Journal of Education for Pure Science- University of Thi-Qar
Vol.9, No.2 (June, 2019)
Website: jceps.utq.edu.iq *Email: jceps@eps.utq.edu.iq*

DOI: http://doi.org/10.32792/utq.jceps.09.02.17

A New Double Sumudu Transform Iterative Method for Solving Some of Fractional Partial Differential Equations

Hasan Shather Kadhem[1], Samear Qasim Hasan[2]

[1] Mathematics Department, Baghdad University, Baghdad, Iraq.
[2] Mathematics Department, Al-Mustansiriyah University, Baghdad, Iraq.

Received 22/01/2019, Accepted 15/04/2019, Published 02/06/2019

Abstract:
In this paper, we apply double sumudu transform coupled with a numerical method ”a new iterative method” to solve some of fractional partial differential equations which cannot be solved by applying double Sumudu transform only. Several examples illustrate with plotting their solutions are given.

Keywords: Fractional Calculus, Double Sumudu transform, Fractional Partial Differential Equation.

1. Introduction

Fractional partial differential equations are found to be an effective tool to describe certain physical phenomena, such as diffusion processes [1], electrical materials properties [2], however, there exist no method that yields an exact solution for non-linear fractional partial differential equations. Some different and powerful methods for solving fractional partial differential equations have been proposed in order to obtain the approximate and numerical solutions. In [3], the authors established the double Laplace transform for partial fractional derivatives, and apply this technique to solve initial and boundary fractional heat equation. In [4], they investigate the transport equations in fractal porous media by using the fractional complex transform method. While in [5], proposed a numerical method that is new iteration method NIM to solve linear and non-linear integral and differential equation.

In this paper, we generalized the NIM to another form by coupled the double Sumudu transform with that iterative method (NDSIM) for obtaining analytical and numerical solutions of some types of fractional partial differential equations. The advantage of this method is to enable us to solve more complicated problems than in [6] such as space-time non-linear fractional partial differential equations with highly accurate to approximate the exact solution, here space fractional derivative means the fractional derivative is with respect to the space x, and time fractional derivative means that the derivative is with respect to time t.

2. Fundamental properties of Fractional Calculus and Double Sumudu transform (DSTM)

In this section, we give some important definitions and notations which are needed in our work.

2.1 Important Facts of the Fractional Calculus

Definition 1 [6] The Riemann–Liouville fractional integral of order a for a function f is defined as

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-x)^{\alpha-1} f(x)dx \qquad \alpha, t > 0$$

The linear operator R-L fractional integral has the following properties:

- $J^0 = I$
- $J^{\alpha}.J^{\beta} = J^{\beta}.J^{\alpha} = J^{\alpha+\beta}$

Definition 2[6] The Caputo fractional derivative of positive order $\alpha$for a function f is defined as

$$D_t^{\alpha} f(t) = J^{n-\alpha} D^n f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \frac{f^{(n)}(x)}{(t-x)^{n-\alpha+1}}\, dx & n \neq \alpha \\ f^{(n)}(x) & n = \alpha \end{cases}$$

Some properties of fractional Caputo derivative and fractional R-L integral are:

- $J^{\alpha} t^{\beta} = \frac{\Gamma(1+\beta)}{\Gamma(1+\beta+\alpha} t^{\alpha+\beta} \qquad \alpha > 0,\ \beta > -1,\ t > 0$
- $\frac{d^{\alpha}}{dt^{\alpha}} t^{\beta} = \begin{cases} \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha} t^{\beta-\alpha} & n-1 < \alpha < n,\ \ \beta > n-1, \quad \beta \in R \\ 0 & n-1 < \alpha < n,\ \ \beta \leq n-1,\ \beta \in N \end{cases}$
- $\frac{\partial^{\alpha}}{\partial t^{\alpha}} (J^{\alpha}) = I$
- $J^{\alpha} \left(\frac{d^{\alpha}}{dt^{\alpha}} f(t) \right) = f(t) - \sum_{i=0}^{n-1} \frac{t^i}{i!} f^{(i)}(0)$
- $\frac{\partial^{\alpha}}{\partial t^{\alpha}} \left(\frac{\partial^{\beta}}{\partial t^{\beta}} f(t) \right) = \frac{\partial^{\beta}}{\partial t^{\beta}} \left(\frac{\partial^{\alpha}}{\partial t^{\alpha}} f(t) \right) = \frac{\partial^{\alpha+\beta}}{\partial t^{\alpha+\beta}} f(t) \qquad provided\ that\ \ f^{(i)} = 0, i = 0,1, \dots, n-1, \alpha + \beta \leq n,\ n \in N$

Definition 3 [6] A two parameters Mittag-Leffler function is defined as:

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \qquad \alpha, \beta \in C, \qquad R(\alpha), \qquad R(\beta) > 0$$

Lemma 1 A Mittag-Leffler function has an interesting properties [6]:

- $E_{\alpha,1}(x) = E_{\alpha}(x)$
- $E_{2,1}(x^2) = cosh(x)$
- $E_{2,1}(x^2) = cos(x)$
- $E_{1,1}(x) = e^x$
- $xE_{2,1}(x^2) = sinh(x)$
- $xE_{2,1}(x^2) = sin(x)$

Lemma 2 The Caputo derivatives of Mittag-Leffler are given as [7]:

- $E_{\alpha,\beta}^{(n)}(x) = \sum_{k=0}^{\infty} \frac{(k+n)!x^k}{k!\Gamma(\alpha k+\alpha n+\beta)} \qquad n \in N$
- $\frac{d^\alpha}{dt^\alpha}\left(E_\alpha(at^\alpha)\right) = aE_\alpha(at^\alpha) \quad \alpha > 0, \quad a \in R$
- $\frac{d^\gamma}{dt^\gamma}\left(t^{\beta-1}E_{\alpha,\beta}(at^\alpha)\right) = t^{\beta-\gamma-1}E_{\alpha,\beta-\gamma}(at^\alpha) \qquad \gamma > 0$

2.2 Fundamental Facts of the Single and Double Sumudu transform

The following definitions and properties of single and double Sumudu transform are necessary for our work. For
all, we consider Sumudu transform of a function and it's inverse is exist.
The following definition is introduced by [8].
Definition 4 The Sumudu transform for the exponent order function $f(x)$ is given by

$$S\{f(x)\} = T(u) = \frac{1}{u}\int_0^\infty e^{\frac{-x}{u}} f(x) \quad dx \qquad x>0$$

And the inverse Sumudu transform of T(u) is defined by:

5? 5 <6 :Q ;=L B :T; L 5/6 ∫ 6 fi – 7 fi ª ˝-˘ 8 ˝ ˆ _ ˇH ª ¯: ; I ˆ–: ;‹

The Sumudu transform of the important function "Mittag-Leffler function " is given by

5 [P ? 5 ' Æ :ª P;_ L Q ? 5 / s F ªQ

In table (2), Sumudu transform for some famous functions are given.
Definition 5 The double Sumudu transform for the function B :PÆT; is given by [9]

5 6 <B :TÆP;= L 6 :QÆR; L 5/ŁØ ∫ fi ∫ fi 7 ª 6 Æ A B :TÆP;@P@T

We state here some of important properties of double Sumudu transform which are need

- 5 6 <B :=T;C :> P;= L 6 :=Q;) :>R;
- 5 6 <+ B :TÆP;= L Q 6 :QÆR;

Theorem 1 [10] If the double Sumudu transform of the function f (x; t) given by 5 6 <B :TÆP; L 6 :QÆR; then

- 5 6 <T Æ B :TÆP;= L Q Æ ^Æ @4 = Æ Q ! / !Ł 6 :QÆR;
- 5 6 <T Æ P B :TÆP;= L Q Æ R ^Æ @4 = Æ > Q R ! 6 / !Ł Ø 6 :QÆR; where = Æ 4 L sÆ = Æ Æ L sÆ = Æ 5 L JŁJÆ = Æ L = Æ?5 ?5 E :I E G;= Æ?5 ; similarly for >

The double Sumudu transform of the partial Caputo fractional derivatives are given in the following theorem

Theorem 2 [9] Let the exponent order function $f\ (x,t)$ has a continuous partial derivatives on $R^+ \times R^+$ and $n-1 < a\ < n,\ m-1\ < b\ < m$ then:

- $S_2\{D_x^\alpha f(x,t)\} = u^{-\alpha}\left[T(u,v) - \sum_{i=0}^{n-1} u^i\, T_i(0,v)\right]$
- $S_2\left\{D_t^\beta f(x,t)\right\} = v^{-\beta}\left[T(u,v) - \sum_{j=0}^{m-1} v^j\, T_j(u,0)\right]$
- $S_2\left\{D_t^\beta D_x^\alpha f(x,t)\right\} = u^{-\alpha}v^{-\beta}\left[T(u,v) - \sum_{i=0}^{n-1} u^i\, T_i(0,v) - \sum_{j=0}^{m-1} v^j\, T_j(u,0) + \right.$

$\left.\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\ u^i v^j\, \frac{\partial^{i+j}}{\partial t^j \partial x^i} f(0,0)\right]$

Where $T_i(0,v) = S_2\left\{\frac{\partial^i}{\partial x^i} f(0,t)\right\}$ and $T_j(u,0) = S_2\left\{\frac{\partial^j}{\partial t^j} f(x,0)\right\}$

3. The New Double Sumudu transform Iterative Method (NDSTIM)

To illustrate the basic idea of the NDSTIM for the fractional partial differential equation, we consider the following equation

$$D_t^\alpha f(x,t) + D_x^\beta f(x,t) + L\, f(x,t) + Rf(x,t) = g(x,t) \qquad n-1 < \alpha \le n, \quad m-1 < \beta \le m \qquad (1)$$

Where D_t^α, D_x^β is the is the Caputo fractional derivative operator with respect to t, x respectively, L is a linear operator, R is general non-linear operator and $g(x,t)$ is a continuous function.

Applying double Sumudu transform on both sides of Eq.(1), we obtain

$$v^{-\alpha}\left[T(u,v) - \sum_{j=0}^{n-1} v^j T_j(u,0)\right] + u^{-\beta}\left[T(u,v) - \sum_{i=0}^{m-1} u^i T_i(0,v)\right] = S_2[g(x,t) - L\{f(x,t)\} - R\{f(x,t)\}] \qquad (2)$$

By simplify Eq.(2), we get

$$\left(v^\alpha + u^\beta\right)T(u,v) = u^\beta \sum_{j=0}^{n-1} v^j T_j(u,0) + v^\alpha \sum_{i=0}^{m-1} u^i T_i(0,v) + u^\beta v^\alpha S_2[g(x,t) - L\{f(x,t)\} - R\{f(x,t)\}]$$

$$T(u,v) = \frac{1}{u^\beta + v^\alpha}\left[u^\beta \sum_{j=0}^{n-1} v^j T_j(u,0) + v^\alpha \sum_{i=0}^{m-1} u^i T_i(0,v) + u^\beta v^\alpha S_2[g(x,t) - L\{f(x,t)\} - R\{f(x,t)\}]\right] \qquad (3)$$

By taking the inverse double Sumudu transform of both sides of Eq. (3), we have

$$f(x,t) = S_2^{-1}\left[\frac{u^\beta \sum_{j=0}^{n-1} v^j T_j(u,0) + v^\alpha \sum_{i=0}^{m-1} u^i T_i(0,v) + u^\beta v^\alpha S_2[g(x,t) - L\{f(x,t)\} - R\{f(x,t)\}]}{u^\beta + v^\alpha}\right] \qquad (4)$$

Now, by assuming that

$$u = S_2^{-1}\left[\frac{u^\beta \sum_{j=0}^{n-1} v^j T_j(u,0) + v^\alpha \sum_{i=0}^{m-1} u^i T_i(0,v) + u^\beta v^\alpha S_2\{g(x,t)\}}{u^\beta + v^\alpha}\right]$$

$$N\{f(x,t)\} = -S_2^{-1}\left[\frac{u^\beta v^\alpha R\{f(x,t)\}}{u^\beta + v^\alpha}\right]$$

$$K\{f(x,t)\} = -S_2^{-1}\left[\frac{u^\beta v^\alpha L\{f(x,t)\}}{u^\beta + v^\alpha}\right]$$

So, we can rewrite Eq.(3) in the following form

$$f(x,t) = u(x,t) + K\{f(x,t)\} + N\{f(x,t)\} \qquad (5)$$

Where f is a known function, K, N are given linear and non-linear operators of f , respectively

Solution of equation(5) in the series form is:

$$f(x,t) = \sum_{n=0}^{\infty} f_n(x,t)$$

Thus, since K is linear operator, we get

$$K\left(\sum_{n=0}^{\infty} f_n(x,t)\right) = \sum_{n=0}^{\infty} k_n(f_n)$$

And the non-linear operator N is decomposed as [11]

$$N\left(\sum_{n=0}^{\infty} f_n\right) = N(f_0) + \sum_{n=0}^{\infty}\left[N\left(\sum_{n=0}^{n} f_n\right) - N\left(\sum_{n=0}^{n-1} f_n\right)\right]$$

Defining the following recurrence relation

$$\begin{gathered} f_0 = u \\ f_1 = K(f_0) + N(f_0) \\ \vdots = \qquad \vdots \\ f_{m+1} = K(f_m) + N(f_0 + f_1 + \cdots + f_m) - N(f_0 + f_1 + \cdots + f_{m-1}) \end{gathered}$$

Then we have

$$\sum_{n=0}^{\infty} f_n = u + K\left(\sum_{n=0}^{\infty} f_n\right) + N\left(\sum_{n=0}^{\infty} f_n\right)$$

For convergence, we refer to [11]

4. Numerical Examples

Example 1. Consider the following non-linear fractional Cauchy reaction- diffusion equation:

$$D_t^{\alpha} f(x,t) - D_x^{\beta} f(x,t) = f(x,t) - f_x(x,t) - f^2(x,t) + f(x,t) f_{xx}(x,t) \qquad 0 < \alpha \leq 1,\ 1 < \beta \leq 2 \tag{6}$$

With conditions

$$f(x,0) = f_t(x,0) = E_{\beta}\left(x^{\beta}\right) + x E_{\beta,2}\left(x^{\beta}\right), \qquad f_x(0,t) = E_{\alpha}(t^{\alpha})$$

Solution:

Operating the double Sumudu transform on both sides of Eq.(6) and applying the property of double Sumudu transform for fractional derivatives and by using Theorem (2.2), we get

$$v^{-\alpha}(T(u,v) - T_0(u,0)) - u^{-\beta}\left(T(u,v) - T_0(0,v) - uT_1(0,v)\right) = S_2(f - f_x + f f_{xx} - f^2)$$

By applying the single sumudu transform of the conditions, we have

$T_0(u,0) = T_1(u,0) = \frac{1+u}{1-u^{\beta}}$, $\qquad T_0(0,v) = T_1(0,v) = \frac{1}{1-v^{\alpha}}$

By simplify, we have

$$\left(u^{\beta}-v^{\alpha}\right)T(u,v)=\frac{u^{\beta}(1+u)}{1-u^{\beta}}-\frac{v^{\alpha}(1+u)}{1-v^{\alpha}}+u^{\beta}v^{\alpha}[S_2(f-f_x)+S_2(ff_{xx}-f^2)]$$
$$T(u,v)=\frac{1+u}{(1-u^{\beta})(1-v^{\alpha})}+\frac{u^{\beta}v^{\alpha}}{u^{\beta}-v^{\alpha}}[S_2(f-f_x)+S_2(ff_{xx}-f^2)]$$

Using inverse double Sumudu transform technique, we get

$$f(x,t)=\left(E_{\beta}\left(x^{\beta}\right)+x\,E_{\beta,2}\left(x^{\beta}\right)\right)E_{\alpha}(t^{\alpha})+S_2^{-1}\left[\frac{u^{\beta}v^{\alpha}}{u^{\beta}-v^{\alpha}}[S_2(f-f_x)+S_2(ff_{xx}-f^2)]\right]$$

According to the NDSTIM, we obtain

$$f_0=\left(E_{\beta}\left(x^{\beta}\right)+x\,E_{\beta,2}\left(x^{\beta}\right)\right)E_{\alpha}(t^{\alpha})$$
$$K\{f(x,t)\}=S_2^{-1}\left[\frac{u^{\beta}v^{\alpha}}{u^{\beta}-v^{\alpha}}\,S_2(f-f_x)\right]$$
$$N\{f(x,t)\}=S_2^{-1}\left[\frac{u^{\beta}v^{\alpha}}{u^{\beta}-v^{\alpha}}\,S_2(ff_{xx}-f^2)\right]$$

Then by Lemma (2.2) and using properties of Mittag-Lefler function, it is obvious that $f_0=f_{0_x}$ and $f_0f_{0_{xx}}=f_0^2$, therefore, $K[\,f\,]=0, N[\,f\,]=0$, and iteration formula of NDSIM reads

$$f_1=K(f_0)+N(f_0)=0$$
$$f_2=K(f_1)+N(f_0+f_1)-N(f_0)=0$$
$$\vdots \quad = \quad \vdots$$
$$f_n=0$$

Thus, the solution of Eq. (6) is given by

$$f(x,t)=f_0(x,t)=\left(E_{\beta}\left(x^{\beta}\right)+x\,E_{\beta,2}\left(x^{\beta}\right)\right)E_{\alpha}(t^{\alpha})$$

Note that this solution is exact solution in this case.

Table 1 showing the absolutely error of exact solution and 10th order approximate solutions for various values of α,β .

Figure 1 show the plot of exact solution and 10-th order approximate of some solutions for Eq.(6).

Table 1 Absolutely Error of 10-th order approach solutions for Eq.(6)

(x,t)	$U_{exac.}$	$U_{appr.}$	|Error|
		$\alpha=\beta=2$	
(0.3, 0.1)	1.4918246976	1. 49182469764	3×10^{-19}
(0.6, 0.2)	2. 2255409285	2. 2255409285	9.507×10^{-16}
(0.4, 0.8)	3. 3201169227	3. 3201169193	$3.4383352825\times 10^{-9}$
(0.9, 0.9)	6. 0496474644	6. 0496474435	$2.08939575638\times 10^{-8}$
		$\alpha=0.8,\beta=1.3$	
(0.3, 0.1)	1.8024905441	1.8024905441	1.06378×10^{-14}
(0.6, 0.2)	3.0505558344	3.0505558344	7.1743864×10^{-12}
(0.4, 0.8)	4.6306328568	4.6306316430	$1.2137436787\times 10^{-6}$
(0.9, 0.9)	9.6056869664	9.6056804816	6.4848×10^{-6}

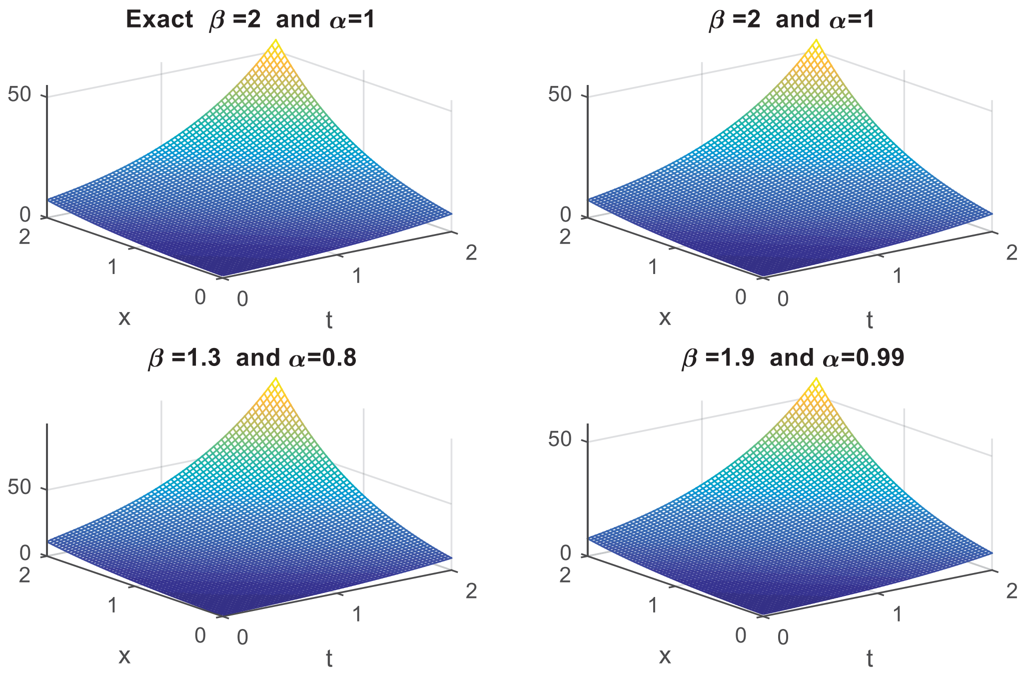

Figure 1-Exact solution and 10-th order approach solutions for Eq.(6)

Example 2 - Consider the following nonlinear parabolic-hyperbolic differential equation

$$(D_t - D_x^2)(D_t^2 - D_x^2)f = (D_t^2 f)^2 - (D_x^2 f)^2 - 2f^2 \qquad (7)$$

With conditions

$$f(x,0) = f_t(x,0) = f_{tt}(x,0) = e^x, \qquad f(0,t) = f_x(0,t) = f_{xx}(0,t) = f_{xxx}(0,t) = e^t$$

Solution:

Single Sumudu transformation of the conditions yields:

$$T(u,0) = T_0(u,0) = T_1(u,0) = \frac{1}{1-u}, \qquad T(0,v) = T_0(0,v) = T_1(0,v) = \frac{1}{1-v}$$

And double Sumudu transformation of the integer order partial derivatives are:

$$S(D_t^3 f) = v^{-3}\left(T(u,v) - \frac{1+v+v^2}{1-u}\right), \; S(D_t D_x^2 f) = v^{-1}u^{-1}\left(T(u,v) - \frac{1+u}{1-v} - \frac{u^2}{1-u}\right),$$

$$S(D_x^2 D_t^2 f) = v^{-2}u^{-2}\left(T(u,v) - \frac{1+u}{1-v} - \frac{1+v}{1-u} + (1+u)(1+v)\right),$$

$$S(D_x^4 f) = u^{-4}\left(T(u,v) - \frac{(1+u)(1+v)}{1-v}\right)$$

Then, operating double Sumudu transform of (7) we obtain:

$$(u^4 - u^2v^2 - u^2v + v^3)T(u,v) = \frac{u^4(1+v+v^2)}{1-u} - u^2v^2\left(\frac{1+u}{1-v} + \frac{u^2}{1-u}\right) - u^2v\left(\frac{1+u}{1-v} + \frac{1+v}{1-u} - (1+u)(1+v)\right) + v^3\left(\frac{(1+u)(1+v)}{1-v}\right) + v^3u^4S((f_{tt})^2 + (f_{xx})^2 - 2f^2)$$

By simplify, we have:

$T(u,v)=\frac{1}{(1-u)(1-v)}+\frac{u^4v^3}{u^4-u^2v^2-u^2v+v^3}S((f_{tt})^2+(f_{xx})^2-2f^2)$

Taking inverse double Sumudu transformation, we get:

$f(x,t)=e^{x+t}+S^{-1}\left[\frac{u^4v^3}{u^4-u^2v^2-u^2v+v^3}S((f_{tt})^2+(f_{xx})^2-2f^2)\right]$

Consequently, NDSIM reads

$f_0=e^{x+t}$

$K[f]=0$

$N[f]=S^{-1}\left[\frac{u^4v^3}{u^4-u^2v^2-u^2v+v^3}S((f_{tt})^2+(f_{xx})^2-2f^2)\right]$

Since we know that $(f_{0_{tt}})^2+(f_{0_{xx}})^2-2{f_0}^2=0$, then by iteration formula

$f_{n+1}=K[f_n]+N(f_0+\cdots+f_n)-N(f_0+\cdots+f_{n-1})$, we have

$f_{n+1}=K[f_0]+N[f_0]=N[f_0]=S^{-1}\left[\frac{u^4v^3}{u^4-u^2v^2-u^2v+v^3}S(0)\right]=0,$

$f_2=N[f_0+f_1]-N[f_0]=0,$

$\vdots \qquad \vdots$

$f_n=0$

So, $\sum_{n=0}^{\infty}f_n=e^{x+t}+K(\sum_{n=0}^{\infty}f_n)+N(\sum_{n=0}^{\infty}f_n)=f_0=e^{x+t}$

And in this case, we obtain the exact solution for Eq. (7).

Example 3 Consider the following non-linear fractional differential equation

$$D_t^{\alpha}f(x,t)=\frac{2xt^{2-\alpha}}{\Gamma(3-\alpha)}+x^2t^4-f^2(x,t) \qquad 0<\alpha\le 1 \quad (8)$$

With the conditions

$$f(x,0)=0,\ f_t(x,0)=0$$

Solution

Operating double Sumudu transform of Eq.(8), we have:

$v^{-\alpha}(T(u,v)-T_0(u,0))=2uv^{2-\alpha}+2.\ 4!\ u^2v^4-S\{f^2\}$ where $T_0(u,0)=S\{f(x,0)\}=0$

$$T=2uv^2+2.\ 4!\ u^2v^{4+\alpha}-S\{f^2\}$$

Inverse double Sumudu transform, yields

$f(x,t)=xt^2+\frac{4!}{\Gamma(\alpha+5)}x^2t^{\alpha+4}-S^{-1}[v^{\alpha}\ S\{f^2\}]$

NDSIM, reads

$f_0=xt^2+\frac{4!}{\Gamma(\alpha+5)}x^2t^{\alpha+4}$

$K[f]=0$

$N[f]=-S^{-1}[v^{\alpha}\ S\{f^2\}]$

$f_1=N[f_0]=-S^{-1}[v^{\alpha}\ S\{{f_0}^2\}]=-S^{1-}\Big[2.\ 4!\ u^2\ v^{\alpha+4}+\left(\frac{4!}{\Gamma(\alpha+5)}\right)^2 4!\ \Gamma(2\alpha+9)v^{3\alpha+8}u^4+$
$\frac{12.\ \ 4!}{\Gamma(\alpha+5)\Gamma(\alpha+7)}u^3v^{2\alpha+6}\Big]$

$$= -\left[\frac{4!}{\Gamma(\alpha+5)}x^2t^{\alpha+4} + \left(\frac{4!}{\Gamma(\alpha+5)}\right)^2 \frac{\Gamma(2\alpha+9)}{\Gamma(3\alpha+9)}x^4t^{3\alpha+8} + \frac{2.\ 4!}{\Gamma(\alpha+5)\Gamma(\alpha+7)\Gamma(2\alpha+7)}x^3t^{2\alpha+6}\right]$$

$$f_2 = N[f_0 + f_1] - N[f_0]$$

$$= -\frac{4!}{\Gamma(\alpha+5)}x^2t^{\alpha+4} - \frac{a^2\Gamma(6\alpha+17)}{\Gamma(7\alpha+17)}x^8t^{7\alpha+16} + \frac{2a\Gamma(3\alpha+11)}{\Gamma(4\alpha+11)}x^5t^{4\alpha+10} + \frac{b^2\Gamma(4\alpha+13)}{\Gamma(5\alpha+13)}x^6t^{5\alpha+12} - \frac{2b\Gamma(2\alpha+9)}{\Gamma(3\alpha+9)}x^4t^{3\alpha+8} - \frac{2ab\Gamma(5\alpha+15)}{\Gamma(6\alpha+15)}x^7t^{6\alpha+14} + \frac{4!}{\Gamma(\alpha+5)}x^2t^{\alpha+4} + ax^4t^{3\alpha+8} + bx^3t^{2\alpha+6}$$

Where $a = \left(\frac{4!}{\Gamma(\alpha+5)}\right)^2 \frac{\Gamma(2\alpha+9)}{\Gamma(3\alpha+9)}$, $\quad b = \frac{2.\ 4!}{\Gamma(\alpha+5)\Gamma(\alpha+7)\Gamma(2\alpha+7)}$

By iteration formula

$$f_{n+1} = N(f_0 + \cdots + f_n) - N(f_0 + \cdots + f_{n-1})$$

One can obtained another terms to approximate the exact solution $f\ (x,t) = xt^2$ when $\alpha = 1$, after cancelling the noise terms.

In Figures 2, 3, we plot the exact solution of equation (8) and the four approximate solutions for fixed value of x, α and various values of t.

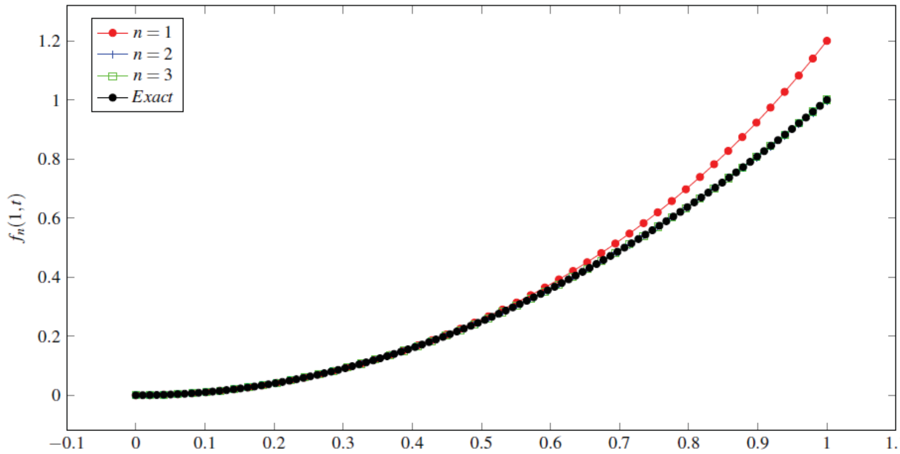

Figure 2 The exact solution and the numerical solutions for equation (8) obtained by the NDSIM with $\boldsymbol{x = 1, \alpha = 1}$ for various values of $\boldsymbol{t}$.

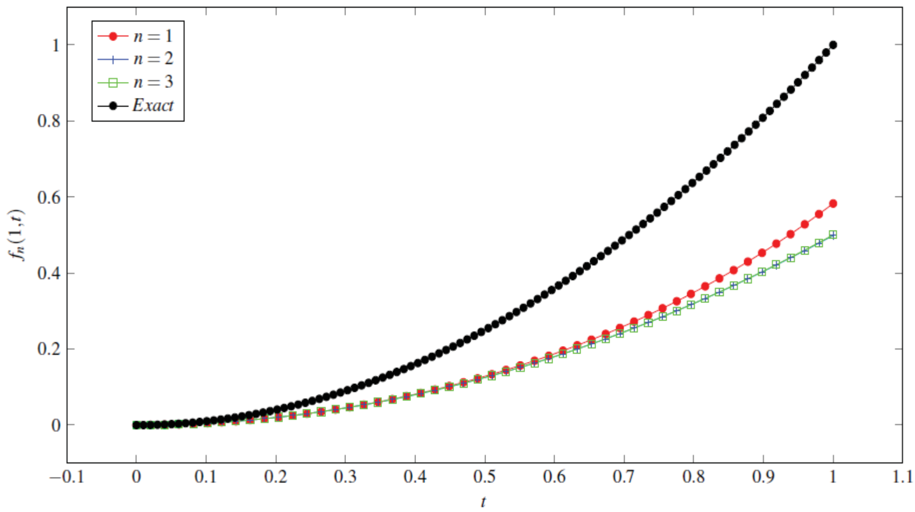

Figure 3 The exact solution and the numerical solutions for equation (8) obtained by the NDSIM with $x = 0.5, \alpha = 0.7$ for various values of t.

Example 4 Consider the one-dimensional linear inhomogeneous fractional Klein-Gordon equation

$$D_t^{\alpha} f(x,t) = 6x^3 t + (x^3 - 6x)t^3 - f(x,t) + f_{xx}(x,t) \qquad 1 < \alpha \leq 2 \qquad (9)$$

With the conditions

$$f(x,0) = 0, \quad f_t(x,0) = 0$$

Solution

Single Sumudu transform of the initial conditions are:

$$T_0(u,0) = 0, \quad T_1(u,0) = 0$$

Operating double Sumudu transform for Eq.(9)

$$v^{-\alpha}[T(u,v) - T_0(u,0) - vT_1(u,0)] = 36u^3 v + 36(u^3 - u)v^3 - S(f - f_{xx})$$

$$T(u,v) = 36u^3 v^{1+\alpha} + 36(u^3 - u)v^{3+\alpha} - v^{\alpha} S(f - f_{xx})$$

Taking inverse double Sumudu transform

$$f(x,t) = \frac{6x^3 t^{1+\alpha}}{\Gamma(\alpha+2)} + \frac{6x^3 t^{3+\alpha}}{\Gamma(\alpha+4)} - \frac{36xt^{3+\alpha}}{\Gamma(\alpha+4)} - S^{-1}[v^{\alpha} S(f - f_{xx})]$$

According NDSIM, we have

$$f_0 = \frac{6x^3 t^{1+\alpha}}{\Gamma(\alpha+2)} + \frac{6x^3 t^{3+\alpha}}{\Gamma(\alpha+4)} - \frac{36xt^{3+\alpha}}{\Gamma(\alpha+4)}$$

$$K[f] = -S^{-1}[v^{\alpha} S(f - f_{xx})], \quad N[f] = 0$$

$$\therefore\ f_1 = K[f_0] + N[f_0] = -S^{-1}[v^{\alpha} S(f - f_{xx})]$$

$$= -S^{-1}\left[v^{\alpha} S\left(\frac{6x^3 t^{1+\alpha}}{\Gamma(\alpha+2)} + \frac{6x^3 t^{3+\alpha}}{\Gamma(\alpha+4)} - \frac{36xt^{3+\alpha}}{\Gamma(\alpha+4)} - \frac{36xt^{1+\alpha}}{\Gamma(\alpha+2)} - \frac{36xt^{3+\alpha}}{\Gamma(\alpha+4)}\right)\right]$$

$$= -S^{-1}\left[v^{\alpha} S\left(\frac{6x^3 t^{1+\alpha}}{\Gamma(\alpha+2)} + \frac{6x^3 t^{3+\alpha}}{\Gamma(\alpha+4)} - \frac{36xt^{1+\alpha}}{\Gamma(\alpha+2)} - \frac{72xt^{3+\alpha}}{\Gamma(\alpha+4)}\right)\right]$$

$$= -S^{-1}[v^{\alpha}(36u^3 v^{1+\alpha} + 36u^3 v^{3+\alpha} - 36uv^{1+\alpha} - 72uv^{3+\alpha})]$$

$$= -S^{-1}(36u^3 v^{1+2\alpha} + 36u^3 v^{3+2\alpha} - 36uv^{1+2\alpha} - 72uv^{3+2\alpha})$$

$$= -\left[\frac{6x^3 t^{1+2\alpha}}{\Gamma(2\alpha+2)} + \frac{6x^3 t^{3+2\alpha}}{\Gamma(2\alpha+4)} - \frac{36xt^{1+2\alpha}}{\Gamma(2\alpha+2)} - \frac{72xt^{3+2\alpha}}{\Gamma(2\alpha+4)}\right]$$

By the same manipulation, we have:

$$f_2 = \frac{6x^3 t^{1+3\alpha}}{\Gamma(3\alpha+2)} + \frac{6x^3 t^{3+3\alpha}}{\Gamma(3\alpha+4)} - \frac{72xt^{1+3\alpha}}{\Gamma(3\alpha+2)} - \frac{108xt^{3+3\alpha}}{\Gamma(3\alpha+4)}$$

$$f_3 = -\left[\frac{6x^3 t^{1+4\alpha}}{\Gamma(4\alpha+2)} + \frac{6x^3 t^{3+4\alpha}}{\Gamma(4\alpha+4)} - \frac{108xt^{1+4\alpha}}{\Gamma(4\alpha+2)} - \frac{144xt^{3+4\alpha}}{\Gamma(4\alpha+4)}\right]$$

$$f_4 = \frac{6x^3 t^{1+5\alpha}}{\Gamma(5\alpha+2)} + \frac{6x^3 t^{3+5\alpha}}{\Gamma(5\alpha+4)} - \frac{144xt^{1+5\alpha}}{\Gamma(5\alpha+2)} - \frac{180xt^{3+5\alpha}}{\Gamma(5\alpha+4)}$$

After cancelling the noise terms, we get the approximate solution of equation (9) when $\alpha = 2$ as

$$f_0 + f_1 + f_2 + f_3 + f_4 + \cdots = x^3 t^3 + \frac{t^{13} x^3}{1037836800} - \frac{t^{13} x}{34594560} + \cdots$$

$$= x^3 t^3 + |small\ amount| \qquad t \leq 1$$

So, one can obtain the exact solution as

$$f(x,t) = \sum_{n=0}^{\infty} f_n = x^3 t^3$$

In Figures 4, 5 , we plot the exact solution of equation (9) and some of numerical solutions obtained by the NDSIM. As we see from Figures , the numerical solutions obtained by NDSIM are in good agreement with the exact solution as the values of n increased.

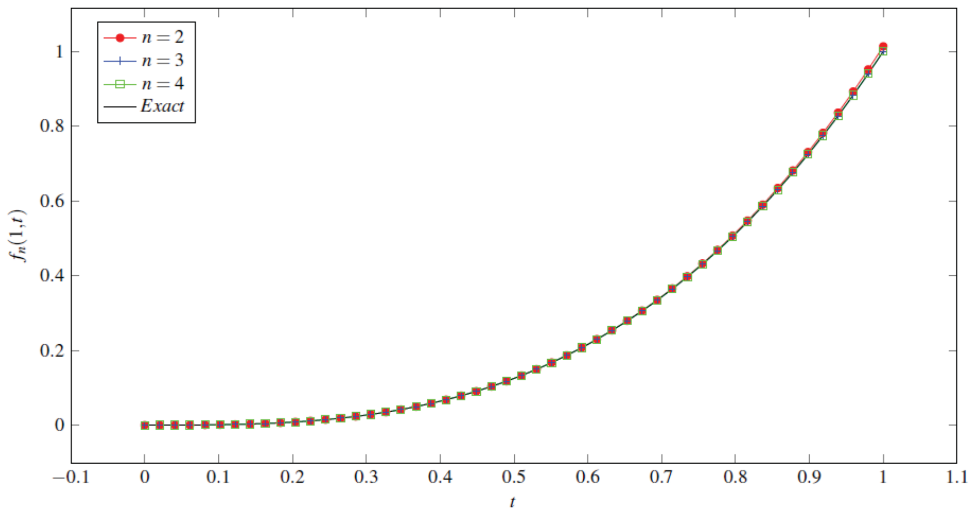

Figure 4 The exact solution and the numerical solutions for equation (9) obtained by the NDSIM with $x = 1, \alpha = 2$ for various values of t.

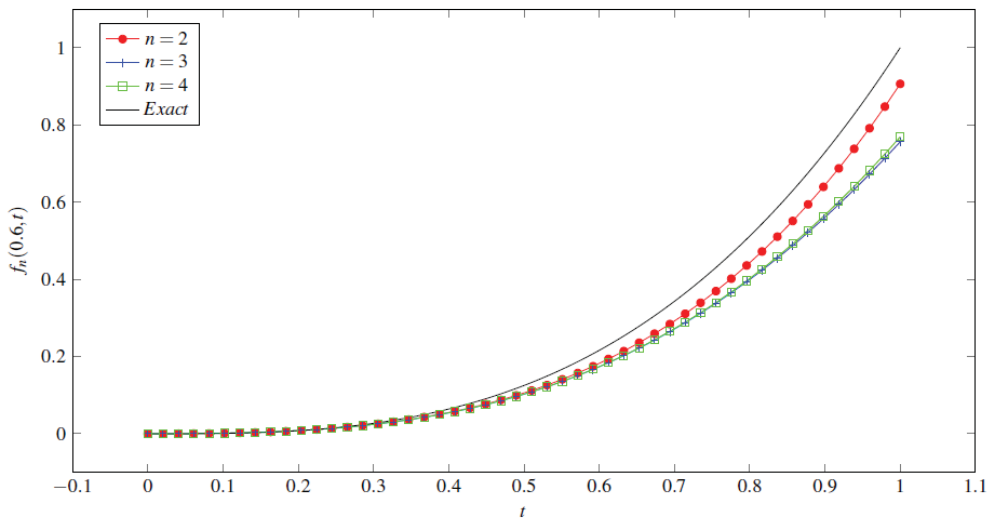

Figure 5 The exact solution and the numerical solutions for equation (9) obtained by the NDSIM with $x = 0.6, \alpha = 1.5$ for various values of t.

Example 5 Consider the one-dimensional linear inhomogeneous fractional equation

$$D_t^{\alpha} f(x,t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \sin(x) + t\cos(x) - f_x(x,t) \qquad 0 < \alpha \leq 1 \qquad (10)$$

With condition $f(x,0) = 0$

Solution

Operating double Sumudu transform for Eq.(10), we get

$$v^{-\alpha}[T(u,v) - T_0(u,0)] = \frac{uv^{1-\alpha}}{1-u^2} + \frac{v}{1+u^2} - S\{f_x(t,x)\}$$

$$T(u,v) = \frac{uv}{1-u^2} + \frac{v^{1+\alpha}}{1+u^2}$$

Taking inverse double Sumudu transform for the last equation, we have

$$f(x,t) = t\sin(x) + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}\cos(x) - S^{-1}[v^{\alpha}S\{f_x(t,x)\}]$$

So, according to NDSIM, we have

$$f_0 = t\sin(x) + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}cos(x)$$

$$f_1 = K[f_0] = -S^{-1}[v^{\alpha}S\{f_{0_x}(t,x)\}] = -S^{-1}\left[v^{\alpha}S\left\{t\cos(x) - \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}\sin(x)\right\}\right]$$

$$= -S^{-1}\left[v^{\alpha}\left(\frac{v}{1+u^2} - \frac{u\,v^{1+\alpha}}{1-u^2}\right)\right] = \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}\sin(x) - \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}\cos(x)$$

$$f_2 = K[f_1] = -S^{-1}[v^{\alpha}S\{f_{1_x}(t,x)\}] = -S^{-1}\left[v^{\alpha}S\left\{\frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}cos(x) + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}sin(x)\right\}\right]$$

$$= -S^{-1}\left[v^{\alpha}\left(\frac{v^{1+3\alpha}}{1+u^2} - \frac{u\,v^{1+2\alpha}}{1-u^2}\right)\right] = -\frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}\sin(x) - \frac{t^{1+3\alpha}}{\Gamma(2+3\alpha)}cos(x)$$

$$f_3 = \frac{t^{1+3\alpha}}{\Gamma(2+3\alpha)}cos(x) - \frac{t^{1+4\alpha}}{\Gamma(2+4\alpha)}\sin(x)$$

$\vdots \qquad \vdots$

$$f_1 + f_2 + f_3 + \cdots = t\sin(x) + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}cos(x) + \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}sin(x) - \frac{t^{1+\alpha}}{\Gamma(2+\alpha)}cos(x) - \frac{t^{1+2\alpha}}{\Gamma(2+2\alpha)}sin(x) - \frac{t^{1+3\alpha}}{\Gamma(2+3\alpha)}cos(x) + \cdots$$

By cancelling the noise terms, we get the exact solution as $\sum_{n=0}^{\infty} f_n = t\sin(x)$.

Figures 6, 7 show the exact solution for equation (10) and some of approximate solutions obtained by the NDSIM

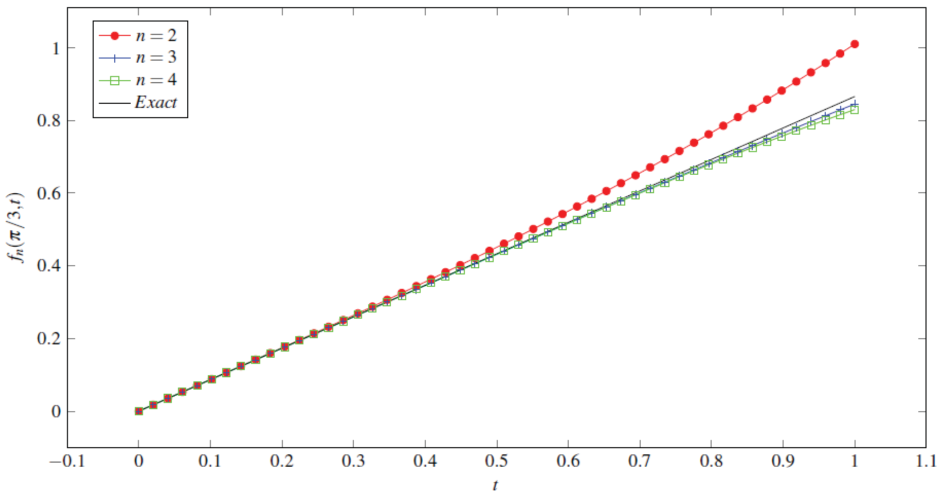

Figure 6 The exact solution and the numerical solutions for equation (10) obtained by the NDSIM with $x = \pi/3, \alpha = 1$ for various values of t.

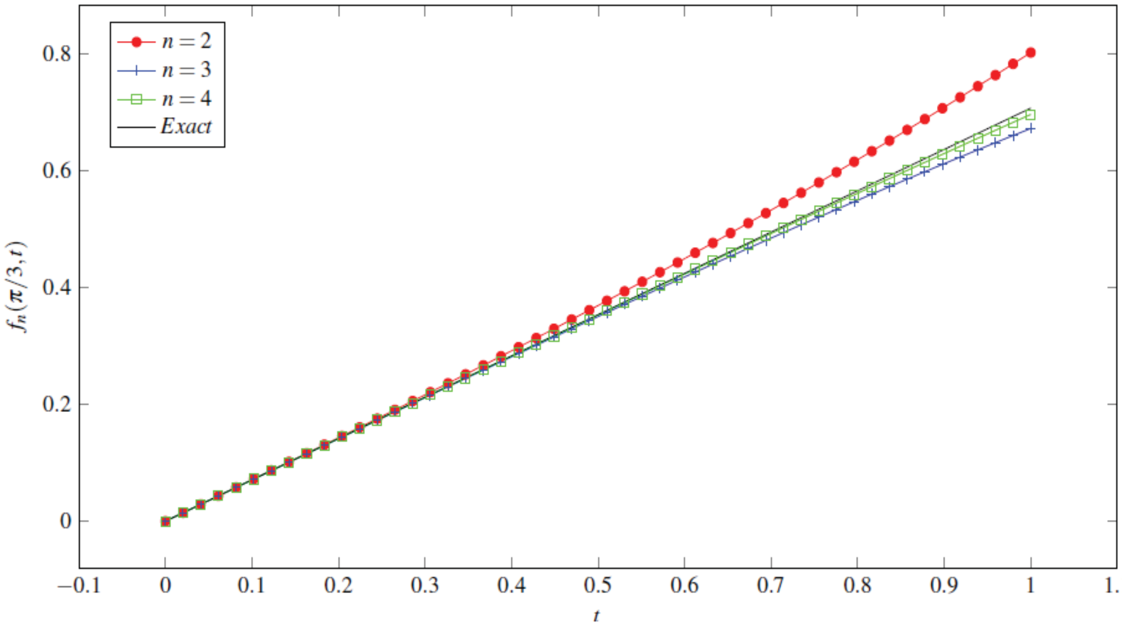

Figure 7 The exact solution and the numerical solutions for equation (10) obtained by the NDSIM with $x = \frac{\pi}{4}, \alpha = 0.8$ for various values of t.

Table 2 Sumudu transform for some speceial functions

$f(x)$	$S\{f(x)\} = T(u)$	$f(x)$	$S\{f(x)\} = T(u)$
1	1	x^α	$\Gamma(1+\alpha)u^\alpha$
e^x	$\frac{1}{1-au}$	$\sin ax$	$\frac{au}{1+a^2u^2}$
$\cos ax$	$\frac{1}{1+a^2u^2}$	$\sinh ax$	$\frac{au}{1-a^2u^2}$
$\frac{x^\alpha e^{ax}}{\Gamma(1+\alpha)}$	$\frac{u^\alpha}{(1-au)^{1+\alpha}}$	$\cosh ax$	$\frac{1}{1-a^2u^2}$

5. Conclusion

Double Sumudu transform method has been known as a powerful tool for solving many fractional partial differential equations, integral equations and so many other equations. In this paper, we have presented a new method, that is coupled the double Sumudu transform with a numerical iteration for solving some of fractional partial differential equations such as (linear, non-linear) parabolic-hyperbolic fractional equations, non-linear fractional Cauchy reaction-diffusion equation.

All of the examples show that the numerical solutions obtained by NDSIM are in good agreement with the exact solution as the values of n iterative are increased. Also, when the initial and boundary conditions are compatible with the assuming problem, the NDSIM gives the exact solution directly. We are plotting the solutions by help of Matlab and Mathcad programs.

References:

[1] Ikeda, Nobuyuki and Watanabe, Shinzo. "*Stochastic differential equations and diffusion processes "Elsevier, volume 24, 2014.*

[2] Zhang, Jiawei and Song, Lirong and Borup, Kasper Andersen and J.rgensen, Mads Ry Vogel and Iversen, Bo Brummerstedt. "*New Insight on Tuning Electrical Transport Properties via Chalcogen Doping in n-type Mg3Sb2-Based Thermoelectric Materials",* Advanced Energy Materials, Wiley Online Library,volume 8, 16,2018.

[3] Elagan, SK and Sayed, M and Higazy, M, "An Analytical Study on Fractional Partial Differential Equations by Laplace Transform Operator Method ", International Journal of Applied Engineering Research, volume 13, 2018

[4 Hosseini, K and Ayati, Z," Exact solutions of space-time fractional EW and modified EW equations using Kudryashov method", Nonlinear Science Letters A Mathematics, Physics and Mechanics, volume 7, 2016.

[5] Daftardar-Gejji, Varsha and Sukale, Yogita and Bhalekar, Sachin," Solving fractional delay differential equations: A new approach", Fractional Calculus and Applied Analysis, De Gruyter Open, volume 18, 2015.

[6] Stamova, Ivanka," Mittag-Leffler stability of impulsive differential equations of fractional order", Quarterly of Applied Mathematics, volume 73, 2015.

[7] Eslami, M., Vajargah, B.F., Mirzazadeh, M. and Biswas," *Applications of first integral method to fractional",* s.l. : Indian J. Phys, 2014.

[8] Al-Omari, Shrideh K Qasem and Agarwal, Praveen," Some general properties of a fractional Sumudu transform in the class of Boehmians", Kuwait Journal of Science, volume 43, 2016.

[9] F. Jarad, " *Application of Sumudu and double Sumudu transforms to Caputo-Fractional differential equations",* s.l. : Journal of Computational Analysis and Applications, 2012.

[10] Gadain, A. Kilicman and H. E. 2010. "*On the application of Laplace and Sumudu transform",* s.l. : Jornal of the Franklin, 2010.

[11] Hong, Mingyi and Wang, Xiangfeng and Razaviyayn, Meisam and Luo, Zhi-Quan," Iteration complexity analysis of block coordinate descent methods", Springer, volume 163, 2017.