التقريب العددي لمسألة القطع المكافئ المضطرب بشكل فردي من نوع انتشار الحمل الحراري مع حالة اولية متقطعة

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المستخلص:

تم فحص مشكلة القطع المكافئ المضطرب بشكل فردي من نوع انتشار الحمل الحراري مع حالة أولية متقطعة. يتم تحديد دالة خطأ مجانية معينة تتوافق مع الانقطاع في الحالة الأولية. يتم تقريب الفرق بين هذه الوظيفة التحليلية وحل مشكلة القطع المكافئ عدديًا. يتم استخدام التحويل الإحداثي بحيث يمكن محاذاة الشبكة المكيفة للطبقة مع الطبقة الداخلية الموجودة في المحلول. تم تقديم التحليل العددي للطريقة العددية المرتبطة بها، والتي تثبت أن الطريقة العددية هي طريقة عددية موحدة المعلمة. تم عرض النتائج العددية لتوضيح حدود الخطأ النقطية المحددة في البحث.

الكلمات المفتاحية: انتشار الحمل الحراري – الحالة الأولية المتقطعة – تقريب المعلمة الموحدة.

Numerical approximations for a singularly perturbed parabolic problem of convection-diffusion type with a discontinuous initial condition

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Abstract:

A individually turbulent parabola problem of convective diffusion type with a discontinuous initial state is investigated. A certain free error function corresponding to the discontinuity in the initial state is determined. The

difference between this analytical function and the solution of the parabola problem is approximated numerically. A coordinate transformation is used so that the layer-adapted mesh can be aligned with the inner layer in the solution. Numerical analysis of the associated numerical method is presented, which proves that the numerical method is a unified numerical method. Numerical results are presented to illustrate the point error limits determined in the research .

Keywords: Convection diffusion- Discontinuous initial condition - Parameter-uniform approximations.

1. Partial differential equations

In the real world, many physical processes and phenomena are studied by mathematical models, often by partially derivative equations. A differential equation containing more than one independent variable is called a partial differential equation. In fact, partial differential equations appear in problems that contain the tracks of changing functions with several independent variables. The general form of a partial differential equation for the function $u(x_1, x_2, \ldots, x_n)$ is as follows:

$$F\left(x_1, \ldots, x_n, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1^2}, \ldots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \ldots\right) = 0 \quad \ldots \quad (1)$$

Where, independent variables and dependent $x_1, x_2, \ldots, x_n u$ variables. If in equation (1) The dependent variable and its derivatives appear linearly, then the equation is linear and otherwise it is called nonlinear. The highest derivative order in these equations, the order of the equation and the number of independent variables are called equation dimensions.[1]

Definition 1. An n-order linear differential operator is a function that is generally one-dimensional as

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(1) A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, N.J., 1964.

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$$L = \frac{d^n}{dx^n} + P_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + P_{n-1}(x)\frac{d}{dx} + P_n(x) \quad \ldots \quad (2)$$

And in two-dimensional mode,

$$L = \frac{d^n}{dx^n} + P_1(x,y)\frac{d^{n-1}}{dx^{n-1}} + q_0(x,y)\frac{\partial^n}{\partial y^n} + q_1(x,y)\frac{\partial^{n-1}}{\partial y^{n-1}}$$

$$+ r_1(x,y)\frac{\partial^{n-1}}{\partial x^m \partial y^{n-1-m}} + \cdots + r_{n-2}(x,y)\frac{\partial^2}{\partial x \partial y}$$

$$+ P_{n-1}(x,y)\frac{d}{dx} + q_{n-1}(x,y)\frac{\partial}{\partial y} + P_n(x,y) \quad \ldots \quad (3)$$

Defined. If differential operators are defined in (1) and (2) Act on a function, respectively, producing a differential equation with partial linear derivatives of one-dimensional order and a partially linear M-order equation nn.

The linear operator has fixed numbers and functions and has $L\alpha\beta uv$ the following properties:

$$L(\alpha u + \beta v) = \alpha L(u) + \beta L(v).$$

Definition 2. If a partial differential equation is linear only relative to the highest available derivative order, it is called pseudolinear.

Now, due to the importance of second-order partial differential equations, we investigate them. The general form of a differential equation with partial linear derivatives is two-dimensional second order as follows:

$$a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + c\frac{\partial u}{\partial y} + fu = g$$

where the functions of the variables are $a, b, c, d, e, f, g x, y$. The equation above is a homogeneous equation, whenever it is zero per and equal $g(x,y) xy.^2$

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A classification for partially differential equations of the second order two-dimensional order, which is performed in terms of abc coefficients, is as follows:

- If the equation is called elliptical $b^2 - 4ac < 0$.
- If the equation is parabolic $b^2 - 4ac = 0$.
- If, the equation is called hyperbolic $b^2 - 4ac > 0$.

2. Elliptical Equations

Elliptical equations usually correspond to stable state behavior. The general shape of an elliptical equation is as follows:

$$-\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu = f \quad \ldots \quad (4)$$

In which the matrix prevalences are the opposite of zero and also symptomatic $(i, j = 1,2, \ldots, n) A = (a_{ij})$.

In order to solve the elliptical equation (4. 1) In a physical amplitude in which the equation is established, a series of boundary conditions are applied to the equation in order for a boundary value problem to be obtained 4. In this case, if the function is found so that it applies to the desired domain in the equation and establishes the boundary conditions, then an analytical answer to the question will be the assumed boundary value. For the equation with the amplitude enclosed with the border, the types of boundary conditions are defined as follows:

- Dirichlet's bet:

u on the border has a known value, i.e., Γ

$$u = \bar{u} \; on \; \Gamma.$$

- Newman's:

A normal derivative means that on the border it has a known value, i.e., $u\frac{\partial u}{\partial n} = n. \nabla u \Gamma$

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$$q = \frac{\partial u}{\partial n} = \bar{q} \; on \; \Gamma.$$

• Rubin's bet or mixed:

The combination of and its normal derivative on the border has a known value $u\frac{\partial u}{\partial n} + \alpha u \Gamma$.

Poisson and Helmholtz equations are specific scenarios of elliptical equations that are as follows:

$$\nabla^2 u = f, \; f \neq 0, \; and \; \nabla^2 u + \lambda u = f, \; f \neq 0,$$

In which the operator is called a laplasine operator ∇^2 and acts as follows in the Cartesian coordinates on the function (an open set is in $u: \Omega \to R \Omega R^n$ the door):

for one-dimensional mode, i.e., for $n = 1$

$$\nabla^2 u = \frac{d^2 u}{dx^2},$$

for 2D mode, i.e., for $n = 2$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2},$$

And for the next general state, that is, to $Nn = N$ and $\nabla^2 u = \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}.$

definition 3 . If a regular one is integers, it is insurmountic $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)d.$ Then it's called a few andys α. We show the size and $\alpha |\alpha|$ it is defined as follows:

$$|\alpha| = \sum_{i=1}^{d} \alpha_i.$$

Also, if they belong and, then, respectively, are defined as $\alpha, z R^d \alpha = (\alpha_1, \dots, \alpha_d), z = (z_1, z_2, \dots, z_d) \alpha! z^\alpha$ follows:

$$\alpha! = \alpha_1! \, \alpha_2! \dots \alpha_d!,$$
$$z^\alpha = z_1^{\alpha_1} z_1^{\alpha_2} \dots z_1^{\alpha_d}.$$

Also, for - Amin is a partial derivative as follows $|\alpha| \le m\alpha \; D^\alpha v(x) = \frac{\partial^{|\alpha|} v(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_{1d}^{\alpha_d}}.$

Consider where the partial elliptical differential operator is linear from order as $L2m$ follows:

$$Lu = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha \left(\sum_{|\beta| \le m} a_\alpha \beta(x) D^\beta u \right), \qquad x \in \Omega \subset R^n \dots (6)$$

Border operators $B_0, B_1, \dots, B_{m-1}$ are also defined as follows:

$$B_j u = \sum_{|\alpha| \le q_j} b_\alpha^{(j)} D^\alpha \quad . \quad . \quad . \quad (7)$$

In second order mode, (6) and (7) Shape (4) With only border condition

$$Bu = \sum_{j=1}^{n} b_j \frac{\partial u}{\partial x_j} + cu = g \; on \; \Gamma,$$

convert. The differential operator displays the adjunct with the symbol and we have LL^*:

$$L^* u = \sum_{|\alpha| \le m} (-1)^{|\alpha|} D^\alpha \left(\sum_{|\beta| \le m} a_{\beta\alpha}(x) \, D^\beta u \right),$$

As we know, we have green formula:

$$\int_\Omega (D^\alpha u) v dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha v dx + \int_\Gamma h(u, v) \, ds.$$

Now, if we apply Greene's formula to the integral, then $\int_\Omega v L u dx$

$$\int_\Omega vLudx = \int_\Omega vL^*vdx + \int_\Gamma h(u,v)\,ds \quad \dots \quad (8)$$

In which the $h(u,v)$ sentences are borderline, which is achieved by applying the green formula. If it is, the operator is called annexation itself $L = L^*a_{\alpha\beta} = a_{\beta\alpha}L$.

Example 1. Two-order differential operator $L = -\dfrac{d^2}{dx^2} + 1$,

Consider. Using component-to-component integration we have for $\Omega \in (0,1)$:

$$\int_0^1 vlu = \int_0^1 \left(-v\frac{d^2u}{dx^2} + vu\right)dx = -\left[v\frac{du}{dx}\right] + \int_0^1 \left(\frac{dv}{dx}\frac{du}{dx} + vu\right)dx$$

$$= -\left[v\frac{du}{dx}\right]_0^1 + \left[\frac{dv}{dx}u\right]_0^1 - \int_0^1 \left(\frac{d^2v}{dx^2} + v\right)udx.$$

$$\int_0^1 v\left(-\frac{d^2u}{dx^2} + u\right)dx = \underbrace{\left[-v\frac{du}{dx} + \frac{dv}{dx}u\right]_0^1}_{h(v,u)} + \int_0^1 \underbrace{\left(-\frac{d^2v}{dx^2} + v\right)}_{L^*v}udx,$$

Consequently, it means the operation of the appendix itself $L = L^*$.

$$F\mathscr{L} = -\varepsilon F_{xx} + \kappa F_x + gF_t = -\varepsilon g\hat{F}_{ss} + \left(\sqrt{g\kappa} + g\frac{\partial s}{\partial t}\right)\hat{F}_s + g\hat{F}_t$$

$$= g\hat{L}_d\hat{F} + g\left(\frac{\kappa}{\sqrt{g}} + \frac{\partial s}{\partial t} - a(d,t)\right)\hat{F}_s$$

Where

$$\hat{L}_d\hat{F} := -\varepsilon\hat{F}_{ss} + \hat{a}(d(t),t)\hat{F}_s + \hat{F}_t.$$

Hence, from (3.5) and using $\hat{a}(d(t),t) = a(d,t)$ we have:

$$F\mathscr{L}(x,t) = g\hat{L}_d\hat{F} + \sqrt{g}(a(x,t) - a(d,t))\frac{\partial F}{\partial x}.$$

We will define a set of functions as $\left\{\hat{\psi}_i\right\}_{i=0}^{4}$ such that $\hat{L}_d\hat{\psi}_i = 0; \hat{\psi}_i \in C^{i-1}(\overline{\hat{Q}}), i \geq 1$. Every function in $\hat{\psi}_i$ region $\hat{Q} \setminus \Gamma^*$ is open smooth. Define two individual functions as follows [3]:

3. Parabolic equations

A set of parabolic equations depends on the time of parabolic equations. The general shape of a parabolic equation is as follows:

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left[c_{ij}(x_1, x_2, \ldots, x_n, t)\frac{\partial u}{\partial x_j}\right] = \frac{\partial u}{\partial t},$$

In which t represents the time and matrix with the valleys C of c_{ij} a true symmetric matrix in terms of place and time. The equation above can be written briefly as follows:

$\nabla.\left[C.\nabla u\right] = \dot{u},$ That represents the derivative relative to time $\dot{u}u$.

4. Hyperbolic equations

Hyperbolic equations are another class of dependent equations when they originate from vibration equations and in general, vibrating devices and wave motion are described by hyperbolic equations.

In order to solve parabolic and hyperbolic equations, in addition to applying a series of boundary conditions, the conditions of the equation at a specific time, for example□ the start time of the desired process are required. By applying these conditions to parabolic and hyperbolic equations, a primary-boundary value problem is achieved.

In this thesis, we examine the issue of dissemination-convection. Therefore, we will introduce this issue in the following. The simplest parabolic equation is the one-dimensional heat transfer equation, which is as follows:

$$\frac{\partial u}{\partial t} = \frac{k}{\sigma \rho}\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0$$
$$u(x, 0) = f(x), \qquad 0 \leq x \leq L$$

[3] A. Friedman, Partial Differential Equations of Parabolic Type, Prentice-Hall, Englewood Cliffs, N.J., 1964.

$$u(0,t) = p(t), \qquad u(L,t) = q(t), \qquad t \geq 0.$$

Where the density of the object and the constant coefficient is proportional to the special heat, the initial temperature, and the temperature at the two end of the rod and the temperature of the rod at the point of length and at the moment $\rho \sigma f(x) p(t) q(t) u(x,t) x t$. Although the above problem is known as heat problem because of the modeling of heat transfer problem, but this problem has various applications such as solving infiltration problems such as water penetration inside the earth, measuring the amount of existing water and oil mines in underground resources, etc. so such issues are called intrusion problems. An infiltration equation is a partial differential equation that describes the change in density in a material that has been intrusive. If the area is our intended area as well as the boundary of this region, then the linear standard form of an infiltration equation can also be expressed as follows: $\Omega \ \partial\Omega$

According to the problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 \leq x \leq 1, t \geq 0.$, the initial and boundary conditions can be considered as follows:

$$u(x,0) = g(x), \qquad 0 \leq x \leq 0,$$
$$u(0,t) = \varphi_0(t), \qquad u(1,t) = \varphi_1(t), \qquad t \geq 0,$$

where $g(0) = \varphi_0(0), g(1) = \varphi_1(0)$. Equation (9) The simplest form of an infiltration equation can be generalized in different ways. As an example,

1. By adding multiple spatial variables instead of a spatial variable

$$\frac{\partial u}{\partial t} = \nabla^2 u,$$

In which it can change in a way that actually determines the dimension of the problem $uu = u(x,y,z,\ldots,)$.

2. By adding the phrase as follows

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x,t)$$

3. Considering the diff propagation coefficient in the equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[D(x) \frac{\partial u}{\partial x} \right]$$

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That is a derivative function so that, $\text{per} D(x) 0 < D(x) < \infty, x \in \Omega$. If the propagation coefficient of the equation depends on the density value, that is, then the equation is nonlinear and otherwise the equation will be linear $D(x, u)$. If the propagation coefficient of the equation is constant, then the equation is reduced to the following linear equation.

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}.$$

If convection is used in the equation in partial differential equations in addition to diffusion expression, the convection equation becomes the diffusion-convection equation. The diffusion-convection equation is a partial parabolic differential equation describing those physical phenomena in which the energy converted into a physical system is caused by two processes of propagation and convection. Convection means the movement of molecules within the fluid, while the word diffusion describes the expansion of particles from high concentration regions to low-concentration regions.

The simplest mathematical model is a diffraction-convection problem of the boundary value problem as follows:

$$-\epsilon u''(x) - b(x)u' + a(x)u(x) = f(x), \qquad 0 < x < l$$

With boundary conditions in which the positive parameter is too small and the given functions correspond to the convection factor, the answers are differential equations corresponding to the propagation factor and coefficient $u(0) = u(l) = 0 \epsilon (0 < \epsilon \le 1) abf uuuu'$. The coefficient of function is known to the right side f. When there are both penetration and convection factors in a differential equation, we say that the problem is diffusion-convection. In a diffusion -convection problem, the parameter that describes the relative impact of convection and propagation components is called the packet number, i.e., we have: $Pe = \frac{bl}{\epsilon}$.

Applications of partial differential equations, in addition to radiate, are abundant in engineering, economics and other sciences. There are different methods for solving differential equations with partial derivatives, including

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finite difference method, finite elements method, boundary elements method and non-network methods. Using finite difference method and finite elements to solve equations requires amplitude segmentation, which increases the computational volume, especially in 3D mode. These divisions make it difficult   to implement them on equations with irregular geometric regions with two and higher dimensions, and the matrix obtained from internalization in these methods is much larger and the volume of operation increases. On the other hand, inappropriate discrete selection can lead to a big error in the answer and also the convergence of the answer will be slow. In contrast to amplitude methods, methods such as the border elements method, first founded by Berbia in 1978, have a border nature and only rely on border division.   This feature reduces the dimension of the problem to the size of a unit and greatly reduces the volume of the device of the equations and significantly   reduces the number of computational operations.   In the boundary elements method, the given differential equation is first converted to an integral equation using certain functions under the title of the basic solution, and then the integral equation is solved by numerical   methods. The specific form of the basic answer causes the creation of individual   integrals, which their calculation requires the use of numerical integration rules of high order.   The method of boundary elements in solving Poisson equations causes amplitude integral   in integral equations.   Therefore, one of the most general methods without the need for domain networking   called the Dual Reciprocity Method was proposed by Berbia and Nardini in 1982. Then, a number of researchers have been used in various fields of engineering in which the amplitude integral of heterogeneous sentence is converted to several boundary integrals, and by finding the approximate private answer, the Poisson equation is converted to a Laplace equation and the Laplace equation is solved by discrete on the boundary with the boundary elements method which requires calculating individual integrals along the boundary.   Solving these single integrals is easy for two-dimensional problems, but it is costly for 3D problems. Novak and Berbia also proposed a multidimensional reciprocity method that could more effectively convert amplitude integrals   into boundary   integrals

and then generalized to unstable problems as well as Helmholtz equation. One of the disadvantages of the boundary elements method is that it requires solving individual integrals. In addition, the boundary elements method is suitable only for equations for which the basic answer (weight function) exists, while calculating the weight function is not possible for all equations. In 1964, basic answer methods were proposed by Coprodaz and Alexidez to deal with this problem. This method is basically a network less method and belongs to the general category of congested border methods in which there is no border or domain segmentation. In general , non-network methods are divided into two categories: amplitude and boundary methods, the basic response method belongs to the second category. In the basic answer method, introducing an artificial boundary including the main boundary and selecting the source points on it prevented the single baseline from being singled out, and the answer to the problem is expressed as a combination of the basic unspeakable answers, which is precisely the case in the main differential equation, in which the calculation of individual integrals used in the method of boundary elements and the dual confrontation method is avoided. This method is an efficient numerical method for solving two and three-dimensional Laplace equations. In recent years, this method has also been used to solve Poisson equations by finding private answers for heterogeneous sentences. In this method, the answer is divided into two parts: homogeneous and private answer. By approximation of heterogeneous sentence by radial basis functions, an approximate private answer is obtained. Depending on the type of basic radial functions selected on the basic solutions for solving the nonlinear Poisson equation, the answer is divided into two parts: homogeneous solution and private solution, or by quasilinear techniques for approximation of heterogeneous sentence, they convert the Poisson equation into homogeneous equations for which the basic answer exists. The basic solutions method for solving nonlinear Poissen equations on which Newman's conditions dominate has low accuracy. Therefore, to improve this method Ramachandran and Carver in 1998, they used the concept of Bosani internation using radial basis functions to approximate heterogeneous sentence. This method uses two

categories of independent linear radial functions to approximate heterogeneous sentence and in addition to the value of the function, derivative information of the function is also used.   Methods without a domain network operate based on the internalization of the unknown function at boundary and amplitude points. A variety of methods without a domain network   can be referred to as symmetric local method and asymmetric co-location method.   The asymmetric co-local method was proposed by Kanza in 1990.   In this method, the answer function is expressed as a linear combination of radial base   functions.   Although asymmetric local method is a very simple method, the resulting internode matrix may be individual.   To solve this problem, a symmetric method was proposed by Faschayor in 1997, which is based on hermit intern mum, and the resulting internode matrix is symmetrical and specifically positive and thus inhospitable.   In this method, the answer function is expressed in terms of   a set of radial base functions and their modified findings under the influence of differential operators in the equation. Therefore, this method will have better results than asymmetric local method for equations governed by Newman's conditions.[4]

5. Radial basic function and their properties

Interlimency using radial base functions as a method without network is a powerful tool in multivariate approximation theory. In this section, radial base functions, its properties and internalization are investigated using these functions.

Definition 4.   The radial function is called a radial function$\phi: R^d \rightarrow R$ provided that there is a one-variable function such as that$\varphi: [0, \infty) \rightarrow R$

$$\phi(x) = \varphi(r), \qquad r = \|x\|$$

And the common soft zinc is normally$\|.\| R^d$ considered Euclidean soft. So, we have a radial function for

$$\|x_1\| = \|x_2\| \Longrightarrow \phi(x_1) = \phi(x_2), \qquad x_1, x_2 \in R^2.$$

(4) G.I. Shishkin, L.P. Shishkina, Difference Methods for Singular Perturbation Problems, CRC Press, 2009.

Definition 5. (Radial basic functions). Radial basic functions are a particular type of radial functions and are generally defined as independent linear functions and are called center points $\phi_{j(x)} = \varphi(\|x - x_j\|)\phi_j = j = 1,2, \dots, N, \{x_j\}$. One of the properties of these functions is being reliable under transitions, rotations and reflections. Also, the usual variable $r = \|x - x_j\|\square$ is used in radial base functions and is soft Euclidean distance. Radial basis functions are the natural generalization of one-variable polynomial splines to multivariate state. When radial base functions were used as basic functions for intermingling multivariate data, they showed favorable characteristics, such as high performance and good quality of radial base functions that inherently have the ability to work with scattered data. Another advantage is that they have a higher accuracy order than traditional methods on a distributed distribution of points. Radial base functions in terms of positivity are classified into two categories of positive functions and conditional positive functions, in terms of smoothness to infinitely smooth functions and smooth piece functions, and in terms of type of support to functions with a global support and compact support. Common radial functions include fully$\square$ smooth radial basis functions such as gaussian, reverse square, multiple inverse squares, multiple square, and smooth piece radial functions such as linear, cubic and spline with narrow bar.

Radial function	Bailiff $\varphi(r)$
Gossip	$e^{-(\epsilon r)^2}$
Square Gaussian	$\dfrac{1}{1 + (\epsilon r)^2}$
Inverse multiple squares	$\dfrac{1}{\sqrt{1 + (\epsilon r)^2}}$
Multiple squares	$\sqrt{1 + (\epsilon r)^2}$

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Linear	r
Cubic	r^3
Spline with slim bar	$r^2 \log(r)$

Definition: (Conditional number). Quantity is called a conditional matrix number in which the function is soft$\|A\|.\|A^{-1}\|A\|.\|$. The larger the conditional number of matrix A, the weaker the device is, and the closer this number gets to one, the closer the device becomes from the primitive state and closer to the good condition.

Oppah: (Positive semi-modal matrix and positive modal matrix). The symmetric matrix A is called semi-specific positive, whenever we have a zero-fitr for each vector$x \in C^n: x^T AX \geq 0$

That's the vector$x^T x'$ song. Whenever it is strictly established, a given matrix A is positive.

Definition 8. (Smooth function). The function is called smoothly$\phi: R^s \to R$ whenever its derivatives are available from any order.

Definition 9. (Function pressed with support). If the function on the region is defined as being the opposite of zero only at points and is outside of zero, i.e., it is closed and bound, then it is a function with a compact support$u\Omega k \subset \Omega, kk\bar{k}k\Omega u.$

Definition: (Certain positive radial function). The radial function on a given is called positive$\phi: [0, \infty) \to RR^d$ if and only if separated for each finite set and from two-to-two points, the matrix$X = \{x_1, x_2, \ldots, x_N\}X \subset R^d$ and

$$A_{\phi,X} = \left(\phi\|x_j - x_k\|\right)_{1 \leq j,k \leq N} \in R^{N \times N}$$

The specified set of functions is positively displayed PD_d with.

As we know, a suitable solution for solving scattered data problems is to construct a function that is a linear combination of certain pie functions. Radial base interlimency for assumed values at points of a linear composition in the form of$f_i = f(x_i)x_i, i = 1,2,\ldots,N$

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$$s(x) = \sum_{j=1}^{N} c_j \phi\left(\|x - x_j\|\right), \qquad x \in R^2 \quad \ldots \quad (10)$$

It is in which coefficients are achieved by applying internalization conditions $c_j s(x_j) = f(x_j)$. Relationship (10) Can be written as linear equations machine $Ac = f$:

$$\begin{pmatrix} \phi(\|x_1 - x_1\|) & \phi(\|x_1 - x_2\|) & \ldots & \phi(\|x_1 - x_N\|) \\ \phi(\|x_2 - x_1\|) & \phi(\|x_2 - x_2\|) & \ldots & \phi(\|x_2 - x_N\|) \\ \vdots & \vdots & \ddots & \ddots \\ \phi(\|x_N - x_1\|) & \phi(\|x_N - x_2\|) & \ldots & \phi(\|x_N - x_N\|) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix}$$

$$= \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_N) \end{pmatrix} \ldots (11)$$

In which the matrix is the internalization and vector and the unknown Afc vector. Now, if the modal interning function is positive, then the internode matrix is positive for any finite number of distinct points and therefore inhospitable to the solvability of the device (11). Guarantees. Therefore, we are looking for functions that have the properties listed above. In the following, we will discuss the definitions and theories that help us to obtain such functions.

Definition: (Fully function) function on the interval is called completely the same whenever $\psi: [0, \infty) \to R[0, \infty)$

1. $\psi \in C[0, \infty)$
2. $\psi \in C^{\infty}[0, \infty)$
3. $l \in N \cup \{0\}, \; r > 0, (-1)^l \psi^{(l)}(r) \geq 0$

Theorem 1. If it is completely uneven but not fixed, it is positive for each set of separate matrix points with certain doors $\psi(r) = \phi(\sqrt{r})[0, \infty)\{x_i\}_{i=1}^{N} A a_{i,j} = \phi(\|x_i - x_j\|)$.

From the above theorem, it is concluded that gaussian radial base functions, inverse squares and multiple squares are positive inverse because they are positive for each

$$l = 1, 2, \ldots, \qquad r > 0,$$

$$\psi(r) = \phi(\sqrt{r}) = e^{-\varepsilon^2 r} \Rightarrow (-1)^l \psi^{(l)}(r) = \varepsilon^{2l} e^{-\varepsilon^2 r} > 0,$$

$$\psi(r) = \phi(\sqrt{r}) = \frac{1}{1 + \varepsilon^2 r} \Rightarrow (-1)^l \psi^{(l)}(r) = \frac{l!\, \varepsilon^{2l}}{(1 + \varepsilon^2 r)^{l+1}} > 0.$$

$$\psi(r) = \phi(\sqrt{r}) = \frac{1}{\sqrt{1 + \varepsilon^2 r}} \Rightarrow (-1)^l \psi^{(l)}(r) = \frac{\Gamma\left(l + \frac{1}{2}\right) \varepsilon^{2l}}{\sqrt{\pi}(1 + \varepsilon^2 r)^{l + \frac{1}{2}}} > 0.$$

Also, for multiple and linear square radial basis functions, since we have for and therefore from the above theorem, it $r \geq 0 \psi(r) > 0 \psi'(r) > 0$ cannot be concluded that these two specific functions are positive.

Theorem 2. If it is completely uneven and not fixed on the surface, it is positive for each set of separate points of the matrix with certain doo $\psi(r) = \phi(\sqrt{r}) \in C^0[0, \infty] \psi(r) > 0 \, r > 0 \psi'(r)(0, \infty)\{x_i\}_{i=1}^N A a_{i,j} = \phi(\|x_i - x_j\|)$. In addition, for the matrix, it has a negative specific value and a positive value $N \geq 2N - 1$.

With the help of this theorem, multiple square radial basis functions and certain linearity are positive. Now, despite all the cases and tools expressed above, there are functional and useful radial basis functions that do not apply to the said conditions and therefore are not certain to be positive, such as the Radial Basis Function of Spline with a narrow bar. In this case to ensure the reversibility of the device (11) Should be a polynomial with a specific maximal degree in the phrase

$$s(x) = \sum_{j=1}^{N} c_j \phi \left(\|x - x_j\| \right) \ . \ . \ (12)$$

Add. Suppose it represents the space of the variable polynomials P_m^d d and the maximum degree, which is called the order of the pie function $m - 1 m = m(\phi)\phi$. By choosing the base for this space, then the relationship $\{p_1, p_2, \ldots, p_q\} q$

$$q = \binom{m - 1 + d}{d} \ . \ . \ . \ . \ (13)$$

it is obtained. Now by adding new sentences, the phrase (12. 1) Converts as follows:

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$$s(x) = \sum_{j=1}^{N} c_j \phi\left(\|x - x_j\|\right) + P_{m-1}(x), \quad p_{m-1} \in P_m^d \dots (14)$$

$$P_{m-1}(x) = \sum_{k=1}^{q} \gamma_k p_k(x).$$

Now there is a degree of freedom that is homogeneous with the equation qq

$$\sum_{j=1}^{N} c_j \, p_k(x_j) = 0, \qquad k = 1, 2, \dots, N,$$

This degree of freedom of question q disappears. Therefore, coefficients by international conditions and constraints c_j, γ_k, (15) It is determined that the symmetric linear system concludes the following. $\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} c \\ \gamma \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}$,

Where the matrix is in (11) Given and matrix with doors for and. $PN \times q p_k(x_j) j = 1, 2, \dots, N k = 1, 2, \dots, q$ If it is, we will have certain radial basis functions positive because if it is based on relationship $(13 m = 0 m = 0)$. 1) We have the second sentence of the phrase $(14 q = 0. 1)$ Will be removed and will have the same primary relationship radial base functions (12). 1) will lead. To check the existence and uniqueness of an answer for (16) First, we consider the corresponding homogeneous device:

$A. c + P. \gamma = 0 \quad$ and $\quad P^T. c = 0$

Here are two parts, interstming conditions and torque conditions (18. 1) Segregated. If Multiply from left in, It results immediately: $c^T. A. c = 0$. Now to guarantee the existence of an answer We need the matrix for each finite set of interning points on the subspace containing all the vectors in $(18 A X = \{x_1, x_2, \dots, N\} R^d c \in R^N)$. 1) It applies, moein is positive. It can be reinstated as follows $c^T. A. c > 0$

For X all and with written $c \in R^N - \{0\} P^T. c = 0$.

Definition: (Certain radial function of the conditional). The function on a given positive is called conditional order, and if and only if $(20 \phi: [0, \infty) \to R R^d m \phi \in CPD_d(m))$. 1) Establish for all possible selections of finite points $X \subset R^d$. CPD Represents a set of conditional positive modal functions.

Now it must be proved that we have a radial basis for each function or $\phi\phi \in CPD_d(m) - \phi \in CPD_d(m)$. Note that we have an arithmetic number set for each pair $m_1, m_2 \in N_0 N_0 m_1 \leq m_2$:

$$CPD_d(m_1) \subset CPD(m_2)$$

That is, for every positive conditional, minimal that's in $(20\phi m = m(\phi)$. $1)\phi$ Applies to the order called. Therefore, according to the order of each positive deputy, it will be zero $PD_d = CPD(0)\phi \in PD_d$.

Had to talk about solvability of devices (17. 1) and (18. 1) We'll be back. For each or directly from $(19\phi \in CPD_d(m) - \phi \in CPD_d(m)$. 1) We conclude that $c = 0$. So (17) Converts to relationship $P. \gamma = 0$. Consequently, to secure a one-way answer (17) and (18) Just need matrix dispersion P. But this property depends on the geometry of the inner points X. In fact, we should note that the matrix is sprinkled if and only for

the relationship $Pp \in P_m^d$ $P(x_j) = 0$ $1 \leq j \leq N$ $for~each$ $\Rightarrow P \equiv 0$

to be established. In this case, although an internalized P_m^d sentence can be constructed uniquely from its function values in points X. Then it is said that the set points are the ystada solver $X - P_m^d$.

Theorem : For each international problem under adverbs $(15\phi \in CPD_d(m)$. A unique answer in the form of (14). It has. Provided that the internalization points got in with the truth in $(21X. 1)$, be a monolith solver $-p_m^d$.

Note that the internalization of radial basic functions is network less. This key property is inconsistent with many other methods for interpolating scattered data, such as splines on triangular regions. Therefore, internalization of radial basic functions for network generation does not require additional data structures of the algorithm.

In fact, the implementation of the design of radial basic functions interlimency methods, for the well-distributed X data set with average size N, is a straightforward task that requires only a few standard methods of numerical linear algebra. For uneven and very large X distributed sets, a detailed preprocessing of X data communication is required. In this case, multi-stage approximation designs are suitable tools.

6-Galerkin method - three-stage wavelet for the convection equation:

For the convection equation, $\mathcal{L} = a(\partial/\partial_x)$ and therefore we have $(u_1 = au_x)$ Where a is a positive constant. We keep the spatial variable x continuous and discretize the time variable with a three-step method, we will have $(u(t + \frac{\Delta t}{3} = u(t) + \frac{\Delta t}{3}(au_2^n))$

$$u\left(t + \frac{\Delta t}{3} = u(t) + \frac{\Delta t}{3}(au_2^n\right) \text{ and } u(t + \Delta t) = u(t) + \Delta t(au_x^{n+\frac{1}{2}}) \ldots (15)$$

We put the wavelet approximation $u_j(x,t)$ in the equation (15) in the above equation and take the inner product from both sides of the equation with $\emptyset_{JI}(x)$,

$$\int_0^1 \sum_{k\cdot l} u_k(t_{n+\frac{1}{3}})\emptyset_{JK}(x)\emptyset_{JK}(x)dx = \int_0^1 \sum_{k\cdot l} u_k(t_n)\emptyset_{JK}(x)\emptyset_{Jl}(x)dx + \frac{\Delta t}{3}\left[a \int_0^1 \sum_{k\cdot l} u_k(t_n)\emptyset_{JK}^1(x)\emptyset_{Jl}(x)dx\right]$$

$$\int_0^1 \sum_{k\cdot l} u_k(t_{n+\frac{1}{2}}\emptyset_{JK}(x)\emptyset_{JK}(x)dx$$

$$= \int_0^1 \sum_{k\cdot l} u_k(t_n)\emptyset_{JK}(x)\emptyset_{Jl}(x)dx$$

$$+ \frac{\Delta t}{2}\left[a \int_0^1 \sum_{k\cdot l} u_k(t_{n+\frac{1}{2}})\emptyset_{JK}^1(x)\emptyset_{Jl}(x)dx\right] \ldots (16)$$

$$\int_0^1 \sum_{k\cdot l} u_k(t_{n+1})\emptyset_{JK}(x)\emptyset_{Jl}(x)dx$$

$$= \int_0^1 \sum_{k\cdot l} u_k(t_n)\emptyset_{JK}(x)\emptyset_{Jl}(x)dx$$

$$+ \Delta t\left[a \int_0^1 \sum_{k\cdot l} u_k(t_{n+\frac{1}{2}})\emptyset_{JK}^1(x)\emptyset_{Jl}(x)dx\right]$$

$$d_u^I := \int_0^1 \sum_k u_k(t_i)\emptyset_{JK}(x)\emptyset_{Jl}(x)dx$$

Using equation (a), equations (16) can be written as follows:

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$$d_u^{n+\frac{1}{3}} = d_u^n + \frac{\Delta t}{3}(a\, D_l^{(1)} d_u^n) \ \text{ and } \ d_u^{n+\frac{1}{2}} = d_u^n + \frac{\Delta t}{2}(a\, D_l^{(1)} d_u^{n+\frac{1}{3}}) \ \ d_u^{n+1} = d_u^n +$$

$$\Delta t(a\, D_l^{(1)} d_u^{n+\frac{1}{2}})$$

7- Galerkin method - three-stage wavelet for Berger's equation:

Here $\mathcal{L} = u(\partial/\partial_x)$ and therefore we have ($u_l + u\, u_x = v\, u_{xx}$)

Where v is a positive constant. In this part, we first keep the spatial variable x continuous and discretize the time variable with a three-step method, we have:

$$u(t + \frac{\Delta t}{3}) = u(t) + \frac{\Delta t}{3}(-u^n u_x^n + v\, u_{xx}^n) \ and \ u(t + \frac{\Delta t}{2})$$

$$= u(t) + \frac{\Delta t}{2}(-u^{n+\frac{1}{3}} u_x^{n+\frac{1}{3}} + v\, u_{xx}^{n+\frac{1}{3}})$$

$$u(t + \Delta t) = u(t) + \Delta t(-u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} + v\, u_{xx}^{n+\frac{1}{2}})$$

Now we discretize the variable x with the Galerkin-wavelet method (WGM), the result is

$$\int_0^1 \sum_{k\cdot l} u_k\left(t_{n+\frac{1}{3}}\right) \emptyset_{JK}(x)\emptyset_{Jl}(x)dx = \int_0^1 \sum_{k\cdot l} u_k(t_n)\emptyset_{Jk}(x)\emptyset_{Jl}(x)dx$$

$$+ \frac{\Delta t}{3}\left[-\int_0^1 \left(\sum_k u_k(t_n)\,\emptyset_{JK}(x)\right) \left(\sum_m u_m(t_n)\emptyset'_{Jm}(x)\right) \cdot \emptyset_{Jl}(x)dx \right.$$

$$\left. + v \int_0^1 \sum_k u_k(t_n)\emptyset''_{Jk}(x)\emptyset_{Jl}(x)dx \right] \ \dots (17)$$

$$\int_0^1 \sum_{k\cdot l} u_k\left(t_{n+\frac{1}{2}}\right) \emptyset_{JK}(x)\emptyset_{Jl}(x)dx = \int_0^1 \sum_{k\cdot l} u_k(t_n)\emptyset_{Jk}(x)\emptyset_{Jl}(x)dx$$

$$+ \frac{\Delta t}{2}\left[-\int_0^1 (\sum_k u_k\left(t_{n+\frac{1}{3}}\right)\emptyset_{JK}(x)) (\sum_m u_m(t_{n+\frac{1}{3}})\emptyset'_{Jm}(x)) \cdot \emptyset_{Jl}(x)dx + \right.$$

$$\left. v \int_0^1 \sum_k u_k(t_{n+\frac{1}{3}})\emptyset''_{Jk}(x)\emptyset_{Jl}(x)dx \right]$$

$$\int_0^1 \sum_{k\cdot l} u_k(t_{n+1})\,\emptyset_{JK}(x)\emptyset_{Jl}(x)dx = \int_0^1 \sum_{k\cdot l} u_k(t_n)\emptyset_{Jk}(x)\emptyset_{Jl}(x)dx$$

$$+\Delta t\left[-\int_0^1 (\sum_k u_k\left(t_{n+\frac{1}{2}}\right)\emptyset_{JK}(x)) (\sum_m u_m(t_{n+\frac{1}{2}})\emptyset'_{Jm}(x)) \cdot \emptyset_{Jl}(x)dx + \right.$$

$$\left. v \int_0^1 \sum_k u_k(t_{n+\frac{1}{2}})\emptyset''_{Jk}(x)\emptyset_{Jl}(x)dx \right]$$

can be written as follows:

$$d_u^{n+\frac{1}{3}} = d_u^n + \frac{\Delta t}{3}\left(-2^J \sum_k \sum_m (u_k)^n (u_m)^n \int_0^1 \emptyset_{Jl}(x)\emptyset_{Jk}(x)\emptyset'_{Jm}(x)dx\right.$$

$$+ v(2')^2 \sum_k (u_k)^n \int_0^1 \emptyset_{Jl}(x)\emptyset''_{Jk}(x)dx$$

$$= d_u^n$$

$$+ \frac{\Delta t}{3}\left[-2' \sum_k \sum_m (u_k)^n (u_m)^n \Lambda_{lkm}^{001} + v(2^j)^2 \sum_k (u_k)^n \Lambda_{lk}^{02}\right]$$

8- Order reduction method for linear chaotic single problems:

To describe this method, we first consider a single chaotic two-point boundary value problem in the following form:

$$\varepsilon\, y''(x) + a(x)\, y'(x) + b(x)y(x) = f(x) \qquad , \qquad x \in [p, q] \dots (18)$$

Where ε is a small positive parameter and β and α are known constants. Suppose $a(x\ and\ f(x)\ and\ b(x)$ are continuously differentiable functions in $[p\ ,\ q]$. In addition, we assume that $a(x) \geq M > 0$ and M are constant throughout the $[p\ ,\ q]$ interval.

9- Order reduction method for nonlinear chaotic singular problems:

In this section, we extend the order reduction method to a class of nonlinear perturbed singular two-point boundary value problems.

For this purpose, we consider a group of two-point turbulent singular nonlinear boundary value problems in the following form:

$$\varepsilon\, y''(x) + \left[a\big(y(x)\big)\right]' + b\big(x\,,y(x)\big) = f(x) \qquad , \quad x \in [p\,,q] \qquad (19)$$

Where ε is a small positive parameter and β and α are known constants. We assume that $f(x)\ and\ b(x,y)\ and\ a(y(x))$ are continuously differentiable functions in the interval $[p, q]$.

We assume: Equations (19) have the same solution, so that a boundary layer appears with a width at $x = p$.

10-Description of the order reduction method with an example:

To demonstrate the order reduction, we apply this method to solve two discrete linear and non-linear perturbed problems. These examples have been chosen because they have been extensively discussed in the cited references.

Example: Consider the following homogeneous chaotic singular problem:

$$\varepsilon\, y''(x) + y'(x) - y(x) = \circ \quad , \quad x \in [\circ\,, 1] \quad \text{and}\, y(\circ) = 1 \quad , \quad y(1) = 1$$

The exact answer to this question:

$$y(x) = \frac{[(e^{m_2} - 1)e^{m_1 x} + (1 - e^{m_1})e^{m_2 x}]}{}[e^{m_2} - e^{m_1}]$$

Which is $m_2 = (-1 - \sqrt{1 + 4\,\varepsilon})/(2\,\varepsilon)$ و $m_1 = (-1 + \sqrt{1 + 4\,\varepsilon}/(2\,\varepsilon)$.

From step 1, the simplified problem is as follows:

$$y_\circ'(x) - y_\circ(x) = \circ \quad , \quad y_\circ(1) = 1$$

The solution of the reduced problem is $y_\circ(x) = e^{x-1}$.

From step 2, two first-order equations equivalent to equation (29-2),

$$z'(x) - y(x) = \circ \quad \text{and}\, \varepsilon y'(x) + y(x) = z(x)$$

From step 3, we have: $z(1) = \varepsilon y_\circ'(1) + a(1)y_\circ(1)$

That's means: $z(1) = \varepsilon + 1$ We replace y(x) with $y_\circ(x)$ in relation , we get

$$z'(x) = y_\circ(x) = \circ$$

Therefore, the pair of initial value problems corresponding as follows:

$(i) \quad z'(x) = e^{x-1}$

with the condition $\quad z(1) = \varepsilon + 1$

$(ii) \quad \varepsilon y'(x) + y(x) = z(x)$

with the condition $\quad y(\circ) = 1$

11-Numerical results

In this section, we present numerical results for two test samples. The exact answer to both examples is unknown. We estimate the global convergence orders of $P_\varepsilon^{N,M}$ and the global parameter-uniformity convergence orders of $P^{N,M}$ using his second method [2, Chapter 8]: for each $\varepsilon \in S := \left\{ \text{٢}^\text{٠}, \text{٢}^{-\text{١}}, \dots, \text{٢}^{-\text{٢٦}} \right\}$, calculate the solutions $Y^{N,M}$ and $Y^{\text{٢}N,\text{٢}M}$ with

(3.18) on Shishkin[5] meshes $\bar{Q}^{N,M}$ and $\bar{Q}^{2N,2M}$. Then, calculate the maximum two-mesh global difference as follows:

$$D_\varepsilon^{N,M} := \left\| \bar{Y}^{N,M} - \bar{Y}^{2N,2M} \right\|_{\bar{Q}^{N,M} \cup \bar{Q}^{2N,2M}}, \forall \varepsilon \in S$$

When $\bar{Y}^{N,M}$ represents the bilinear interpolation of the discrete solution $Y^{N,M}$ on the mesh $\bar{Q}^{N,M}$. For each $\varepsilon \in S$, the order of global convergence $P_\varepsilon^{N,M}$ is estimated based on $P_\varepsilon^{N,M} := \log_2 \left(\frac{D_\varepsilon^{N,M}}{D_\varepsilon^{2N,2M}} \right), \forall \varepsilon \in S$ The uniform two-mesh global differences $D^{N,M}$ and the uniform orders of global convergence with $P^{N,M}$ are calculated $D^{N,M} := \max_{\varepsilon \in S} D_\varepsilon^{N,M}$, $P^{N,M} := \log_2 \left(\frac{D^{N,M}}{D^{2N,2M}} \right)$

Table 1 Example 1: Maximum two-mesh global differences and convergence order using scheme [6[6]], where coordinate transformation (3.5) is not used.

	$N = M = 32$	$N = M = 64$	$N = M = 128$	$N = M = 256$	$N = M = 512$	$N = M = 1024$	$N = M = 2048$
$D^{N,M}$	4.322E−02	4.546E−02	1.531E−02	3.916E−02	1.966E−02	4.338E−02	
$P^{N,M}$	−0.040	1.570	−1.355	0.994	−1.178	1.744	1.328E−02

Shishkin [5]

[6] J.L. Gracia, E. O'Riordan, Numerical approximation of solution derivatives of singularly perturbed parabolic problems of convection–diffusion type, Math. Comput. 85 (2016) 581–599.
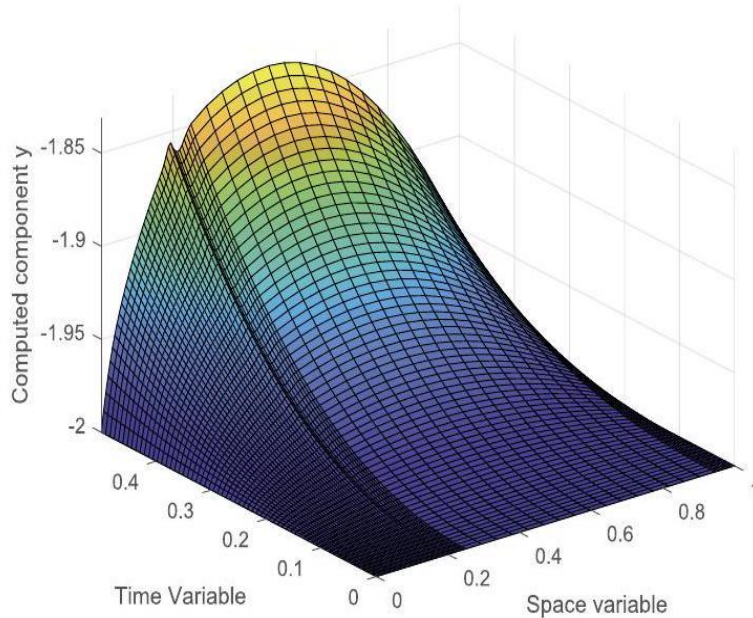
(a) Approximation for y.
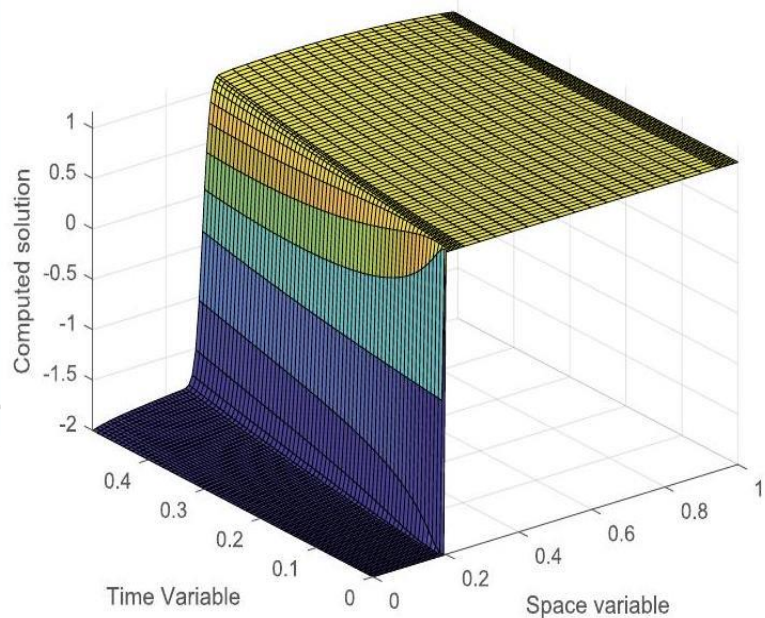
(b) approximation for $\hat{u}$.

Figure 1. Example 1: Numerical approximation to y and $\hat{u}$ with $\varepsilon = 2^{-12}$ and N=M=64.

Example 1. Consider the following test problem:

$$-\varepsilon \hat{u}_{ss} + \hat{a}(s,t)\hat{u}_s + \hat{u}_t = \mathfrak{f}s(\mathfrak{1} - s)t + t^{\mathfrak{r}}, \ (s,t) \in (\cdot,\mathfrak{1}) \times (\cdot,\cdot.\mathfrak{d}],$$

$$\hat{u}(s,\cdot) = -\mathfrak{r}, \cdot \leq x < \cdot.\mathfrak{r}, \ \hat{u}(s,\cdot) = \mathfrak{1}, \cdot.\mathfrak{r} \leq s \leq \mathfrak{1},$$

$$\hat{u}(\cdot,t) = -\mathfrak{r}, \ \hat{u}(\mathfrak{1},t) = \mathfrak{1}, \cdot < t \leq \cdot.\mathfrak{d},$$

$$\hat{a}(s,t) = \frac{\left(\cdot.\mathfrak{q}^{\mathfrak{r}} - (s-\cdot.\mathfrak{r})^{\mathfrak{r}}\right)}{\mathfrak{r}}.$$ Note that $\hat{a}_s(d,0) = 0$ is the characteristic curve:

$$\hat{d}(t) = \frac{\mathfrak{1}.\mathfrak{1} - \cdot.\mathsf{V}e^{-\frac{\mathfrak{q}t}{\mathfrak{r}\cdot}}}{\mathfrak{1}+e^{-\frac{\mathfrak{q}t}{\mathfrak{r}\cdot}}}$$ It has been proved in [6] that coordinate transformation

(3.5) is not required to obtain a global approximation when it depends only on $\hat{a}$ variable t. Hence, we first check whether this transformation $\hat{a} = \hat{a}(s,t)$ is needed or not. In Table 1, we see that without mapping, the method is not parameter uniform.

Example. Consider the test problem:

$$-\varepsilon \hat{u}_{ss} + (\mathfrak{1} + s^{\mathfrak{r}})\hat{u}_s + (s+t)\hat{u} + \hat{u}_t = \mathfrak{f}s(\mathfrak{1} - s)t + t^{\mathfrak{r}}, \ (s,t) \in (\cdot,\mathfrak{1}) \times (\cdot,\cdot.\mathfrak{d}],$$

$$u(s,\cdot) = -\mathfrak{r}, \cdot \leq s < \cdot.\mathfrak{1}, \ u(s,\cdot) = \mathfrak{1}, \cdot.\mathfrak{1} \leq s \leq \mathfrak{1},$$

$$u(\cdot,t) = -\mathfrak{r}, \ u(\mathfrak{1},t) = \mathfrak{1}, \cdot < t \leq \cdot.\mathfrak{d}.$$

Note that the source term is present in this example, and then the problem (3.25) is approximated by the numerical method (3.18) on the Shishkin mesh

$\bar{Q}^{N,M}$. For this example we have $I(t) = (\cos t - \cdot.\mathfrak{1}\sin t)e^{-\frac{t^{\mathfrak{r}}}{\mathfrak{r}}}$.

Moreover, observe $\hat{a}_s(d,0) \neq 0$ and $\hat{b}_s(d,0) \neq 0$. In Table 3, we see that the numerical approximations converge almost to the first order.

Table 2 Example 1: global differences of two-uniform mesh and order of convergence using numerical method(3-18).

	N=M=32	N=M=64	N=M=128	N=M=256	N=M=512	N=M=1024	N=M=2048
$\varepsilon = 2^0$	3.503E-02	4.546E-02	1.531E-02	5.169E-03	2.067E-03	1.005E-03	4.955E-04
	-0.376	1.570	1.567	1.322	1.041	1.020	
$\varepsilon = 2^{-1}$	4.422E-02	1.495E-02	5.041E-03	2.017E-03	9.795E-04	4.827E-04	2.396E-04
	1.564	1.569	1.322	1.042	1.021	1.010	
$\varepsilon = 2^{-2}$	1.426E-02	4.795E-03	1.927E-03	9.318E-04	4.585E-04	2.274E-04	1.132E-04
	1.573	1.315	1.048	1.023	1.012	1.006	
$\varepsilon = 2^{-4}$	1.986E-03	7.580E-04	3.886E-04	1.967E-04	9.897E-05	4.964E-05	2.486E-05
	1.390	0.964	0.982	0.991	0.996	0.998	
$\varepsilon = 2^{-6}$	8.317E-03	3.022E-03	9.091E-04	3.251E-04	1.625E-04	8.126E-05	4.063E-05
	1.461	1.733	1.483	1.000	1.000	1.000	
$\varepsilon = 2^{-8}$	1.610E-02	8.733E-03	3.419E-03	1.081E-03	3.076E-04	1.008E-04	4.369E-05
	0.882	1.353	1.662	1.813	1.610	1.206	
$\varepsilon = 2^{-10}$	1.325E-02	9.919E-03	5.841E-03	2.769E-03	1.111E-03	4.467E-04	1.897E-04
	0.418	0.764	1.077	1.317	1.315	1.236	
$\varepsilon = 2^{-12}$	9.178E-03	5.996E-03	3.206E-03	1.437E-03	6.355E-04	3.306E-04	1.718E-04
	0.614	0.903	1.158	1.177	0.943	0.945	
$\varepsilon = 2^{-14}$	6.754E-03	4.265E-03	2.232E-03	1.121E-03	6.165E-04	3.434E-04	1.895E-04
	0.663	0.934	0.994	0.863	0.844	0.858	
	⋮	⋮	⋮	⋮	⋮	⋮	⋮
$\varepsilon = 2^{-24}$	5.397E-03	3.502E-03	1.916E-03	1.130E-03	6.769E-04	3.823E-04	2.149E-04
	0.624	0.870	0.762	0.739	0.824	0.831	
$\varepsilon = 2^{-26}$	5.396E-03	3.501E-03	1.916E-03	1.130E-03	6.770E-04	3.823E-04	2.149E-04
	0.624	0.870	0.761	0.739	0.824	0.831	
$D^{N,M}$	4.422E-02	4.546E-02	1.531E-02	5.169E-03	2.067E-03	1.005E-03	4.955E-04
$P^{N,M}$	-0.040	1.570	1.567	1.322	1.041	1.020	

Conclusion:

In this thesis, a single turbulent parabolic problem of convection-diffusion type with discontinuous initial conditions was investigated. A special complementary error function is identified that corresponds to the discontinuity in the initial conditions. The difference between the analytical function and solving the parabolic problem is numerical approximation. In this research, a coordinate transformation was used, so that a layer compatible network can be aligned with the inner layer in the solution. Numerical analysis was presented for the related numerical method, which proves that the numerical method is a uniform parameter numerical method. Numerical results were presented to show the boundaries of point error created in the thesis.

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