Ibn Al Haitham Journal for Pure and Applied Science

Journal homepage: http://jih.uobaghdad.edu.iq/index.php/j/index

Using Entropy and Linear Exponential Loos Function Estimators the Parameter and Reliability Function of Inverse Rayleigh Distribution

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Article history: Received 12,October,2019, Accepted8,January,2020, Published in January 2021

Doi: 10.30526/34.1.2563

Abstract

This paper is devoted to compare the performance of non-Bayesian estimators represented by the Maximum likelihood estimator of the scale parameter and reliability function of inverse Rayleigh distribution with Bayesian estimators obtained under two types of loss function specifically; the linear, exponential (LINEX) loss function and Entropy loss function, taking into consideration the informative and non-informative priors. The performance of such estimators assessed on the basis of mean square error (MSE) criterion. The Monte Carlo simulation experiments are conducted in order to obtain the required results.

Keyword: Inverse Rayleigh distribution, Entropy loss function, LINEX loss function, Prior information.

1. Introduction

Inverse Rayleigh distribution (IRD) is one of the comprehensive and relevant lifetime model, and its applications are in reliability and survival data sets .A numerous work has been done in the literature concerning IRD. The distribution was supported by Voda in 1972,who considered its properties and consider MLE estimator for estimate its scale parameter [1]. Next, Gharraph in 1993 developed closed form expressions for the mean, mode, median, harmonic mean and geometric mean of IRD [2]. Furthermore, Soliman, Amin, and Abd-EI Aziz in 2010 estimated the parameter using different traditional and Bayesian estimation's methods [3].

The probability density function of inverse Rayleigh distribution is defined as follows [4]:

$$f(t,\theta)=\frac{2\theta}{t^3}\exp\left(-\frac{\theta}{t^2}\right), \quad t>0,\theta>0 \tag{1}$$

Where (t) is a random variable that follows IRD and θ is the scale parameter. The cumulative distribution function is given by [4]:

$$F(t,\theta)=\exp\left(-\frac{\theta}{t^2}\right), \quad t>0,\theta>0 \tag{2}$$

The reliability function of IRD is therefore defined as [4]:

$$R(t,\theta)=1-F(t,\theta)=1-\exp\left(-\frac{\theta}{t^2}\right), \quad t>0,\theta>0 \tag{3}$$

It is worth mentioning here that the variance and higher order moments not exists in this distribution.

In this section, Maximum Likelihood Estimators, Posterior density of the inverse Rayleigh parameter based on (Jeffrey's prior information, exponential prior distribution) and Types of loss functions (Entropy loss function, linear Exponential loss function) will be considered .

2. Maximum Likelihood Estimators

Let $t_1, t_2, \ldots\ldots\ldots, t_n$ be random samples drawn from the density given in equation (1), then the likelihood function is defined as

$$L(\underline{t}|\theta)=\prod_{i=1}^{n} f(t_i,\theta)=2^n\theta^n\prod_{i=1}^{n}\frac{1}{t_i^3}e^{\left[-\theta\sum_i^n\frac{1}{t_i^2}\right]} \tag{4}$$

Taking the natural logarithm for the likelihood function, we get

$$\ln L(\underline{t}|\theta)=n\ln 2+n\ln\theta+\sum_{i=1}^{n}\ln\frac{1}{t_i^3}-\theta\sum_{i=1}^{n}\frac{1}{t_i^2}$$

By differentiating the log likelihood function with respect to θ and then equating the resultant derivative to zero, we get

$$\frac{\partial \ln L(\underline{t}|\theta)}{\partial\theta}=\frac{n}{\theta}-\sum_{i=1}^{n}\frac{1}{t_i^2}=0$$

Hence, the MLE for θ denoted by $\hat{\theta}_{MLE}$ is

$$\hat{\theta}_{MLE}=\frac{n}{\sum_{i=1}^{n}\frac{1}{t_i^2}}=\frac{n}{T} \tag{5}$$

where $T=\sum_{i=1}^{n}\frac{1}{t_i^2}$

3. Posterior Density of Inverse Rayleigh Parameter Based on Jeffrey's Prior Information

Assume that θ has a non-informative prior. Applying Jeffrey's rule [5], we get

$$g(\theta)\propto\sqrt{I(\theta)} \quad \text{or} \quad g(\theta)=c\sqrt{I(\theta)}$$

Where $g(\theta)$ represents Jeffrey's prior information, C is the constant of proportionality and $I(\theta)$ represents Fisher information, defined as follows:

$$I(\theta) = -nE\left[\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right] \quad (6)$$

Therefore,

$$g_1(\theta) = c\sqrt{-nE\left(\frac{\partial^2 \ln f(t,\theta)}{\partial\theta^2}\right)} \quad (7)$$

By taking the logarithm of equation (1), we get

$$\ln f(t_i,\theta) = \ln(2) + \ln(\theta) + \ln(\frac{1}{t^3}) - \frac{\theta}{t^2}$$

$$\frac{\partial \ln f(t_i,\theta)}{\partial\theta} = \frac{1}{\theta} - \frac{1}{t_i^{\,2}}$$

Thus, the second derivative is

$$\frac{\partial^2 \ln f(t_i, \theta)}{\partial\, \theta^2} = -\frac{1}{\theta^2}$$

Hence, we get:

$$E\left(\frac{\partial^2 \ln f\,(t_i\,,\theta)}{\partial\,\theta^2}\right) = -\frac{1}{\theta^2}$$

After substitution into (7), we get

$$g_1(\theta) = \frac{c}{\theta}\sqrt{n} \qquad , \theta > 0 \quad (8)$$

The posterior density function is defined as:

$$h(\theta|\underline{t}) = \frac{g(\theta)L(\underline{t}|\theta)}{\int_0^\infty g(\theta)L(\underline{t}|\theta)} \quad (9)$$

Hence, the posterior density function for θ based on Jeffreys prior will be

$$\mathrm{h}_1(\theta|\underline{t}) = \frac{\frac{c}{\theta}\sqrt{n}\;2^n\theta^n\prod_{i=1}^n\frac{1}{t_i^{\,3}}\exp\left(-\theta\sum_{i=1}^n\frac{1}{t_i^{\,2}}\right)}{\int_0^\infty\frac{c}{\theta}\sqrt{n}\,2^n\theta^n\prod_{i=1}^n\frac{1}{t_i^{\,3}}\exp\left(-\theta\sum_{i=1}^n\frac{1}{t_i^{\,2}}\right)\,d\theta}$$

$$\mathrm{h}_1(\theta|\underline{t}) = \frac{\theta^{n-1}e^{-\theta T}}{\int_0^\infty\theta^{n-1}e^{-\theta T}d\theta} \quad ,T = \sum_{i=1}^n\frac{1}{t^2}\,, \quad \theta > 0$$

Hence, the posterior density function of θ with Jeffreys prior can be written as

$$= \frac{T^n \theta^{n-1} e^{-\theta T}}{\Gamma(n)} \tag{10}$$

The posterior density function is recognized as the density of the Gamma distribution, i.e. $(\theta|\underline{t}) \sim Gamma\left(n, \frac{1}{T}\right)$, with

$$E(\theta) = \frac{n}{T}, \quad Var(\theta) = \frac{n}{T^2} \tag{11}$$

4. Posterior Density of Inverse Rayleigh Parameter Based on Exponential Prior Distribution

Assuming that the inverse Rayleigh parameter θ follows exponential prior distribution with parameter λ [6], that is

$$g_2(\theta) = \lambda e^{-\lambda\theta}, \ \lambda > 0, \theta > 0 \tag{12}$$

Where $g_2(\theta)$ denotes the exponential prior distribution of the inverse Rayleigh parameter θ.

From Bayesian theorem the posterior density function of θ denoted by $h_2(\theta|\underline{t})$ can be obtained as

$$h_2(\theta|\underline{t}) = \frac{\theta^n e^{-\theta(T+\lambda)}}{\int_0^\infty \theta^n e^{-\theta(T+\lambda)}\, d\theta} \tag{13}$$

where $T = \sum_{i=1}^{n} \frac{1}{t_i^2}$

$$h_2(\theta|\underline{t}) = \frac{(T+\lambda)^{n+1} \theta^n e^{-\theta(T+\lambda)}}{\Gamma(n+1)} \qquad \theta > 0 \tag{14}$$

It can easily be noted that $\theta|\underline{x} \sim \text{Gamma}\left(n+1, \frac{1}{p}\right)$ where $P = T + \lambda$ with

$$\text{E}(\theta|\underline{t}) = \frac{n+1}{P}, \quad Var\,(\theta|\underline{t}) = \frac{n+1}{P^2} \tag{15}$$

5. Types of Loss Functions [7]

From the Bayesian viewpoint, the essential step in the estimation and prediction problems was represented by choosing the loss function. In fact, there is no specific analytical procedure to determine the suitable loss function to be employed. In this paper, we consider two types of loss function, the Entropy loss function and Linear Exponential loss function (LINEX), as follows:

i) Entropy Squared Loss Function which is defined as below

$$L(\hat{\theta}, \theta) = \frac{\hat{\theta}}{\theta} - ln\frac{\hat{\theta}}{\theta} - 1 \tag{16}$$

ii) Linear Exponential Loss Function (LINEX) which is defined as below

$$L(\hat{\theta}, \theta) = e^{(\hat{\theta}-\theta)} - (\hat{\theta} - \theta) - 1 \tag{17}$$

6. Bayesian Estimation

The Bayes estimator of the parameter θ is the value of θ that minimize the posterior expectation known as the risk function denoted by $R(\hat{\theta},\theta)$, that is

$$R(\hat{\theta},\theta) = E[L(\hat{\theta},\theta)] = \int_0^\infty L(\hat{\theta},\theta)h(\theta|\underline{t})d\theta \tag{18}$$

Where $h(\theta|\underline{t})$ is the posterior density of $\theta|t$

7. Bayes Estimator of Parameter θ and Reliability Function of IRD under Entropy Loss Function [8, 9].

If entropy loss function is chosen, then according to equation (18), we have

$$R(\hat{\theta},\theta) = \int_0^\infty [\frac{\hat{\theta}}{\theta} - ln\frac{\hat{\theta}}{\theta} - 1]\, h(\theta|\underline{t})d\theta$$

By differentiating $R(\hat{\theta},\theta)$ with respect to $\hat{\theta}$ and setting the resultant, derivative equal to zero, then solving for $\hat{\theta}$, we get

$$\hat{\theta}_{En} = \frac{1}{\int_0^\infty \frac{1}{\theta} h(\theta|\underline{t})d\theta} \tag{19}$$

On the basis of non-informative prior and according to equation (10), the Bayes estimator of inverse Rayleigh parameter θ denoted as $\hat{\theta}_{En_{(J)}}$ is given by

$$\hat{\theta}_{En_{(J)}} = \frac{n-1}{T} \tag{20}$$

If the inverse Rayleigh parameter follows the exponential prior distribution, then by equation (14) we conclude that

$$\hat{\theta}_{En_{(E)}} = \frac{n}{p}, \quad \text{where p=T+λ} \tag{21}$$

The estimator of the reliability function based on Jeffrey's prior can be approximated as

$$\hat{R}(t)_{En_{(J)}} \cong 1 - e^{-\frac{\hat{\theta}_{En_{(J)}}}{t^2}}$$

$$\hat{R}(t)_{En_{(J)}} \cong 1 - e^{-\frac{n-1}{Tt^2}} \tag{22}$$

The estimator of the reliability function based on exponential prior can be approximated as

$$\hat{R}(t)_{En_{(E)}} \cong 1 - e^{-\frac{\hat{\theta}_{En}}{t^2}}$$

$$\hat{R}(t)_{En_{(E)}} \cong 1 - e^{-\frac{n}{pt^2}} \tag{23}$$

8. Bayes estimator of the parameter θ and reliability function under LINEX

By substituting from $L(\hat{\theta},\theta)$ given in equation (17) into equation (18), we get

$$R(\hat{\theta},\theta) = \int_0^{\infty} [e^{(\hat{\theta}-\theta)} - (\hat{\theta}-\theta) - 1] h(\theta|\underline{t}) d\theta$$

By simplification, we get

$$e^{\hat{\theta}} \int_0^{\infty} e^{-\theta}\, h(\theta|\underline{t}) d\theta = 1$$

By differentiating $R(\hat{\theta},\theta)$ with respect to θ then equating the resultant derivative to zero and solving for $\hat{\theta}$, we get the Bayes estimator of θ under linear Exponential loss function denoted by $\hat{\theta}_L$ as follows

$$\hat{\theta}_{LE} = -ln \int_0^{\infty} e^{-\theta}\, h(\theta|\underline{t}) d\theta$$

on the basis of non-informative prior, the Bayes estimator of the inverse Rayleigh parameter θ denoted as $\hat{\theta}_{LE_{(J)}}$ is given by

$$\hat{\theta}_{LE_{(J)}} = -\ln\left(\frac{T}{1+T}\right)^{n} \quad (24)$$

If the inverse Rayleigh parameter follows the exponential prior distribution $\hat{\theta}_{LE_{(E)}} = -\ln\left(\frac{P}{1+P}\right)^{n+1}$, (25)

the estimator of the reliability function based on Jeffrey's prior can be approximated as $\hat{R}(t)_{LE_{(J)}} \cong 1 - e^{-\frac{\hat{\theta}_{LE_{(J)}}}{t^2}}$

Which implies that

$$\hat{R}(t)_{LE_{(J)}} \cong 1 - e^{\frac{-ln\left(\frac{T}{1+T}\right)^{n}}{t^2}}$$

$$\hat{R}(t)_{LE_{(J)}} \cong 1 - \left(\frac{T}{1+T}\right)^{\frac{n}{t^2}} \quad (26)$$

The estimator of the reliability function based on exponential prior can be approximated as

$$\hat{R}(t)_{LE_{(E)}} \cong 1 - e^{-\frac{\hat{\theta}_{LE_{(E)}}}{t^2}}$$

Which implies that

$$\hat{R}(t)_{LE_{(E)}} \cong 1 - e^{\frac{-ln\left(\frac{P}{1+P}\right)^{n+1}}{t^2}}$$

$$\hat{R}(t)_{LE_{(E)}} \cong 1 - \left(\frac{P}{1+P}\right)^{\frac{n+1}{t^2}} \quad (27)$$

10. Simulation Study

In our simulation study, number of repetitions L=2000 sample of size n=10,50,100 and 200 are generated in order to represent, small, moderate, large and very large sample sizes from inverse Rayleigh distribution with two values of the scale parameter ($\theta = 0.5$, $\theta = 1.5$), the scale parameter λ of exponential prior was chosen to be (λ=0.5, λ=1) and mean square error (MSE) is employed to compare the performance of different methods for estimation of the scale parameter and reliability function of IRD where

$$MSE(\hat{\theta}) = \frac{1}{L}\sum_{i=1}^{L}(\hat{\theta}_i - \theta)^2 \tag{28}$$

$$MSE[\hat{R}(t)] = \frac{1}{L}\sum_{i=1}^{L}[\hat{R}_i(t) - R(t)]^2 \tag{29}$$

The results are presented in the following **Tables (1-8).**

Table 1. MSE for parameter θ by using Jeffrey's prior information at $\theta = 0.5$

n / Estimator	10	50	100	200
MLE	0.0043	0.0001082	0.000026176	0.000006513
Ent	0.0033	0.0001019	0.000025396	0.000006412
Lin	0.0250	0.0050	0.0025	0.0013
Best	Ent	Ent	Ent	Ent

Table 2. MSE values of the Reliability function estimators by using Jeffrey's prior information at $\theta = 0.5$

n / Estimator	10	50	100	200
MLE	0.000321	0.0000101	0.0000024924	0.0000006246
Ent	0.000269	0.0000097	0.0000024460	0.0000006185
Lin	0.000283	0.0000098	0.0000024635	0.0000006209
Best	Ent	Ent	Ent	Ent

Table 3. MSE for parameter θ by using exponential prior information at $\theta = 0.5$

Estimator \ n		10	50	100	200
Ent	$\lambda = 0.5$	0.2810	0.0472	0.0231	0.0114
	$\lambda = 1$	0.2968	0.0479	0.0232	0.0114
Lin	$\lambda = 0.5$	0.0250	0.0050	0.0025	0.0013
	$\lambda = 1$	0.0250	0.0050	0.0025	0.0013
Best		Lin	Lin	Lin	Lin

Table 4. MSE values of the Reliability function estimators by using Exponential prior information at $\theta = 0.5$

Estimator \ n		10	50	100	200
Ent	$\lambda = 0.5$	0.0090	0.0018	0.00087303	0.00043568
	$\lambda = 1$	0.0094	0.0018	0.00087669	0.00043659
Lin	$\lambda = 0.5$	0.000317	0.0000101	0.0000025078	0.0000006267
	$\lambda = 1$	0.000278	0.0000098	0.0000024731	0.0000006223
Best		Lin	Lin	Lin	Lin

Table 5. MSE for parameter θ by using Jeffrey's prior information at $\theta = 1.5$

Estimator \ n	10	50	100	200
MLE	0.0391	0.00097346	0.00023557	0.00005862
Ent	0.0293	0.00091708	0.00022856	0.000057715
Lin	0.2250	0.0450	0.0225	0.0113
Best	Ent	Ent	Ent	Ent

Table 6. MSE values of the Reliability function estimators by using Jeffrey's prior information at $\theta = 1.5$

n / Estimator	10	50	100	200
MLE	0.000691	0.000024581	0.0000061457	0.0000015451
Ent	0.000647	0.000024264	0.0000061043	0.0000015393
Lin	0.000551	0.000023534	0.0000060124	0.0000015278
Best	Lin	Lin	Lin	Lin

Table 7. MSE for parameter θ by using exponential prior information at $\theta = 1.5$

n / Estimator		10	50	100	200
Ent	$\lambda = 0.5$	0.0660	0.0137	0.0069	0.0035
	$\lambda = 1$	0.0584	0.0134	0.0068	0.0034
Lin	$\lambda = 0.5$	0.2250	0.0450	0.0225	0.0113
	$\lambda = 1$	0.2250	0.0450	0.0225	0.0113
Best		Ent	Ent	Ent	Ent

Table 8. MSE values of the Reliability function estimators by using exponential prior information at $\theta = 1.5$

n / Estimator		10	50	100	200
Ent	$\lambda = 0.5$	0.00310	0.00061919	0.00030974	0.00015491
	$\lambda = 1$	0.00260	0.00060029	0.00030498	0.0000015371
Lin	$\lambda = 0.5$	0.000474	0.000022883	0.0000059292	0.000001517
	$\lambda = 1$	0.000423	0.00002227	0.0000058461	0.000001506
Best		Lin	Lin	Lin	Lin

11. Simulation Results and Conclusions

From our simulation study, the following conclusions are pointed out :

1. When $\theta = 0.5$, the Bayes estimators of the scale parameter and reliability function under Entropy loss function with Jeffrey's prior is the best for all cases as shown in

tables (1−4). While the estimators under Linear Exponential loss function (LINEX) are the best when the prior information is exponential.

2. At $\theta = 1.5$, the Bayes estimator of the scale parameter best on Entropy loss function is the best for all cases, while the Bayes estimator of the reliability function under Linear Exponential loss function (LINEX) is the best for all cases as shown in tables (5−8).

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