



Dynamical behavior of the Family of cubic functions

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1. Introduction

The word chaos in modern dictionaries, is defined as “total disorder and confusion”. [3] Edward Lorenz simplified chaos in short words “when the present determines the future , but the approximate present does not approximately determine the future” . Chaos as an property is observed after a period of time which implies a system which could possibly display non-chaos as usual in the initial stages of iteration though highly and easily chaotic after a few iteration .

2.Fixed points.

A point whose iterates are the same point is called a fixed point . In other words it's not changed under the effect of the function motion , therefore fixed points very important in the study of the dynamics of functions.

Definition. 2.1 [5],[1]

Let p be a point in the domain of the function f . Then p is called a fixed point of f if $f(p) = p$. Graphically , the point p is a fixed point of the function f if and only if the graph of f touches (or crosses) the line $y = x$ at (p, p) .

Definition. 2.2 [5]

Let p be a fixed point of the function f . The point p is called attracting fixed point if there exists an interval $(p - \varepsilon, p + \varepsilon)$ such that if x is in the domain of f and $x \in (p - \varepsilon, p + \varepsilon)$ then $f^{[n]}(x) \rightarrow p$ as n increases without bound , such a point also

ABSTRACT

May R.M. gave the example of the family of cubic maps of the interval $[-1,1]$. Rogers T. D. extends the analysis of May beyond that region .

In this paper we are trying to introduce comprehensive study of the cubic family which defined in the form:

$$f_{\alpha}(x) = \alpha x^3 + (1 - \alpha)x \quad (1)$$

The fixed points of the family are determined and described according to the values of the parameter α . Dynamical and chaotic behaviours of the family discussed according to different definitions of chaos and via conjugacy .

called asymptotically stable. i.e. attracting fixed point attracts the iterates of the near points to itself in some interval.

Definition. 2.3 [5]

Let p be a fixed point of the function f . The point p is called repelling fixed point if there exist an interval $(p - \varepsilon, p + \varepsilon)$ such that if x is in the domain of f , $x \in (p - \varepsilon, p + \varepsilon)$ and $x \neq p$ then $|f(x) - p| > |x - p|$. i.e. it repels the iterations of near points to some distance.

3.Dynamical behaviour of the family

Theorem 3.1 [5]

- Suppose that f is differentiable at a fixed point p .
- If $|f'(p)| < 1$, then the point p is called attracting fixed point.
 - If $|f'(p)| > 1$, then the point p is called repelling fixed point.
 - If $|f'(p)| = 1$, then p can be attracting , repelling , or neither.

To find fixed points of any function f we just solve the equation $f(x) = x$.

So , solving the equation

$$f_{\alpha}(x) = x$$

$$\alpha x^3 + (1 - \alpha)x = x$$

We obtain three fixed points $x_1 = 0$, $x_2 = 1$ and $x_3 = -1$

Now we investigate each of them in three cases.

Case1. $x = 0$

$$f_{\alpha}(x) = \alpha x^3 + (1-\alpha)x$$

$$f'_{\alpha}(x) = 3\alpha x^2 + (1-\alpha)$$

$$\Rightarrow f'_{\alpha}(0) = 3\alpha(0) + (1-\alpha)$$

$$\Rightarrow f'_{\alpha}(0) = 1-\alpha$$

$$\Rightarrow |f'_{\alpha}(0)| = |1-\alpha|$$

Then if $\alpha > 2$ or $\alpha < 0$ that leads to make

$|f'_{\alpha}(0)| > 1$ which means that a fixed point $x = 0$ is

a repelling fixed point by theorem (3.1).

If $0 < \alpha < 2$ that leads to make $|f'_{\alpha}(0)| < 1$ which

means that a fixed point $x = 0$ is an attracting fixed point by theorem (3.1).

If $\alpha = 2$ that leads to make $|f'_{\alpha}(0)| = 1$ and the

theorem above doesn't tell us anything about the fixed point, therefore we must use another criterion using Schwarzian derivative which defined as follows :[3]

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2} \left[\frac{f''(x)}{f'(x)} \right]^2$$

Since $x = 0$ and $\alpha = 2$

$$\Rightarrow f'_{\alpha}(x) = 3\alpha x^2 + (1-\alpha)$$

$$\Rightarrow f''_{\alpha}(x) = 6\alpha x$$

$$\Rightarrow f'''_{\alpha}(x) = 6\alpha$$

$$Sf(0) = -f'''(0) - \frac{3}{2} [f''(0)]^2$$

$$= -6\alpha - \frac{3}{2} [6\alpha(0)]^2$$

$$= -12$$

Since , $Sf(0) < 0$ then a fixed point $x = 0$ is

attracting when $\alpha = 2$ and it's basin of attracting is $B_0 = (-1, 1)$.

Case2 . $x = 1$

$$f'_{\alpha}(1) = 2\alpha + 1$$

$$|f'_{\alpha}(1)| = |2\alpha + 1|$$

so, if $\alpha > 0$ or $\alpha < -1 \Rightarrow |f'_{\alpha}(1)| > 1$ which makes

the fixed point $x = 1$ is a repelling fixed point.

But, if $-1 < \alpha < 0$ then $|f'_{\alpha}(1)| < 1$ which makes

the fixed point $x = 1$ is an attracting fixed point.

If $\alpha = -1$ then $|f'_{\alpha}(1)| = 1$ and since $f'_{\alpha}(1) = -1$ we

need to use Swarzian derivative again to determine the character of the point.

Calculating the value of the derivative we obtain:

$$Sf(1) = -48$$

Since $Sf < 0$ then , the fixed point $x = 1$ is attracting and the basin of attraction $B_0 = (0, 1]$.

Case3. $x = -1$

We obtain that $|f'_{\alpha}(-1)| = |2\alpha + 1|$, therefore the same discussion repeated as in case 2 .

so, if $\alpha > 0$ or $\alpha < -1 \Rightarrow |f'_{\alpha}(1)| > 1$ which makes

the fixed point $x = -1$ is a repelling fixed point.

But, if $-1 < \alpha < 0$ then $|f'_{\alpha}(1)| < 1$ which makes the

fixed point $x = -1$ is an attracting fixed point.

If $\alpha = -1$ then $|f'_{\alpha}(1)| = 1$ and since $f'_{\alpha}(1) = -1$ we

need to use Swarzian derivative again to determine the character of the point.

Calculating the value of the derivative we obtain:

$$Sf(1) = -48$$

Since $Sf < 0$ then , the fixed point $x = -1$ is attracting and the basin of attracting $B_0 = [-1, 0)$.

4. Chaotic behaviour of the family

Definition . 4.1 [3],[5],[2]

Let J be a bounded interval, and $f : J \rightarrow J$ continuously differentiable on J , then $\lambda(x)$ which defined as follows:

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln |(f^{[n]})'(x)|$$
 , is called Lyapunov

exponent of f .

Definition . 4.2 [5]

Let J be an interval , and suppose that $f : J \rightarrow J$.

Then f has sensitive dependence on initial conditions at X if , there exist $\varepsilon > 0$ such that for each $\delta > 0$, there is $y \in J$ and $n \in \mathbb{N}^+$ such that

$$|x - y| < \delta \text{ and } |f^{[n]}(x) - f^{[n]}(y)| > \varepsilon$$

Definition . 4.3

We will use the next formula to calculate Lyapunov exponent [5]

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |f'_{\alpha}(x)|$$
 then we obtain

$$\lambda(x) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=0}^{n-1} \ln |3x^2 + (1-\alpha)|$$

$$\Rightarrow \lambda(x) > 0$$

Definition . 4.4 [5]

A function f is said to be chaotic if it is satisfied at least one of the following conditions:

- i. f has positive Lyapunov exponent at each point in it's domain.
- ii. f has sensitive dependence on initial conditions on it's domain.

Then , the cubic family is chaotic by Guilick definition of chaos since it has positive Lyapunov exponent .

5. Conjugacy

In some cases , it's difficult to show if one function has features as transitivity or existence of a dense set of periodic points and it's easier to find other function which has these features and conjugate to our function , specially existence of period - 3 points for the cubic family will give an important clue about the chaotic behaviour of the family, but to find out such

points is not as easy [10]. If two functions are conjugate to one another, then one function inherits such properties as transitivity and the existence of a dense set of periodic points from the other one.

Definition. 5.1 [2],[5],[7]

Let J and K be intervals, and suppose that $f: J \rightarrow J$ and $g: K \rightarrow K$. then f and g are conjugate if there exist a homeomorphism $h: J \rightarrow K$ such that $h \circ f = g \circ h$ and in this case we write $f \underset{h}{\approx} g$.

Theorem. 5.2 Suppose $f: [a, b] \rightarrow [a, b]$ is a continuous map. If f has an orbit of period - 3, then f is chaotic. [7]

Theorem. 5.3 If $f: X \rightarrow X$ and $g: Y \rightarrow Y$ are conjugate maps via conjugacy h , then f is chaotic iff g is chaotic. [4]

Let g be defined as follows:

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$$g(x) = \begin{cases} 3x & , 0 \leq x < \frac{1}{3} \\ 2-3x & , \frac{1}{3} \leq x < \frac{2}{3} \\ 3x-2 & , \frac{2}{3} \leq x \leq 1 \end{cases}$$

We'll show that $f \underset{h}{\approx} g$ in case if $\alpha = 4$, i.e.

$$f(x) = 4x^3 + 3x \quad \text{and} \quad h(x) = \cos(\pi x).$$

Since $f \circ h(x) = \cos(3\pi x)$

And $h \circ g(x) = \cos(3\pi x)$

Then $f \circ h(x) = h \circ g(x)$

To show that f has period-3 point

let $x = \frac{2}{7}$, then $\left\{ \frac{2}{7}, \frac{6}{7}, \frac{4}{7} \right\}$ a 3-cycle for g .

Then g has a periodic orbit [5] and therefore it is chaotic by theorem 5.2.

So the cubic function f is chaotic by theorem 5.3

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السلوك الديناميكي لعائلة الدوال التكريرية

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الملخص

لقد قدم May R.M. مثالا لعائلة الدوال التكريرية على الفترة $[-1, 1]$ ، وقام Rogers T. D. بتوسيع تحليل ماي على فترة أكبر. نقدم في هذه الورقة دراسة شاملة لعائلة الدوال التكريرية المعرفة بالشكل: $f_\alpha(x) = \alpha x^3 + (1-\alpha)x$. حيث تم ايجاد النقاط الثابتة للعائلة وتم وصفها لقيم مختلفة للمعلمة α . كما تم مناقشة السلوك الديناميكي والفوضوي للعائلة نسبة الى مختلف تعاريف الفوضى ومن خلال الدالة المرافقة.