^2 - 3y^2 = 1$.

By Step 4 we have used the continued fraction expression of $\sqrt{3}$ to find $(x_1\,, y_1)$: Assume that $\alpha_0 = \sqrt{d} = \sqrt{3} = 1.7\ and\ a_0 = \lfloor 1.7 \rfloor = 1$

$$\alpha_0 = a_0 + \frac{1}{\alpha_1} \ \Rightarrow\ \sqrt{3} = 1 + \frac{1}{\alpha_1} \ \Rightarrow\ \alpha_1 = \frac{1}{\sqrt{3} - 1}$$

$$\alpha_1 = \frac{1}{\sqrt{3} - 1} * \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{\sqrt{3} + 1}{2} \ \Rightarrow\ a_1 = \lfloor 1.3 \rfloor = 1$$

$$\alpha_1 = a_1 + \frac{1}{\alpha_2} \ \Rightarrow\ \frac{\sqrt{3} + 1}{2} = 1 + \frac{1}{\alpha_2}$$

$$\alpha_2 = \frac{2}{\sqrt{3} - 1} * \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{2\left(\sqrt{3} + 1\right)}{2} = \sqrt{3} + 1 \ \Rightarrow\ a_2 = \lfloor 2.7 \rfloor = 2$$

$$\alpha_2 = a_2 + \frac{1}{\alpha_3} \ \Rightarrow\ \sqrt{3} + 1 = 2 + \frac{1}{\alpha_3}$$

$$\alpha_3 = \frac{1}{\sqrt{3} - 1} = \frac{\sqrt{3} + 1}{2} = \alpha_1 \ \Rightarrow\ a_3 = \lfloor 1.3 \rfloor = 1$$

$$\alpha_3 = a_3 + \frac{1}{\alpha_4} \Rightarrow\ \frac{\sqrt{3} + 1}{2} = 1 + \frac{1}{\alpha_4}$$

$$\alpha_4 = \frac{2(\sqrt{3} + 1)}{2} = \sqrt{3} + 1 = \alpha_2 \ \Rightarrow\ a_4 = \lfloor 2.7 \rfloor = 2$$

And so on then: $1 = a_1 = a_3 = a_5 = a_7 = \cdots$, 2$= a_2 = a_4 = a_6 = \cdots$. Hence $\sqrt{3} = [1; 1,2,1,2,\ldots]$.

The continued fraction expression for $\sqrt{3} = 1 + \frac{1}{1} = \frac{2}{1} = \frac{x_1}{y_1}$

thus, $(x_1, y_1) = (2,1) \;\Rightarrow\; \alpha = 2 + \sqrt{3}$.

By Eq. (4) $\gamma\alpha^i = s_r + \sqrt{3}\; t_r$ where $i = 0,1,2,3, \dots$, and $r = 1,2,3, \dots$

$$\gamma\alpha^i = (4 + 2\sqrt{3})(2 + \sqrt{3})^i = s_r + \sqrt{3}\; t_r$$

For $i = 0,\ r = 1 \;\Rightarrow\; \gamma\alpha^0 = (4 + 2\sqrt{3})(2 + \sqrt{3})^0 = s_1 + \sqrt{3}\; t_1$

$$4 + 2\sqrt{3} = s_1 + t_1\sqrt{3} \;\Rightarrow\; (s_1, t_1) = (4,2)$$

For $i = 1,\ r = 2 \;\Rightarrow\; \gamma\alpha^1 = (4 + 2\sqrt{3})(2 + \sqrt{3})^1 = s_2 + \sqrt{3}\; t_2$

$$= 14 + 8\sqrt{3} = s_2 + t_2\sqrt{3} \;\Rightarrow\; (s_2, t_2) = (14,8)$$

For $i = 2,\ r = 3 \;\Rightarrow\; \gamma\alpha^2 = (4 + 2\sqrt{3})(2 + \sqrt{3})^2 = s_2 + \sqrt{3}\; t_2$

$$\gamma\alpha^2 = (4 + 2\sqrt{3})(7 + 4\sqrt{3}) = s_3 + \sqrt{3}\; t_3$$

$$= 52 + 30\sqrt{3} = s_3 + t_3\sqrt{3} \;\Rightarrow\; (s_3, t_3) = (52,30).$$

And so on for $i = 3,4,5,6, \dots$ and $r = 4,5,6, \dots$ we will get the rest of all natural solutions (s_r, t_r).

Remark 3.3: All natural pairs (m, n) such that u is a natural square number in Eq. (1) have been found by using the new form Eq.(4), the natural solution to Eq. (1) when $u = 1^2$ is only the pair $(m, n) = (1,1)$.

If $u = 2^2$ in Eq. (1), we have

$$2^2 = \frac{m^2 + n^2}{1 + mn}$$

$$4 + 4mn = m^2 + n^2 \;\Rightarrow\; m^2 - 4mn + n^2 = 4$$

Add and subtract $4n^2$, thus

$$m^2 - 4mn + 4n^2 - 4n^2 + n^2 = 4 \;\Rightarrow\; (m - 2n)^2 - 3n^2 = 4$$

This equation has infinitely many solutions (s_r, t_r), $r = 1,2,3, \dots$

for $r = 1,2,3, \dots$ then, $m - 2n = s_r$, and $n = t_r$ so

$$s_r{}^2 - 3t_r{}^2 = 4$$

is a hyperbola equation has a same form in Eq.(2).

If $u = 3^2$ in Eq. (1), thus

$$3^2 = \frac{m^2 + n^2}{1 + mn}$$

$$9 + 9mn = m^2 + n^2 \;\Rightarrow\; m^2 - 9mn + n^2 = 9$$

By multiplying both sides by 4, then $4m^2 - 36mn + 4n^2 = 36$ Add and subtract $81n^2$, gives $4m^2 - 36mn + 4n^2 + 81n^2 - 81n^2 = 36$

$$(2m)^2 - 2(2m)(9n) + 81n^2 - 77n^2 = 36$$

$$(2m - 9n)^2 - 77n^2 = 36$$

Then for $r = 1,2,3, \dots$ we obtain, $2m - 9n = s_r$, $n = t_r$ such that

$$s_r{}^2 - 77t_r{}^2 = 36$$

is a hyperbola equation has a same form Eq.(2).

In order to find all pairs $(m, n) \in N$ such that $u = 2^2, 3^2$ in Eq. (1), we will use Eq.(4). For $u = 4^2, 5^2, 6^2, \dots$ has the same form of Eq. (2).

Conclusion: We give a new form Eq.(4) to find all natural solutions to the hyperbola Eq.(2) using the Pell equation Eq.(3) and continued fraction method . By this form all natural pairs (m, n) such that u is a natural square number in Eq. (1) have been found.

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