Journal of Basrah Researches ((Sciences)) Vol. 36, No.1, 15 February ((2010))

ON THE ORLIK-SOLOMON ALGEBRA OF A HYPERSOLVABLE ARRANGEMENT

ISSN -1817 -2695

Abd Ali Al-Ta'ai[1], Hana' M. Ali[2] and Mayadah A. Majeed[2]

[1]*Ministry of Higher Education/ Iraq.*

[2] *Department of Mathematics, College of Science, University of Basrah.*

((Received 1/9/2009,Accepted 9/12/2009))

Abstract:

This paper is devoted to construct the Orlik-Solomon algebra of a hypersolvable r-arrangement by using the hypersolvable partition analogue. We used the hypersolvable ordering which preserves the hypersolvable structure to define the NBC-basis of the Orlik-Solomon algebra and we embedded it in the partition module as a free graded submodule. We used this ordering also to prove that a hypersolvable arrangement is quadratic if, and only if, it's fiber-type. As a generalization of a result of Jambu and Papadima, we gave the quadratic Orlik-Solomon algebra a structure isomorphic to a partition module and we used this isomorphism to produce a comparison between the Orlik-Solomon algebra and the quadratic Orlik-Solomon algebra as free modules with many applications.

Key words: Hyperplane arrangements, hypersolvable arrangements, the exterior algebra, the Orlik-Solomon algebra and the quadratic Orlik-Solomon algebra.

Introduction:

Let $A=\{H_1,\ldots,H_n\}$ be a complex hyperplane r-arrangement, with complement $M(A)=C^r/\bigcup_{i=1}^{n} H_i$. The cohomology ring for the complement $M(A)$ with arbitrary constant coefficients was given by Arnold [2], 1969 and Brieskorn [3], 1971. Orlik and Solomon [8], 1980 generalized Brieskorn's result to construct a graded algebra $\mathrm{A}^*(A)$ associated to a complex r-arrangement A and their description involves the intersection lattice $L(A)=\{X\subseteq C^r \mid X=\bigcap_{H\in B} H \text{ and } B\subseteq A\}$ of A which is partially ordered by reverse inclusions and ranked by $\mathrm{rk}(X)=\mathrm{codim}(X)=r-\dim(X)$. For a commutative ring K and an arbitrary total order $\trianglelefteq$ on A, they defined $\mathrm{A}^*(A)$ to be the quotient of the exterior algebra $\wedge^{p\geq 0}(\oplus_{i=1}^{n}\mathrm{K}e_{H_i})$ by a homogeneous ideal $I(A)$ generated by relations $\sum_{k=1}^{p}(-1)^{k-1}e_{H_{i_1}}\ldots\hat{e}_{H_{i_k}}\ldots e_{H_{i_p}}$ for all $1\leq i_1<\cdots<i_p\leq n$ such that $\{H_{i_1},\ldots,H_{i_p}\}$ is dependent. They proved that $\mathrm{A}^*(A)$ which is named by Orlik-Solomon algebra isomorphic to the cohomological ring of the complement $H^*(M(A))$ by giving a presentation of the last one in terms of generators and relations.

For a given total order $\trianglelefteq$ on A, if $C\subseteq A$ is a minimal (with respect to inclusion) dependent set, we call C a circuit of A and $\overline{C}=C\setminus\{H\}$ a broken circuit of C, where H is the smallest hyperplane in C via $\trianglelefteq$ and by NBC base $B\subseteq A$ we mean B contains no broken circuit. The set of the NBC bases of A forms an explicit basis of the Orlik-Solomon algebra (see [4] and we refer the reader to [9] as a general reference).

Our aim in this paper is to give a structure of the Orlik-Solomon algebra $\mathrm{A}^*(A)$ of the hypersolvable class of arrangements which was introduced by Jambu and Papadima at 1998 [5] and 2002 [6], as a generalization of a fiber-type (Stanley supersolvable) class ([13] and [9]). We used the hypersolvable partition, the hypersolvable ordering which is defined by Ali in [1], 2007 and their study of the NBC bases of a hypersolvable arrangement to construct $\mathrm{A}^*(A)$. To make our structure more agreement with our analogue we embedded it in section (2) as a graded free submodule of the partition complex $(\Pi)^*$ (definition (2.7)). We study how they are connected and the differences between them.

Section (3) is devoted to prove that the quadratic property on the hypersolvable class is related to the fiber-type subclass only and that is caused by these subarrangements of non fiber-type arrangement A which are not NBC bases of A and contain no broken circuit of rank two. Then we used the hypersolvable ordering which preserves the hypersolvable structure and a result of Jambu and Papadima (3.8) to give the partition structure to the quadratic Orlik-Solomon algebra $\overline{\mathrm{A}}^*(A)$ of a hypersolvable arrangement [5] which is defined as the quotient of the exterior algebra; $\wedge^{p\geq 0}(\oplus_{i=1}^n \mathrm{K}e_{H_i})$ by a homogeneous ideal $J(A)$ generated by relations $e_{H_{i_1}}e_{H_{i_2}}+e_{H_{i_2}}e_{H_{i_3}}+e_{H_{i_3}}e_{H_{i_1}}$, for all $1\leq i_1<i_2<i_3\leq n$ such that $S=\{H_{i_1},H_{i_2},H_{i_3}\}$ is dependent.

Section (4) in this paper reflects Ali and Al-Ta'ai studies of the hypersolvable NBC bases of a hypersolvable arrangement into the structure the Orlik-Solomon algebra for the hypersolvable class of arrangements and we compare it with the structure of the quadratic Orlik-Solomon algebra as applications of the partition complex structure.

(1)THE HYPERSOLVABLE PARTITION OF A HYPERSOLVABLE ARRANGEMENT

A hypersolvable class of arrangements was introduced by Jambu and Papadima (1998 [5], 2002[6]). Ali in ([1], 2007) derived a natural partition from the hypersolvable structure and the goal of this section is to give an impression of the importance of this partition:

Definition (1.1):

Let A be an essential central complex r-arrangement (i.e. $\bigcap_{i=1}^n H_i=T(A)=T\neq\phi$ and $\mathrm{rk}(A)=\mathrm{rk}(T(A))=\mathrm{co}\dim(\bigcap_{H\in A}H)=r=\dim(C^r)$). A partition $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ of A is said to be a *hypersolvable partition* of A with *length* $\ell(A)=\ell$ and denoted by HP, if $|\Pi_1|=1$, (i.e. Π_1 is a singleton) and for fixed $2\leq j\leq\ell$, the block Π_j satisfies the following properties:

(j-*closedness property of* Π_j**)** For each $H_1,H_2\in\Pi_1\cup\cdots\cup\Pi_j$, there is no hyperplane $H\in\Pi_{j+1}\cup\cdots\cup\Pi_\ell$ such that $\mathrm{rk}(H_1,H_2,H)=2$.

(j-*completeness property of* Π_j**)** For each $H_1,H_2\in\Pi_j$, there is a hyperplane $H\in\Pi_1\cup\cdots\cup\Pi_{j-1}$ such that $\mathrm{rk}(H_1,H_2,H)=2$. Note that, from closedness properties of the blocks $\Pi_2,\ldots,\Pi_{j-1}$, the hyperplane H is unique and we denoted it by $H_{1,2}$.

(j-*solvability property of* Π_j**)** If $H_1,H_2,H_3\in\Pi_j$, the hyperplanes $H_{1,2},H_{1,3},H_{2,3}\in\Pi_1\cup\cdots\cup\Pi_{j-1}$ are equal or $\mathrm{rk}(H_{1,2},H_{1,3},H_{2,3})=2$. Observe that, if $\mathrm{rk}(H_1,H_2,H_3)=2$ then from closendess properties of the blocks $\Pi_2,\ldots,\Pi_{j-1}$, $H_{1,2}=H_{1,3}=H_{2,3}$.

The vector of integers $d=(d_1,\ldots,d_\ell)$ is called the *exponent vector* of Π, if for $j=1,\ldots,\ell$,

$d_j = |\Pi_j|$. For $1 \le j \le \ell$ the *rank* of Π_j is defined to be $\mathrm{rk}(\Pi_j) = \mathrm{rk}(\Pi_1 \cup \cdots \cup \Pi_j) = \mathrm{rk}(\bigcap_{H \in \Pi_1 \cup \cdots \cup \Pi_j} H)$. We call the block Π_j, a *singular block* of Π if $\mathrm{rk}(\Pi_j) = \mathrm{rk}(\Pi_{j-1})$ and we call it *non-singular block* otherwise. Notice that, in general $\mathrm{rk}(\Pi_j) \le \mathrm{rk}(\Pi_{j-1}) + 1$.

Remark (1.2):

If $\ell \ge 3$, any three different blocks $\Pi_{i_1}, \Pi_{i_2}, \Pi_{i_3} \in \Pi$ such that $1 \le i_1 < i_2 < i_3 \le \ell$ are independent from i_2-closedness property of the block Π_{i_2}, i.e. for $k = 1,2,3$, if $H_{i_k} \in \Pi_{i_k}$, then $\mathrm{rk}(H_{i_1}, H_{i_2}, H_{i_3}) = 3$.

Definition (1.3):

A hypersolvable arrangement is an essential central complex r-arrangement with a composition series $A_1 \subset \cdots \subset A_j \subset A_{j+1} \subset \cdots \subset A_\ell$ with length $\ell(A) = \ell$ such that $\mathrm{rk}(A_1) = 1$, (i.e. A_1 is singleton), $A_\ell = A$, and (A_j, A_{j-1}) is solvable, for each $2 \le j \le \ell$, i.e. the pair (A_j, A_{j-1}) satisfies the following properties:

1- A_{j-1} is closed in A_j, if for each distinct hyperplanes $H_1, H_2 \in A_j$, there is no hyperplane $H \in A_j \setminus A_{j-1}$ such that $\mathrm{rk}(H_1, H_2, H) = 2$.

2- A_{j-1} is complete in A_j, if for each distinct hyperplanes $H_1, H_2 \in A_j \setminus A_{j-1}$, there exists a unique hyperplane $H_{1,2} \in A_{j-1}$ such that $\mathrm{rk}(H_1, H_2, H_{1,2}) = 2$.

3- If $H_1, H_2, H_3 \in A_j \setminus A_{j-1}$, the hyperplanes $H_{1,2}, H_{1,3}, H_{2,3} \in A_{j-1}$ are equal or $\mathrm{rk}(H_{1,2}, H_{1,3}, H_{2,3}) = 2$.

Proposition (1.4):

Let A be an essential central complex r-arrangement. A is hypersolvable if, and only if, A has a HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$.

Proof: If A is a hypersolvable r-arrangement with a composition series $A_1 \subset \cdots \subset A_{i-1} \subset A_i \subset \cdots \subset A_\ell$, then the partition $\Pi = (\Pi_1, \cdots, \Pi_\ell)$ which is defined by $\Pi_1 = A_1$ and for $i = 2, \ldots, \ell$, $\Pi_i = A_i \setminus A_{i-1}$ forms a HP of A (we refer the reader to [1] for more details). Conversely, if $\Pi = (\Pi_1, \cdots, \Pi_\ell)$ forms a HP of A, then the composition series $A_1 \subset \cdots \subset A_{i-1} \subset A_i \subset \cdots \subset A_\ell$ which is defined by $A_i = \bigcup_{H \in \Pi_1 \cup \cdots \cup \Pi_i} H$, $1 \le i \le \ell$ gives A a structure of hypersolvable arrangement [5]. □

Definition (1.5):

Let A be a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$. For fixed $1 \le j \le \ell$, the properties of the hypersolvable partition give rise to a natural partition to the block Π_j as follows:

1- Let $\Pi_{j*1} = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq \Pi_j$, such that $\mathrm{rk}(H_{i_1}, \ldots, H_{i_k}) = 2$.

2- Let $\Pi_{j*2} = \Pi_j \setminus \Pi_{j*1}$

Define the *hypersolvable ordering of A induced by the HP* Π and denoted by $\trianglelefteq$, as follows:

1- If $H \in \Pi_i$ and $H' \in \Pi_j$ such that $1 \le i < j \le \ell$, put $H \trianglelefteq H'$.

2- For fixed $1 < j \le \ell$, give the hyperplanes of the subblock Π_{j*1} of Π_j an arbitrary total order with preserving the order of Π_i in Π for each $1 \le i \le j-1$ and preserving the order of Π_{j*2} as if $H_1, H_2, H_3 \in \Pi_j$ with $\mathrm{rk}(H_1, H_2, H_3) = 3$, put $H_{i_1} \trianglelefteq H_{i_2} \trianglelefteq H_{i_3}$ if, and only if, $H_{i_1,i_2} \trianglelefteq H_{i_1,i_3} \trianglelefteq H_{i_2,i_3}$ such that $\{H_{i_1}, H_{i_2}, H_{i_3}\} = \{H_1, H_2, H_3\}$. Observe that, since $\mathrm{rk}(H_1, H_2, H_3) = 3$ then there is at least one of H_1, H_2, H_3 is in Π_{j*2}.

Notice that the hypersolvable ordering of a hypersolvable arrangement respects the hypersolvable structure.

Definition (1.6): [13]

Let A be a central complex r-arrangement and $L(A)$ be the intersection lattice of A which is partially ordered by the inclusions and ranked by $\mathrm{rk}(X) = \mathrm{codim}(X) = r - \dim(X)$. An element $X \in L(A)$ is said to be modular if and only if $X + Y \in L(A)$ for all $Y \in L(A)$. We call A *supersolvable* if it has a maximal chain of modular elements $V = X_0 < X_1 < \cdots < X_\ell = T(A)$. Note that such maximal chain of modular elements, if it exists, needs not to be unique.

Proposition (1.7): [5]

An essential central complex r-arrangement A is said to be supersolvable if, and only if, A is a hypersolvable arrangement with $r = \mathrm{rk}(A) = \ell(A) = \ell$.

Note (1.8):

Let A be a supersolvable ℓ-arrangement and $\Pi = (\Pi_1, \cdots, \Pi_\ell)$ be a fixed hypersolvable partition of A. We call your attention that we ordered the hyperplanes of A by the hypersolvable ordering which preserves the supersolvable structure. Notice that, A has a composition series associated with the Hp Π is $A_1 \subset \cdots \subset A_j \subset A_{j+1} \subset \cdots \subset A_\ell$, where for $1 \le j \le \ell$ $A_j = \bigcup_{H \in \Pi_1 \cup \cdots \cup \Pi_j} H$, which gives A a structure of a hypersolvable arrangement, where the maximal chain of the modular elements $V = X_0 < X_1 < \cdots < X_\ell = T(A)$ which is defined as $X_j = \bigcap_{H \in A_j} H$, $1 \le j \le \ell$ obtains on A a structure of supersolvable arrangement.

(2) THE ORLIK-SOLOMON ALGEBRA, THE BROKEN CIRUIT COMPLEX AND THE PARTITION COMPLEX OF A HYPERSOLVABLE ARRANGEMENT

Orlik and Solomon [8], 1980 constructed an algebra $\mathrm{A}^*(A)$ associated with a central arrangement A, named by their names and they proved that it is isomorphic to the cohomological ring of the complement $H^*(M(A))$. This section is motivated to construct $\mathrm{A}^*(A)$ for a hypersolvable arrangement by using the hypersolvable partition analogue and we serve our goal as follows:

- We used a result of Ali to embed NBC basis of $\mathrm{A}^*(A)$ in a basis spanned by the sections of a fixed hypersolvable partition Π of A.

- We constructed a graded free module $(\Pi)^*$ which has a basis spanned by the sections of Π. We call $(\Pi)^*$ a hypersolvable partition module.
- We embedded the Orlik-Solomon algebra as a free submodule in the hypersolvable partition module to produce a simplification of the structure of $\mathrm{A}^*(A)$ and the comparison between these structures leads us to the next section.

Construction (2.1): [9]

Let K be a commutative ring with unit and let $A=\{H_1,\ldots,H_n\}$ be a central r-arrangement in C^r. By $E=E(A)=\wedge^*_{\mathrm{K}}(e_{H_1},\ldots,e_{H_n})$, we denote the exterior K-algebra generated by symbols e_H in one to one correspondence with the hyperplanes $H\in A$, which is graded by exterior degree, i.e. $E=\bigoplus_{p=0}^{n} E_p$ where $E_0=\mathrm{K}$, $E_1=\bigoplus_{H\in A}\mathrm{K}e_H$ and E_p with standard monomial K-basis;

$\mathbf{B}=\{e_S=e_{H_{i_1}}\wedge\ldots\wedge e_{H_{i_p}} \mid 1\le i_1<\cdots<i_p\le n\}$,

i.e. it is spanned by all subarrangements $S=\{H_{i_1},\ldots,H_{i_p}\}\subseteq A$ such that $1\le i_1<\cdots<i_p\le n$. Write $uv=u\wedge v$ and note that $e_H^2=0$ and $e_He_{H'}=-e_{H'}e_H$ for $H',H\in A$. Observe that the construction of the exterior algebra depends of an arbitrary total ordering $\trianglelefteq$ defined on the hyperplanes of A and this ordering can give the monomial K-basis $\mathbf{B}$ the degree lexicographic (DegLex) order. That is; if $p<q$, then $e_{H_{i_1}}\ldots e_{H_{i_p}}<e_{H_{j_1}}\ldots e_{H_{j_q}}$ and if $k_0=\min\{k \mid i_k\ne j_k\}$, then $e_{H_{i_1}}\ldots e_{H_{i_p}}<e_{H_{j_1}}\ldots e_{H_{j_p}}$ if $H_{i_{k_0}}\trianglelefteq H_{j_{k_0}}$.

Define a K-linear map $\partial_E=\partial:E\to E$ by $\partial 1=0$, $\partial e_H=1$ and $\partial e_{H_{i_1}}\ldots e_{H_{i_p}}=\sum_{k=1}^{p}(-1)^{k-1}e_{H_{i_1}}\ldots\hat{e}_{H_{i_k}}\ldots e_{H_{i_p}}$, for all $H_{i_1},\ldots H_{i_p}\in A$. The pair (E,∂_E) forms a chain complex with ∂_E is the unique derivation of E satisfies $\partial e_H=1$.

Define the *Orlik-Solomon algebra* $\mathrm{A}=\mathrm{A}^*_{\mathrm{K}}(A)$ to be the quotient of E by the ideal $I=I(A)$ which is generated by relations of the form $\partial e_S=\partial e_{H_{i_1}}\ldots e_{H_{i_p}}=\sum_{k=1}^{p}(-1)^{k-1}e_{H_{i_1}}\ldots\hat{e}_{H_{i_k}}\ldots e_{H_{i_p}}$ for all $1\le i_1<\cdots<i_p\le n$ such that $S=\{H_{i_1},\ldots,H_{i_p}\}$ is dependent, i.e. $\mathrm{rk}(S)<p$. The gradation of E defines a gradation of the ideal I as follows: Since each subarrangement of A contains one hyperplane or two hyperplanes is independent then $I_0=0$ and $I_1=0$. For each $2\le p\le r$, let

$\mathrm{I}_p=\{e_B \mid B\subseteq A, |B|=p+1 \text{ and } rk(B)<p+1\}$,

i.e. I_p is the collection of those monomials that are related to the dependent subarrangements with cardinality $p+1$. Then I_p is generated by $\partial\mathrm{I}_p$.

By $\varphi:E\to\mathrm{A}=E/I$ we denote the canonical projection and for each $H\in A$, let $a_H=\varphi(e_H)=e_H+I$ and $\varphi(E_p)=\mathrm{A}_p$. That is $\mathrm{A}=\bigoplus_{p=1}^{rk(A)}\mathrm{A}_p$ be a graded anticommutative algebra, where $\mathrm{A}_0=\mathrm{K}$, $\mathrm{A}_1=E_1$. Since $\partial_E I\subseteq I$, then a K-linear map $\partial_{\mathrm{A}}:\mathrm{A}\to\mathrm{A}$ defined by $\partial_{\mathrm{A}}\varphi=\varphi\partial_E$ gives the pair $(\mathrm{A},\partial_{\mathrm{A}})$ a structure of acyclic chain complex, i.e. the following sequence is exact;

$$0 \xrightarrow{\partial_{r+1}^{\mathrm{A}}} \mathrm{A}_r(A) \xrightarrow{\partial_r^{\mathrm{A}}} \mathrm{A}_{r-1}(A) \xrightarrow{\partial_{r-1}^{\mathrm{A}}} \cdots \xrightarrow{\partial_1^{\mathrm{A}}} \mathrm{A}_0(A) \xrightarrow{\partial_0^{\mathrm{A}}} 0 .$$

Definition (2.2) [12]:

Let $L(A) = \{X \subseteq C^r \mid X = \bigcap_{H\in B} H \text{ and } B \subseteq A\}$ be the intersection lattice of a central complex r-arrangement A which is partially ordered by reverse inclusions, i.e. $X \le Y \Leftrightarrow Y \subseteq X$ and ranked by $\mathrm{rk}(X) = \mathrm{codim}(X) = r - \dim(X)$. Then:

1- If $C \subseteq A$ is a minimal (with respect to inclusion) dependent subarrangement, then we call C a *circuit* of A. That is, C forms a circuit of A if, and only, if $C \setminus \{H\}$ is linearly independent for each $H \in C$. Observe that $\mathrm{rk}(C) = |C| - 1$.

2- Put a total order $\trianglelefteq$ on A to distinguish it from the partial order $\le$ of $L(A)$. A corresponding *broken circuit* of a circuit C is $\overline{C} = C \setminus \{H\}$, where H is the smallest hyperplane in C.

3- We call $B \subseteq A$, a *NBC-base* if it contains no broken circuit. Note that, such set must be independent and we denoted B by i-NBC base if $|B| = i$.

4- Let $X \in L(A)$ and $A_X \subseteq A$ be the set of all hyperplanes $H \in A$ such that $H \le X$. We say that the NBC-base $B \subseteq A_X$ is a NBC-base of X if $\bigcap_{H\in B} H = X$ and we call the broken circuit $\overline{C} \subseteq A_X$ is a broken circuit of X if $\bigcap_{H\in \overline{C}} H = X$.

Remark (2.3):

Rota [11] and Sagan [12], showed that the number of the k^{th} NBC-bases of A are independent of our choice of the total ordering $\trianglelefteq$ on the hyperplanes of A. Thus a NBC base of A for a given total order may be a broken circuit in another total order on A. That is, for a fixed total order $\trianglelefteq$ on A, if $B = \{H_{i_1}, \ldots, H_{i_k}\} \subseteq A$ is a k^{th}-broken circuit, then there is a hyperplane $H \in A$ such that $H \trianglelefteq H_{i_j}$, $1 \le j \le k$ and $\{H\} \bigcup B$ is a circuit where $\{H\} \bigcup B \setminus \{H_{i_j}\}$ forms an independent subarrangement for each $1 \le j \le k$ (may be NBC base or broken circuit). Thus $e_{\{H\}\bigcup B} \in \mathrm{I}_k$ and $\partial_{k+1}^{\mathrm{A}} a_{\{H\}\bigcup B} = a_B - a_H \partial_k^{\mathrm{A}} a_B = 0_{\mathrm{A}_k}$. That is;

$$a_B = a_H \partial_k^{\mathrm{A}} a_B = \sum_{j=1}^{k} (-1)^{j-1} a_{\{H\}\bigcup B\setminus\{H_{i_j}\}} ,$$

written as a linear combination of monomials derived by independent subarrangements with DegLex order, $a_{\{H\}\bigcup B\setminus\{H_{i_j}\}} < a_B$ for each $1 \le j \le k$.

We mention, if A is a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$, the total order $\trianglelefteq$ on A is defined to be the hypersolvable order which we described in definition (1.5) and the ordering on the monomial K-basis $\mathbf{B}$ of the exterior algebra E is defined to be the DegLex ordering which is obtained from the hypersolvable ordering.

Notation (2.4):

Let $L(A)$ be the intersection lattice of a central complex r-arrangement A. Then:

1- For each $X \in L(A)$, the set of all NBC bases of X is denoted by $NBC^X(A)$ and the set of all broken circuit of X is denoted by $BC^X(A)$.

2- If $L_i(A) = \{X \in L(A) \mid rk(X) = i\} \subseteq L(A)$, let $NBC_i(A) = \bigcup_{X\in L_i(A)} X \in L_i(A)$, $NBC(A) = \bigcup_{i=1}^{rk(A)} NBC_i(A)$.

Let $BC_i(A)=\bigcup_{X\in L_i(A)} BC^X(A)$ and $BC(A)=\bigcup_{i=1}^{rk(A)} BC_i(A)$. In fact, for $X_1, X_2 \in L_i(A)$, $NBC^{X_1}(A)\cap NBC^{X_2}(A)=\varphi$ and $BC^{X_1}(A)\cap BC^{X_2}(A)=\varphi$. Therefore, $|NBC_i(A)|=\sum_{X\in L_i(A)}|NBC^X(A)|$ and $|BC_i(A)|=\sum_{X\in L_i(A)}|BC^X(A)|$.

3- If $X\in L_i(A)$, the Möbius function $\mu(X)=(-1)^i|NBC^X(A)|$, see [11]. Therefore, the *Poincaré polynomial* of A can be represented as;

$$p(A,t)=\sum_{X\in L(A)}\mu(X)(-t)^{rk(X)}=\sum_{i\geq o}|NBC_i(A)|\,t^i.$$

We call $b_i=|NBC_i(A)|$, the i^{th} *Betti number* of $P(A,t)=\sum_{i\geq 0} b_i t^i$.

Definition (2.5): [9]

A *broken circuit module* $\mathrm{NBC}(A)$ is defined as; $\mathrm{NBC}_0(A)=\mathrm{K}$ and $\mathrm{NBC}_p(A)$ be the free K-module with NBC monomials basis $\{e_B\in E \mid B\in NBC_p(A)\}$, let $\mathrm{NBC}(A)=\bigoplus_{p=0}^{rk(A)}\mathrm{NBC}_p(A)$. Then $\mathrm{NBC}(A)$ is a free K-submodule (not subalgebra) of $E(A)$. Let $\partial_C:\mathrm{NBC}(A)\to\mathrm{NBC}(A)$ be the restriction of $\partial_E: E\to E$. Then the pair $(\mathrm{NBC}(A),\partial_C)$ has a structure of acyclic chain complex, i.e.;

$$0\xrightarrow{\partial^C_{r+1}}\mathrm{NBC}_r(A)\xrightarrow{\partial^C_r}\mathrm{NBC}_{r-1}(A)\xrightarrow{\partial^C_{r-1}}\cdots\xrightarrow{\partial^C_1}\mathrm{NBC}_0(A)\xrightarrow{\partial^C_0}0,$$

is exact.

Remark (2.6): [9]

The restriction $\psi:\mathrm{NBC}(A)\to\mathrm{A}(A)$ of the K-linear map $\varphi: E(A)\to\mathrm{A}(A)$ on $\mathrm{NBC}(A)$ forms an isomorphism as the following commutative diagram:

$$\begin{array}{ccccccccc}
0\xrightarrow{\partial^C_{r+1}} & \mathrm{NBC}_r(A) & \xrightarrow{\partial^C_r} & \mathrm{NBC}_{r-1}(A) & \xrightarrow{\partial^C_{r-1}}\cdots\xrightarrow{\partial^C_1} & \mathrm{NBC}_0(A) & \xrightarrow{\partial^C_0} 0\\
 & \psi_r\downarrow & & \psi_{r-1}\downarrow & & \psi_0\downarrow & \\
0\xrightarrow{\partial^{\mathrm{A}}_{r+1}} & \mathrm{A}_r(A) & \xrightarrow{\partial^{\mathrm{A}}_r} & \mathrm{A}_{r-1}(A) & \xrightarrow{\partial^{\mathrm{A}}_{r-1}}\cdots\xrightarrow{\partial^{\mathrm{A}}_1} & \mathrm{A}_0(A) & \xrightarrow{\partial^{\mathrm{A}}_0} 0
\end{array}$$

The set

$\mathbf{C}=\{e_B+I=a_B\in\mathrm{A}(A)\mid B\in NBC(A)\}$

forms a basis for the Orlik-Solomon algebra $\mathrm{A}(A)$ as a graded free K-module. The DegLex order on the monomial K-basis $\mathbf{B}$ of the exterior algebra E induced a DegLex order on $\mathbf{C}$ as $a_{B_1}<a_{B_2}$ if, and only if, $e_{B_1}<e_{B_2}$, $B_1, B_2\in NBC(A)$.

Define the *Poincaré polynomial* of $\mathrm{A}(A)$ by $p(\mathrm{A}(A),t)=\sum_{k=0}^{r} rk(\mathrm{A}_k(A)).t^k$, where t is an indeterminate. We call $b_k=rk(\mathrm{A}_k(A))=|NBC_k(A)|$, the i^{th} *Betti number of* $P(\mathrm{A}(A),t)=\sum_{k\geq 0} b_k.t^k$.

Definition (2.7): [9]

Let $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ be a partition of a central r-arrangement $A=\{H_1,\ldots,H_n\}$ and let (Π_i) denote the free K-module with basis 1 and the elements of Π_i and it is graded by $\deg(1)=0$ and $\deg(H)=1$. Define the graded *partition* K-*module* $(\Pi)=(\Pi_1)\otimes\cdots\otimes(\Pi_\ell)$. We agree that $(\Pi)=\mathrm{K}$ if $A=\Phi_r$ (the empty r-arrangement).

Let $S=\{H_{i_1},\ldots,H_{i_k}\}$ be a subarrangement of A such that $1\le i_1<\cdots<i_k\le n$. Call S a k-*section* of Π, if for each $1\le j\le k$, $H_{i_j}\in\Pi_{m_j}$ such that $1\le m_1<\cdots<m_k\le\ell$. We agree that the 0-section is $S=\{\}$. By $\mathbf{S}_k$ we denote the set of all k-sections of Π and $\mathbf{S}=\bigcup_{k=0}^{\ell}\mathbf{S}_k$. For each $S\in\mathbf{S}_k$, define $p_S=x_1\otimes\cdots\otimes x_\ell\in(\Pi)$ such that;

$$x_j=\begin{cases}H_{i_l} & if\ \ j=m_l,1\le l\le k;\\ 1 & if\ \ j\notin\{m_1,\ldots,m_k\}.\end{cases}$$

Observe that $p_{\{\}}=1$ and p_S is homogeneous of degree k. Denote by the homogeneous part of degree k of (Π) by $(\Pi)_k$, then $(\Pi)=\bigoplus_{k=0}^{\ell}(\Pi)_k$. Therefore, the graded K-module (Π) is free with K-basis $\{p_S\mid S\in\mathbf{S}\}$. We switch your attention that the construction of the partition module (Π) depends on two orders. The first one is defined on the blocks of the partition Π and the second one is defined on the hyperplanes of A with the fact that each ordering of them preserves the other one. That is for a given total orderings on A and Π, define the DegLex order on the K-basis $\{p_S\mid S\in\mathbf{S}\}$, i.e. if $k_1<k_2$, then $p_{H_{i_1}}\ldots p_{H_{i_{k_1}}}<p_{H_{j_1}}\ldots p_{H_{j_{k_2}}}$ and if $k_0=\min\{m\mid i_m\ne j_m\}$, then $p_{H_{i_1}}\cdots p_{H_{i_k}}<p_{H_{j_1}}\cdots p_{H_{j_k}}$ if $H_{i_{k_0}}\trianglelefteq H_{j_{k_0}}$.

Let $S=\{H_{i_1},\ldots,H_{i_k}\}\in\mathbf{S}_k$ and for each $1\le j\le k$, let S_j denotes the subarrangement of S with H_{i_j} deleted and we call it a *boundary* of S. Define a K-linear map $\partial_k:(\Pi)_k\to(\Pi)_{k-1}$ by $\partial_0(p_{\{\}})=0$, $\partial_1(p_H)=1$ and for $2\le k\le\ell$, $\partial_k(p_S)=\sum_{j=1}^{k}(-1)^{j-1}p_{S_j}$. Then $\partial\partial=0$ and $((\Pi),\partial_{(\Pi)})$ is a chain complex and we call it the *partition complex*.

Lemma (2.8): [9]

Suppose we have the conclusions of definition (2.7). If Π contains a block which is a singleton, then the complex $((\Pi),\partial_{(\Pi)})$ is acyclic, i.e. the sequence

$$0\xrightarrow{\partial^{\Pi}_{\ell+1}}(\Pi)_\ell\xrightarrow{\partial^{\Pi}_{\ell}}(\Pi)_{\ell-1}\xrightarrow{\partial^{\Pi}_{\ell-1}}\cdots\xrightarrow{\partial^{\Pi}_{1}}(\Pi)_0\xrightarrow{\partial^{\Pi}_{0}}0$$ is exact.

Notation(2.9):

Let A be a hypersolvable $r-$arrangement with HP $\prod=(\prod_1,\prod_2,\ldots,\prod_\ell)$ and exponent vector $d=(d_1,\ldots,d_\ell)$. For $1\le k\le\ell$ and $1\le j\le d_k$, by $H^k_j\in\prod_k$ we denote the $((1+d_2+\ldots+d_{k-1})+j))^{th}$-hyperplane of A or it forms the j^{th}-hyperplane of the block $\prod_k$ with respect to a hypersolvable ordering $\trianglelefteq$.

Proposition (2.10):

Let A be a hypersolvable r-arrangement with HP $\Pi=(\Pi_1,\cdots,\Pi_\ell)$ and exponent vector $d=(d_1,\ldots,d_\ell)$. Then the complex $((\Pi),\partial_{(\Pi)})$ is acyclic, for $2\le k\le \ell$;

$$(\Pi)_k=\bigoplus_{S\in \mathbf{S}_k} \mathrm{K}p_S \approx \bigoplus_{i_1=1}^{\ell-k+1}\big(\bigoplus_{\substack{j_1=1\\ H^{i_1}_{j_1}\in\Pi_{i_1}}}^{d_{i_1}}(\cdots(\bigoplus_{i_k=i_{k-1}+1}^{\ell}(\bigoplus_{\substack{j_k=1\\ H^{i_k}_{j_k}\in\Pi_{i_k}}}^{d_{i_k}}\mathrm{K}p_{\{H^{i_1}_{j_1},\ldots,H^{i_k}_{j_k}\}}))\cdots)\big),$$

and $rk((\Pi)_k)=\sum_{i_1=1}^{\ell-k+1}(\sum_{i_2=i_1+1}^{\ell-k+2}(\cdots(\sum_{i_k=i_{k-1}+1}^{\ell}d_{i_1}d_{i_2}\ldots d_{i_k})\cdots))$.

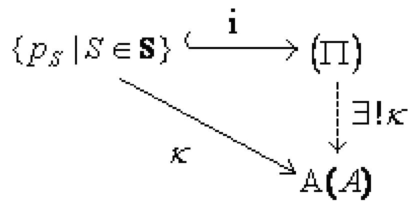
Proof: From the construction of the HP Π, the first block Π_1 is a singleton. Hence the complex $((\Pi),\partial_{(\Pi)})$ is acyclic and the number of all k-sections is

$$|\mathbf{S}_k|=\sum_{i_1=1}^{\ell-k+1}(\sum_{i_2=i_1+1}^{\ell-k+2}(\cdots(\sum_{i_k=i_{k-1}+1}^{\ell}d_{i_1}d_{i_2}\ldots d_{i_k})\cdots))$$

which forms the rank of $(\Pi)_k$ for $2\le k\le \ell$. □

Definition (2.11): [9]

Suppose we have the conclusions of definition (2.7). Define $\kappa:(\Pi)\to \mathrm{A}(A)$ as follows: For $S\in\mathbf{S}$, let $\kappa(p_S)=a_S=e_S+I$ and let κ be the unique homogeneous K-linear map of degree 0 which extends this assignment as the following commutative diagram:



Definition (2.12): [9]

Let $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ be a partition of a central r-arrangement A and let $L(A)$ be the intersection lattice of A.

1- We say Π is *independent* if for every choice of hyperplanes $H_i\in\Pi_i$ for $1\le i\le\ell$, the resulting ℓ- hyperplanes are independent, i.e. $\mathrm{rk}(\bigcap_{i=1}^{\ell}H_i)=\ell$, i.e. every ℓ-section of Π is independent. That is, for $1\le k\le\ell$, every k-section is independent

2- For each $X\in L(A)$, *the induced partition* Π_X is a partition of the arrangement $A_X=\{H\in A\mid X\subseteq H\}$ and its blocks are the non-empty subsets $\Pi_i\cap A_X$.

3- Π is called *nice* if it is independent and if $X\in L(A)\setminus V$, the induced partition Π_X contains a block which is singleton.

Theorem (2.13):

Let A be a hypersolvable r-arrangement with HP $\Pi=(\Pi_1,\cdots,\Pi_\ell)$. Then, A is fiber-type arrangement if, and only if, the homogeneous K-linear map $\kappa:(\Pi)\to\mathrm{A}(A)$ is an isomorphism.

Proof: Ali in [1] proved that "*a hypersolvable r-arrangement is supersolvable if, and only if, the HP Π is nice*", where from [9], the homogeneous K

-linear map $\kappa : (\Pi) \to \mathrm{A}(A)$ is an isomorphism if, and only if, the partition Π is nice. □

Lemma (2.14):

Let A be a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$. Then for $1 \le k \le r$, $NBC_k(A) \subseteq \mathbf{S}_k$ and $NBC(A) \subseteq \mathbf{S}$.

Proof: This is a direct result of Ali's result in [1] which states that, "*in a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$, if the subarrangement $B \subseteq A$ forms a k-NBC base of A then B must contains k-hyperplanes from k-different blocks*". □

Theorem (2.15):

Let A be a hypersolvable r-arrangement with HP, $\Pi = (\Pi_1, \cdots, \Pi_\ell)$ such that $\ell > r$. Then the homogeneous K-linear map $\kappa : (\Pi) \to \mathrm{A}(A)$ is an epimorphism, i.e. $\mathrm{A}(A) \approx (\Pi)/\ker(\kappa)$ and $\ker(\kappa) \xrightarrow{\ i\ } (\Pi) \xrightarrow{\ \kappa\ } \mathrm{A}(A)$ forms a finite K-modules presentation of the free K-module $\mathrm{A}(A)$.

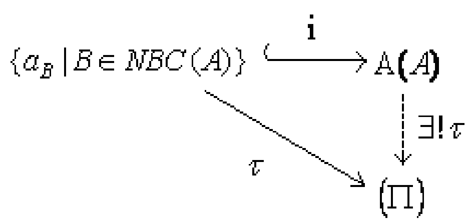
Proof: From lemma (2.14), $NBC(A) \subseteq \mathbf{S}$. Therefore;

$\{a_S \mid S \in NBC(A)\} \subseteq \{\kappa(p_S) = a_S \mid S \in \mathbf{S}\}$.

Thus, $\kappa : (\Pi) \to \mathrm{A}(A)$ is an epimorphism and by applying the first isomorphism theorem of K-linear maps we have $\mathrm{A}(A) = \mathrm{Im}(\kappa) \approx (\Pi)/\ker(\kappa)$.
□

Definition (2.16):

Let A be a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$. Define $\tau : \mathrm{A}(A) \to (\Pi)$ as follows. For $B \in NBC(A)$, let $\tau(e_B + I) = \tau(a_B) = p_B$ and from the universal property of the free K-module $\mathrm{A}(A)$, let τ be the unique homogeneous K-linear map of degree 0 which extends this assignment as the following commutative diagram:



Theorem (2.17):

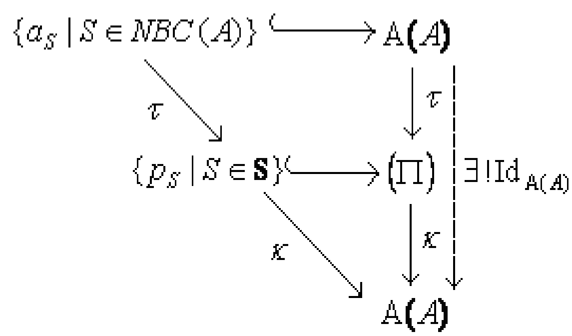
Let A be a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$. Then A is fiber-type arrangement if, and only if, $\tau = \kappa^{-1}$ where κ is the K-linear map defined in (2.11) and for $2 \le k \le \ell$;

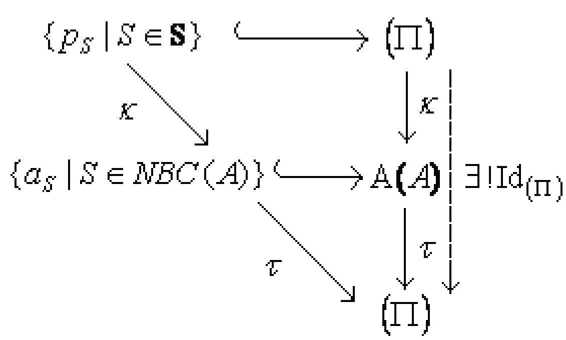
$$\mathrm{A}_k(A) \approx (\Pi)_k = \bigoplus_{S \in \mathbf{S}_k} \mathrm{K} p_S \approx \bigoplus_{\substack{i_1=1}}^{\ell-k+1} \Big(\bigoplus_{\substack{j_1=1 \\ H^{i_1}_{j_1} \in \Pi_{i_1}}}^{d_{i_1}} \Big(\cdots \Big(\bigoplus_{i_k = i_{k-1}+1}^{\ell} \Big(\bigoplus_{\substack{j_k=1 \\ H^{i_k}_{j_k} \in \Pi_{i_k}}}^{d_{i_k}} \mathrm{K} p_{\{H^{i_1}_{j_1}, \ldots, H^{i_k}_{j_k}\}} \Big)\Big) \cdots \Big)\Big).$$

Proof: Suppose A is a fiber-type r-arrangement. Ali showed that "*a hypersolvable r-arrangement is fiber type if, and only if, for each $1 \le k \le r$, every subarrangement $B \subseteq A$ that contains k-hyperplanes from k-different blocks forms a k-NBC base of A*". That is every k-section forms a k-NBC base of A, i.e. for $1 \le k \le r$, $NBC_k(A) = \mathbf{S}_k$. Thus;

$$\{\tau(a_S) \mid S \in NBC_k(A)\} = \{p_S \mid S \in \mathbf{S}_k\},\ \{\kappa(p_S) \mid S \in \mathbf{S}_k\} = \{a_S \mid S \in NBC_k(A)\}$$

and by applying the universal properties of the free K-modules $\mathrm{A}(A)$ and (Π), we have the following two commutative diagrams respectively:





Therefore, $\kappa \circ \tau = \mathrm{Id}_{\mathrm{A}(A)}$ and $\tau \circ \kappa = \mathrm{Id}_{(\Pi)}$. i.e. $\tau = \kappa^{-1}$.

Conversely, assume that $\tau = \kappa^{-1}$. That is $\kappa \circ \tau = \mathrm{Id}_{\mathrm{A}(A)}$ and $\tau \circ \kappa = \mathrm{Id}_{(\Pi)}$. Thus the homogeneous K-linear map $\kappa : (\Pi) \to \mathrm{A}(A)$ is an isomorphism and by applying theorem (2.13), A is a fiber-type arrangement. □

Theorem (2.18):

Let A be a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$ and $r < \ell$. Then the restriction of the homogeneous K-linear map $\tau : \mathrm{A}(A) \to (\Pi)$ on $\mathrm{Im}(\tau)$ of (Π), $\tau' : \mathrm{A}(A) \to \mathrm{Im}(\tau)$ is an isomorphism of the free K-modules, where $\mathrm{Im}(\tau)$ is the free K-submodule of (Π) with basis $\{p_S \mid S \in NBC(A)\}$.

Proof: By applying the first isomorphism theorem of K-linear maps we have $\mathrm{Im}(\tau) \approx \mathrm{A}(A)/\ker(\tau)$. Since $\{a_S \mid S \in NBC(A)\}$ form a K-basis of $\mathrm{A}(A)$,

hence $\{\tau(a_S) = p_S \mid S \in NBC(A)\}$ forms a K-basis of the K-submodule $\mathrm{Im}(\tau)$ of (Π), which inherits a structure of free K-module. So, $\tau' : \mathrm{A}(A) \xrightarrow{\sim} \mathrm{Im}(\tau)$ is an isomorphism of the free K-modules.
□

Corollary (2.19):

Suppose we have the conclusions of theorem (2.20). Then $(\mathrm{Im}(\tau), \partial_{\mathrm{Im}(\tau)})$ is acyclic chain complex, where $\partial_{\mathrm{Im}(\tau)} : \mathrm{Im}(\tau) \to \mathrm{Im}(\tau)$ is the restriction of the boundary operator $\partial_{(\Pi)} : (\Pi) \to (\Pi)$.

Proof: In light of remark (2.3), if $1 \le k \le r$ and $S \in NBC_k(A)$, then

$$\partial_k^{\Pi}\tau_k(a_S) = \partial_k^{\Pi} p_S = \sum_{j=1}^{k}(-1)^{j-1} p_{S_j} = \sum_{j=1}^{k}(-1)^{j-1}\tau_{k-1}(a_j) \in (\Pi)_{k-1}$$

, since we have $S_j \in NBC_{k-1}(A) \subseteq \mathbf{S}_{k-1}$ where S_j is the subarrangement of S with H_{i_j} deleted. Thus $\partial_k^{\Pi}(\mathrm{Im}(\tau_k)) \subseteq \mathrm{Im}(\tau_{k-1})$ and $\partial_{k-1}^{\Pi}\partial_k^{\Pi} = 0$ inherit a structure of a chain complex to $(\mathrm{Im}(\tau), \partial_{\mathrm{Im}(\tau)})$ which is acyclic since $((\Pi), \partial_{(\Pi)})$ is acyclic. □

Lemma (2.20):

Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ such that $r < \ell$. If $S \in \mathbf{S}_k$ such that S is k-broken circuit, then there is $S' \in \mathbf{S}_{k+1}$ such that S be the broken circuit of the circuit S'.

Proof: Suppose $S = \{H_{i_1}, \ldots, H_{i_k}\} \in \mathbf{S}_k$ and $rk(S) = k$. Since S is k-broken circuit, then there exists a hyperplane $H' \in A$ such that $H' \trianglelefteq H_{i_j}$ for $1 \le j \le k$ and $\{H'\} \cup S$ forms a k-circuit. If $H' \in \Pi_{i_0}$ with $1 \le i_0 < i_1 < \cdots < i_k \le \ell$, then our claim is true. On the other hand, if $H', H_{i_1} \in \Pi_{i_1}$, then from completeness property of the block $\Pi_{i_1} \neq \Pi_1$ there exists a unique hyperplane $H'' \in \Pi_1 \cup \ldots \cup \Pi_{i_1 - 1}$ such that $rk\{H'', H', H_{i_1}\} \in \Pi_{i_1}$. That is $H' \cap H_{i_1} = H'' \cap H_{i_1}$. Thus;

$$H' \cap H_{i_1} \cap \ldots \cap H_{i_k} = H'' \cap H_{i_1} \cap \ldots \cap H_{i_k}.$$

So put $S' = \{H'', H_{i_1}, \ldots, H_{i_k}\} \in \mathbf{S}_{k+1}$ which forms a k-circuit and it is broken circuit is S. □

(3) THE QUADRATIC ORLIK-SOLOMON ALGEBRA AND THE PARTITION COMPLEX OF A HYPERSOLVABLE ARRANGEMENT

Ali in [1], classified the hypersolvable class of arrangements into two sub classes which are the fiber type class and the non fiber type class by using the hypersolvable partition. This section is devoted to study the differences between these subclasses by using a quadratic property in order to explain how it reflects into amazing difference between the structures of the Orlik-Solomon algebras in each one of them.

Definition (3.1):

Let A be a central complex r-arrangement. Order the hyperplanes of A via a total order $\trianglelefteq$. Recall the set $BC(A)$ of all broken circuits of A. We say A is quadratic with respect to $\trianglelefteq$, if for each $S \in BC(A)$ there exists $T \in BC(A)$ with $T \subseteq S$ and $|T| = 2$.

Lemma (3.2):

Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$. For each $k \geq 3$, if $S = \{H_{i_1}, \ldots, H_{i_k}\} \in \mathbf{S}_k$. Then:

1- There is no $B = \{H_{i_m}, H_{i_n}, H_{i_p}\} \subseteq S$ such that $\mathrm{rk}(B) = 2$ and $1 \leq m < n < p \leq k$, i.e. S contains no collinear relations among any three hyperplanes of it.

2- There is no hyperplane $H \in A$ with $\mathrm{rk}\{H, H_{i_m}, H_{i_n}\} = 2$, $1 \leq m < n \leq k$ and $H \trianglelefteq H_{i_j}$, $j = m, n$, i.e. S contains no broken circuit of rank two.

Proof:

For (1): The existence of a subarrangement $B = \{H_{i_m}, H_{i_n}, H_{i_p}\} \subseteq S$ such that $\mathrm{rk}(B) = 2$, contradicts remark (1.2).

For (2): The existence of a hyperplane $H \in A$ such that $rk\{H, H_{i_m}, H_{i_n}\} = 2$, contradicts the closedness property of the block Π_{i_m}. □

Remark (3.3):

It is known that every supersolvable arrangement A is quadratic [10]. Ali [1] shows that "*a hypersolvable r-arrangement A is supersolvable if and only if $NBC(A) = \mathbf{S}$ via the hypersolvable order*", we can visualize this fact. Definitely if $S \in BC(A)$, then $S \notin \mathbf{S}$ and $S \cap \Pi_1 = \phi$. Therefore, there are $H, H' \in S$ with $H, H' \in \Pi_j$ for some $1 \leq j \leq \ell$ and from the completeness property of the block Π_j we have $T = \{H, H'\} \subseteq S$ and $T \in BC_2(A)$. Thus A is quadratic via the hypersolvable order which preserves the supersolvable structure.

Since the hypersolvable class forms a generalization of the supersolvable class, there is an accepted question "Is each hypersolvable r-arrangements quadratic?" Our aim is to answer this question. So we start with the following definition:

Definition (3.4):

Let K be a commutative ring with unit and let $A = \{H_1, \ldots, H_n\}$ be a central r-arrangement in C^r and let $E = E(A) = \bigwedge_{\mathrm{K}}^{*}(e_{H_1}, \ldots, e_{H_n})$ be the exterior K-algebra generated by symbols e_H in one to one correspondence with the hyperplanes $H \in A$, recall definition (2.1). Define the *Quadratic Orlik-Solomon algebra* $\overline{\mathrm{A}} = \overline{\mathrm{A}}_{\mathrm{K}}^{*}(A)$ to be the quotient of E by the homogenous ideal $J = J(A)$ which is generated by the following family of quadratic relations of the form $\partial e_S = \partial e_{H_{i_1}} e_{H_{i_2}} e_{H_{i_3}} = e_{H_{i_1}} e_{H_{i_2}} + e_{H_{i_3}} e_{H_{i_1}} + e_{H_{i_2}} e_{H_{i_3}}$, for all $1 \leq i_1 < i_2 < i_3 \leq n$ such that $S = \{H_{i_1}, H_{i_2}, H_{i_3}\}$ is dependent, i.e.

$\mathrm{rk}(S)=2$. The gradation of E defines a gradation of the ideal J as follows; since each subarrangement of A contains one hyperplane or two hyperplanes are independent then $J_0=0$ and $J_1=0$, where $J_2=\{\partial e_B \mid B\subseteq A, |B|=3 \text{ and } rk(B)=2\}=I_2$. Therefore $\overline{\mathrm{A}}_0(A)=\mathrm{K}$, $\overline{\mathrm{A}}_1(A)=E_1$ and $\overline{\mathrm{A}}_2(A)=\mathrm{A}_2(A)$.

By $\overline{\varphi}:E\to\overline{\mathrm{A}}=E/J$ we denote the canonical projection and for each $H\in A$, let $\overline{a}_H=\varphi(e_H)=e_H+J$ and $\overline{\varphi}(E_p)=\overline{\mathrm{A}}_p$ for $p\geq 0$. That is $\overline{\mathrm{A}}(A)=\oplus_{p\geq 0}\overline{\mathrm{A}}_p(A)$ be a graded anticommutative algebra. Since $\partial_E J\subseteq J$ then a K-linear map $\partial_{\overline{\mathrm{A}}}:\overline{\mathrm{A}}\to\overline{\mathrm{A}}$ which is defined by $\partial_{\overline{\mathrm{A}}}\overline{\varphi}=\overline{\varphi}\,\partial_E$ gives the pair $(\overline{\mathrm{A}},\partial_{\overline{\mathrm{A}}})$ a structure of an acyclic chain complex, as the following exact sequence;

$$\cdots\xrightarrow{\partial^{\overline{\mathrm{A}}}_{p+1}}\overline{\mathrm{A}}_p(A)\xrightarrow{\partial^{\overline{\mathrm{A}}}_{p}}\overline{\mathrm{A}}_{p-1}(A)\xrightarrow{\partial^{\overline{\mathrm{A}}}_{p-1}}\cdots\xrightarrow{\partial^{\overline{\mathrm{A}}}_{1}}\overline{\mathrm{A}}_0(A)\xrightarrow{\partial^{\overline{\mathrm{A}}}_{0}}0.$$

In fact if $c\in\overline{\mathrm{A}}_{p-1}(A))$ is a cycle, i.e. $\partial^{\overline{\mathrm{A}}}_{p-1}(c)=0_{\overline{\mathrm{A}}_{p-2}}$, then there is $u\in E_{p-1}(A)$ such that $\overline{\varphi}_{p-1}(u)=c$. Therefore, for $H\in A$ such that $e_H u\neq 0$, we have $e_H u\in E_p(A)$ and;

$$\partial^{\overline{\mathrm{A}}}_p\overline{\varphi}_p(e_H u)=\overline{\varphi}_{p-1}\partial^E_p(e_H u)=\overline{\varphi}_{p-1}(u-e_H\partial^E_{p-1}(u))=c-\overline{a}_H\partial^{\overline{\mathrm{A}}}_{p-1}(c)=c.$$

That is $\ker(\partial^{\overline{\mathrm{A}}}_{p-1})\subseteq\mathrm{Im}(\partial^{\overline{\mathrm{A}}}_{p})$.

Theorem (3.5):

Let A be a hypersolvable r-arrangement with Hp $\Pi=(\Pi_1,\ldots,\Pi_\ell)$ and exponent vector $d=(d_1,\ldots,d_\ell)$ such that $rk(A)=r<\ell$, (i.e. A is not fiber-type arrangement). Let

$$p(A)=\sup\{i\mid b_j(A)=\overline{b}_j(A),\ \forall\ j\leq i\},$$

where for $0\leq j\leq r$, $b_j(A)$ and $\overline{b}_j(A)$ are the j^{th} Betti numbers of the Poincaré polynomial of $\mathrm{A}^*(A)$ and $\overline{\mathrm{A}}^*(A)$ respectively. Then there are $S\in BC_{p(A)+1}(A)$ such that there is no $T\in BC_2(A)$ with $T\subseteq S$, i.e. a hypersolvable r-arrangement A which is not fiber-type is not quadratic.

Proof: Ali proved that for such arrangement "*every j-section of A forms a j-NBC base of A, for $2\leq j\leq p(A)$*", i.e. for every $2\leq j\leq p(A)$ the blocks $\Pi_{l_1},\ldots,\Pi_{l_j}$ are independent for each $1\leq l_1<\cdots<l_j\leq\ell$ and she proved that "*if $q=p(A)+1$, then $L_q(A)$ forms the first level of the intersection lattice $L(A)$ contains a dependent relations among q-different blocks of Π*". Thus there is $S'=\{H_{i_1},\ldots,H_{i_{q+1}}\}\in\mathbf{S}_{q+1}$ such that $rk(S')=q$, i.e. $S=\{H_{i_2},\ldots,H_{i_{q+1}}\}\in\mathbf{S}_q$ forms q-broken circuit of A and as an application of lemma (3.2) our claim was proved. □

Corollary (3.6):

A hypersolvable r-arrangement is quadratic if, and only if, it is fiber-type r-arrangement.

Proof: This is a direct result of remark (3.3) and theorem (3.5). □

Remark (3.7):

For a hypersolvable arrangement A which is not fiber-type arrangement (i.e. it is not quadratic), the collection $BC_{p(A)+1}(A)$ can be partitioned into two parts, where $p(A)$ is defined as in theorem (3.5) and $q = p(A)+1$. The first one is $BC_q^1(A) = \mathbf{S}_q \cap BC_q(A)$, which forms the collection of all q-broken circuits with no sub broken circuit of rank two. The second part is $BC_q^2(A) = BC_2(A) \setminus BC_q^1(A)$, which forms the collection of all q-broken circuits with sub broken circuit of rank two.

In the hypersolvable class of arrangements the quadratic property can be reflected on the construction of "quadratic Orlik-Solomon algebra". So we need to study this notion more precisely for a hypersolvable arrangement. Let us start with the following result of Jambu and Papadima:

Theorem (3.8): [5]

Given any arrangement pair (A,B), i.e. $B \subseteq A$, then B is solvable in A if, and only if, there exists an isomorphism of graded K-modules $\overline{\mathrm{A}}^*(A) \approx \overline{\mathrm{A}}^*(B) \otimes \mathrm{H}^*(\bigvee_{H\in\overline{B}} S^1, \mathrm{K})$, where $\overline{B} = A \setminus B$.

Theorem (3.9):

Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1,\ldots,\Pi_\ell)$ and exponent vector $d = (d_1,\ldots,d_\ell)$. Then there exists an isomorphism of graded K-modules $\overline{\mathrm{A}}^*(A) \approx \bigotimes_{k=1}^{\ell} \mathrm{H}^*(\bigvee_{i=1}^{d_k} S^1, \mathrm{K})$.

Proof: Since A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1,\ldots,\Pi_\ell)$, hence A has a composition series $A_1 \subset \cdots \subset A_{i-1} \subset A_i \subset \cdots \subset A_\ell$ defined by $A_i = \bigcup_{H\in\Pi_1\cup\cdots\cup\Pi_i} H$, $1 \le i \le \ell$. Thus for each $2 \le j \le \ell$, the pair (A_i, A_{i-1}) is solvable and by applying theorem (3.8), we have $\overline{\mathrm{A}}^*(A_i) \approx \overline{\mathrm{A}}^*(A_{i-1}) \otimes \mathrm{H}^*(\bigvee_{H\in\Pi_i} S^1, \mathrm{K})$.

We will inductively prove our result. If $j = 2$, $\overline{\mathrm{A}}^*(A_2) \approx \overline{\mathrm{A}}^*(A_1) \otimes \mathrm{H}^*(\bigvee_{H\in\Pi_2} S)$. But A_1 contains just one hyperplane and the complement $M(A_1) = C^r \setminus A_1$ is of the same homotopy type of $C \setminus \{pt\} \approx S^1$. That is $\mathrm{H}^*(M(A_1),\mathrm{K}) \approx \mathrm{H}^*(S^1,\mathrm{K})$. Thus $\overline{\mathrm{A}}^*(A_1) \approx \mathrm{A}^*(A_1) \approx \mathrm{H}^*(M(A_1),\mathrm{K}) \approx \mathrm{H}^*(S^1,\mathrm{K})$ and $\overline{\mathrm{A}}^*(A_2) \approx \bigotimes_{k=1}^{2} \mathrm{H}^*(\bigvee_{i=1}^{d_k} S^1, \mathrm{K})$.

Consequently by applying the induction rules our claim was shown. □

Construction (3.10):

For $1 \le k \le \ell$, $\mathrm{H}^*(\bigvee_{i=1}^{d_k} S^1, \mathrm{K})$ can be identified with the graded free K-module F_k^* having basis with 1 in degree 0 and d_k elements $\overline{a}_H$, $H \in \Pi_k$ in degree 1. Since such structure is unique (up to isomorphism) and by our notation in (2.7) $(\Pi_k)^* \approx \mathrm{H}^*(\bigvee_{i=1}^{d_k} S^1, \mathrm{K}) = F_k^*$ and;

$$(\Pi)^* = \bigotimes_{k=1}^{\ell} (\Pi_k)^* \approx \bigotimes_{k=1}^{\ell} \mathrm{H}^*(\bigvee_{i=1}^{d_k} S^1, \mathrm{K}) = \bigotimes_{k=1}^{\ell} F_k^* \approx \overline{\mathrm{A}}^*(A).$$

Assume that the K-linear isomorphism

$$\overline{\kappa} : (\Pi_1)^* \otimes \cdots \otimes (\Pi_\ell)^* = (\Pi)^* \xrightarrow{\sim} F_1^* \otimes \cdots \otimes F_\ell^* \xrightarrow{\sim} \overline{\mathrm{A}}^*(A_\ell) = \overline{\mathrm{A}}^*(A),$$

which uniquely extends the assignment $p_{H_1} \mapsto \overline{a}_{H_1}, \ldots, p_{H_{i_{\ell-1}}} \mapsto \overline{a}_{H_{i_{\ell-1}}}$ and $p_{H_{i_\ell}} \mapsto \overline{a}_{H_{i_\ell}}$. This dialogue leads us to the following result:

Corollary (3.11):

Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ and exponent vector $d = (d_1, \ldots, d_\ell)$. Then $\bigoplus_{k=0}^{\ell} (\Pi)_k \approx \overline{\mathrm{A}}^*(A)$ and for $1 \le k \le \ell$, the k th-Betti number of the Poincaré polynomial of $\overline{\mathrm{A}}^*(A)$ is;

$$\overline{b}_k(A) = rk(\mathrm{A}_k(A)) = \sum_{i_k > \cdots > i_1 = 1}^{\ell} d_{i_1} \ldots d_{i_k} = \sum_{i_1=1}^{\ell-k+1} (\sum_{i_2=i_1+1}^{\ell-k+2} (\cdots (\sum_{i_k=i_{k-1}+1}^{\ell} d_{i_1} d_{i_2} \ldots d_{i_k}) \cdots)).$$

Proof: The homogeneous K-isomorphism of degree 0, $\overline{\kappa} : (\Pi) \xrightarrow{\sim} \overline{\mathrm{A}}(A)$ is given in (3.10) inherits to $\overline{\mathrm{A}}^*(A)$ a structure of a graded free K-module as the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \xrightarrow{\partial_{\ell+1}^{\Pi}} & (\Pi)_\ell & \xrightarrow{\partial_\ell^{\Pi}} & (\Pi)_{\ell-1} & \xrightarrow{\partial_{\ell-1}^{\Pi}} \cdots \xrightarrow{\partial_1^{\Pi}} & (\Pi)_0 & \xrightarrow{\partial_0^{\Pi}} 0 \\
 & \overline{\kappa}_\ell \downarrow & & \overline{\kappa}_{\ell-1} \downarrow & & \overline{\kappa}_0 \downarrow & \\
0 \xrightarrow{\partial_{\ell+1}^{\overline{\mathrm{A}}}} & \overline{\mathrm{A}}_\ell(A) & \xrightarrow{\partial_\ell^{\overline{\mathrm{A}}}} & \overline{\mathrm{A}}_{\ell-1}(A) & \xrightarrow{\partial_{\ell-1}^{\overline{\mathrm{A}}}} \cdots \xrightarrow{\partial_1^{\overline{\mathrm{A}}}} & \overline{\mathrm{A}}_0(A) & \xrightarrow{\partial_0^{\overline{\mathrm{A}}}} 0.
\end{array}$$

Then $\bigoplus_{k=0}^{\ell} (\Pi)_k \approx \overline{\mathrm{A}}^*(A)$ and by applying corollary (2.9) our claim is held. □

Remark (3.12):

By applying corollary (3.11), the set $\{\overline{\kappa}(p_S) = \overline{a}_S \mid S \in \mathbf{S}\}$ forms a K-basis to $\overline{\mathrm{A}}^*(A)$. This leads us to the following essential question "What is the property that our driving set $\mathbf{S}$ has to make $\{\overline{a}_S \mid S \in \mathbf{S}\}$ a K-basis to $\overline{\mathrm{A}}^*(A)$?". The answer is given in lemma (3.2). This property is that for each $S \in \mathbf{S}$, S contains no broken circuit of rank two, where if $S \notin \mathbf{S}$, deduces that there exists $T \in BC_2(A)$ with $T \subseteq S$.

The above analogue leads us to the property that the broken circuits of rank two have to be important to our construction. So, For each $k \ge 2$, let

$$\mathrm{J}_k = \{e_B \mid B \subseteq A, |B| = k+1, \exists B' \subseteq B \ni |B'| = 3 \text{ and } rk(B') = 2\}$$

be the set of monomials which is spanned by the collection of all dependent subarrangements with cardinality $k+1$ which contains collinear relations among (at least) three hyperplanes of it. This set plays a role to construct the ideal I_k as follows:

If $B = \{H_{i_2}, H_{i_3}\} \subseteq A$ be a broken circuit of rank two, then there is a hyperplane $H_{i_1} \in A$ such that $H_{i_1} \trianglelefteq H_{i_2}$, $H_{i_1} \trianglelefteq H_{i_3}$ and

$rk(H_{i_1}, H_{i_2}, H_{i_3}) = 2$. In fact if $B' = \{H_{i_1}, H_{i_2}, H_{i_3}\}$ then $e_{B'} \in J_2$. Since $\text{rk}(B') = 2$, hence there are only two possible distributions of the hyperplanes of B' among the blocks of Π as;

i- $H_{i_1} = H_{i_2,i_3} \in \Pi_m$ and $H_{i_2}, H_{i_3} \in \Pi_n$, where $2 \le m < n \le \ell$;

ii- $H_{i_1}, H_{i_2}, H_{i_3} \in \Pi_m$ for some $2 \le m \le \ell$.

We have $\partial_2^E e_{B'} = e_{H_{i_2}} e_{H_{i_3}} - e_{H_{i_1}} e_{H_{i_3}} + e_{H_{i_1}} e_{H_{i_2}} \in J_2$ and the following quadratic relation $\bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} - \bar{a}_{H_{i_1}} \bar{a}_{H_{i_3}} + \bar{a}_{H_{i_1}} \bar{a}_{H_{i_2}} = 0_{\bar{A}_2}$ is held in $\bar{A}^*(A)$. Thus:

For (i): we have $\bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} = \bar{a}_{H_{i_2,i_3}} \bar{a}_{H_{i_3}} - \bar{a}_{H_{i_2,i_3}} \bar{a}_{H_{i_2}} = \bar{a}_{H_{i_2,i_3}} \partial_2^{\bar{A}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}}$, that is our broken circuit can be written as a linear combination of NBC bases of rank two $\bar{a}_{H_{i_2,i_3}} \bar{a}_{H_{i_3}}$ and $\bar{a}_{H_{i_1,i_2}} \bar{a}_{H_{i_2}}$. Observe that, $\bar{a}_{B'} = \bar{a}_{H_{i_1}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} = \bar{a}_{H_{i_2,i_3}} \bar{a}_{H_{i_2,i_3}} \partial_2^{\bar{A}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} = 0_{\bar{A}_3}$ and for every $H \in A$ with $H \notin B'$ we have $\pm e_H e_{B'} \in J_4$, i.e. $\partial_4^E(\pm e_H e_{B'}) = \pm e_{B'} - (\pm e_H \partial_3^E(e_{B'}))$ and $\partial_4^{\bar{A}}(\pm \bar{a}_H \bar{a}_{B'}) = \pm \bar{a}_{B'} = 0_{\bar{A}_3}$.

For (ii): since $H_{i_1}, H_{i_2}, H_{i_3} \in \Pi_m$ are collinear, hence there is a unique hyperplane say $H \in \Pi_1 \cup \cdots \cup \Pi_{m-1}$ with $\text{rk}(H, H_{i_1}, H_{i_2}, H_{i_3}) = 2$. Thus for each $1 \le k < j \le 3$ we have $\bar{a}_{H_{i_k}} \bar{a}_{H_{i_j}} = \bar{a}_H \bar{a}_{H_{i_j}} - \bar{a}_H \bar{a}_{H_{i_k}} = \bar{a}_H \partial_2^{\bar{A}} \bar{a}_{H_{i_k}} \bar{a}_{H_{i_j}}$. Thus we write our broken circuit as a linear combination of NBC bases of rank two $\bar{a}_H \bar{a}_{H_{i_3}}$ and $\bar{a}_H \bar{a}_{H_{i_2}}$. And we have;

$$\partial_4^{\bar{A}} \bar{a}_H \bar{a}_{H_{i_1}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} = \bar{a}_{H_{i_1}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} - \bar{a}_H \partial_3^{\bar{A}} \bar{a}_{H_{i_1}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} = \bar{a}_{H_{i_1}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} = \bar{a}_{H_{i_1}} \bar{a}_{H_{i_1}} \partial_2^{\bar{A}} \bar{a}_{H_{i_2}} \bar{a}_{H_{i_3}} = 0_{\bar{A}_3}.$$

This conversation classifies the collection of all subarrangements of a hypersolvable arrangement A into two disjoint classes. The first one is $\mathbf{S}$ and the second one is the complement of $\mathbf{S}$. That is for a fixed $3 \le k \le \ell$, the k^{th} monomial basis $\mathbf{B}_k = \{e_{H_{i_1}} \ldots e_{H_{i_k}} \mid 1 \le i_1 < \cdots < i_k \le n\}$ of the k^{th}-exterior algebra E_k in definition (2.1) which is spanned by the collection of all subarrangement of A with cardinality k, is partitioned into $\{e_S \mid S \in \mathbf{S}_k\}$ and $\mathbf{B}_k \setminus \{e_S \mid S \in \mathbf{S}_k\}$, where $J_k \subseteq \mathbf{B}_k \setminus \{e_S \mid S \in \mathbf{S}_k\}$. Let $e_{B'} \in \mathbf{B}_k \setminus \{e_S \mid S \in \mathbf{S}_k\}$. Thus either $e_{B'} \in J_{k-1}$ or $e_{B'} \notin J_{k-1}$.

For $e_{B'} \in J_{k-1}$:

$$\partial_k^E(e_{B'}) = \pm(\partial_3^E e_T) e_{B' \setminus T} \pm e_T(\partial_{k-3}^E e_{B' \setminus T}) \Rightarrow \pm(\partial_3^E e_T) e_{B' \setminus T} = \partial_k^E(e_{B'}) \pm e_T(\partial_{k-3}^E e_{B' \setminus T}) \in J_{k-1},$$

where T be any 2-circuit of B'. That is;

$$\pm(\partial_3^{\bar{A}} \bar{a}_T) \bar{a}_{B' \setminus T} = \partial_k^{\bar{A}}(\bar{a}_{B'}) \pm \bar{a}_T(\partial_{k-3}^{\bar{A}} \bar{a}_{B' \setminus T}) = \partial_k^{\bar{A}}(\bar{a}_{B'}) = 0_{\bar{A}_{k-1}}, \text{ i.e. } \partial_k^E(e_{B'}) \in \ker(\bar{\varphi}_k).$$

For $e_{B'} \notin J_{k-1}$: That is B' contains a broken circuit of rank two say $\{H_{i_1}, H_{i_2}\} \subseteq B'$ and $H_{i_1}, H_{i_2} \in \Pi_m$ for some $2 \le m \le \ell$ with $H_{i_1,i_2} \in A$ be the unique hyperplane such that $\text{rk}(H_{i_1,i_2}, H_{i_1}, H_{i_2}) = 2$. Thus;

$$\partial^E_{k+1}(\pm e_{H_{i_1,i_2}} e_{B'}) = \pm((\partial^E_3 e_{\{H_{i_1,i_2},H_{i_1},H_{i_2}\}})e_{B'\setminus\{H_{i_1},H_{i_2}\}} - e_{\{H_{i_1,i_2},H_{i_1},H_{i_2}\}}\partial^E_{k-2} e_{B'\setminus\{H_{i_1},H_{i_2}\}})$$, i.e.

$$\partial^E_{k+1}(\pm e_{H_{i_1,i_2}} e_{B'}) \pm e_{\{H_{i_1,i_2},H_{i_1},H_{i_2}\}}\partial^E_{k-1} e_{B'\setminus\{H_{i_1},H_{i_2}\}}) = \pm((\partial^E_3 e_{\{H_{i_1,i_2},H_{i_1},H_{i_2}\}})e_{B'\setminus\{H_{i_1},H_{i_2}\}} \in J_k$$. Therefore,

$$\partial^{\bar{A}}_{k+1}(\pm \bar{a}_{H_{i_1,i_2}} \bar{a}_{B'}) \pm \bar{a}_{\{H_{i_1,i_2},H_{i_1},H_{i_2}\}}\partial^{\bar{A}}_{k-2} \bar{a}_{B'\setminus\{H_{i_1},H_{i_2}\}}) = \pm((\partial^{\bar{A}}_3 \bar{a}_{\{H_{i_1,i_2},H_{i_1},H_{i_2}\}})\bar{a}_{B'\setminus\{H_{i_1},H_{i_2}\}} = 0_{\bar{A}_k}$$

and $\partial^{\bar{A}}_{k+1}(\pm \bar{a}_{H_{i_1,i_2}} \bar{a}_{B'}) = 0_{\bar{A}_k}$, $\partial^E_{k+1}(\pm e_{H_{i_1,i_2}} e_{B'}) \in \ker(\bar{\varphi}_k)$. On the other hand $\bar{a}_{B'} = \partial^{\bar{A}}_{k+1}(\pm \bar{a}_{H_{i_1,i_2}} \bar{a}_{B'}) \pm \bar{a}_{H_{i_1,i_2}}\partial^{\bar{A}}_k \bar{a}_{B'} = \bar{a}_{H_{i_1,i_2}}\partial^{\bar{A}}_k \bar{a}_{B'} = \bar{a}_{H_{i_1,i_2}}\partial^{\bar{A}}_k(\pm \bar{a}_{H_{i_1}}\bar{a}_{H_{i_2}}\bar{a}_{B'\setminus\{H_{i_1},H_{i_2}\}})$. That is, $\bar{a}_{B'} = \pm\bar{a}_{H_{i_1,i_2}}\partial^{\bar{A}}_2(\bar{a}_{H_{i_1}}\bar{a}_{H_{i_2}})\bar{a}_{B'\setminus\{H_{i_1},H_{i_2}\}}$. By continuing this procedure we can write $\bar{a}_{B'}$ as a linear combination of monomials spanned by subarrangements that contain no two broken circuits.

Our aim now is to explain how this classification constructs the quadratic Orlik-Solomon algebra as follows:

Definition (3.13):

Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$. Define $\mathrm{NC}(A)$ as follows, $\mathrm{NC}_0(A) = \mathrm{K}$, for $1 \le k \le \ell$ let $\mathrm{NC}_k(A)$ be the free K-submodule of $E(A)$ with basis $\{e_S \mid S \in \mathbf{S}_k\}$ and $\mathrm{NC}(A) = \bigoplus_{k=0}^{\ell} \mathrm{NC}_k(A)$. It is clear that $\mathrm{NC}(A)$ forms a free submodule of $E(A)$ (not algebra since $\mathrm{NC}(A)$ not closed under the multiplication).

In fact for each $S \in \mathbf{S}_k$, the boundaries $S_j \in \mathbf{S}_{k-1}$, $1 \le j \le k$. Therefore $\partial\mathrm{NC(A)} \subseteq \mathrm{NC(A)}$ and $(\mathrm{NC}, \partial_{\mathrm{NC}})$ form a chain complex which is acyclic. Indeed if $c \in \mathrm{NC}(A)$ is a cycle, i.e. $\partial c = 0$. We have $e_{H_1}c \in \mathrm{NC}(A)$ and $\partial(e_{H_1}c) = c - e_{H_1}\partial c = c$. Thus c is a boundary.

Let $\bar{\varphi}' : \mathrm{NC}(A) \to \bar{\mathrm{A}}(A)$ be the restriction of the canonical projection $\bar{\varphi} : E(A) \to \bar{\mathrm{A}}(A)$, (definition (3.7)).

Theorem (3.14):

Suppose we have the conclusions of definition (3.13). Then the K-linear map $\bar{\varphi}' : \mathrm{NC}(A) \to \bar{\mathrm{A}}(A)$ forms an isomorphism.

Proof: It is clear that $\bar{\varphi}'$ forms an isomorphism, indeed $\mathrm{NC}_0(A) \approx \bar{\mathrm{A}}_0(A)$, $\mathrm{NC}_1(A) \approx \bar{\mathrm{A}}_1(A)$ and for $2 \le k \le \ell$, $\bar{\varphi}'_k : \mathrm{NC_k}(A) \to \bar{\mathrm{A}}_k(A)$ is an isomorphism since $\{\bar{\varphi}'_k(e_S) = \bar{a}_S \mid S \in \mathbf{S}_k\}$ forms a basis of $\bar{\mathrm{A}}\ (A)$ as the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{\partial^{\mathrm{NC}}_{\ell+1}} & NC_\ell(A) & \xrightarrow{\partial^{\mathrm{NC}}_{\ell}} & \mathrm{NC}_{\ell-1}(A) & \xrightarrow{\partial^{\mathrm{NC}}_{\ell-1}} \cdots \xrightarrow{\partial^{\mathrm{NC}}_{1}} & NC_0(A) & \xrightarrow{\partial^{\mathrm{NC}}_{0}} & 0 \\
 & & \bar{\varphi}'_\ell \downarrow & & \bar{\varphi}'_{\ell-1} \downarrow & & \bar{\varphi}'_0 \downarrow & & \\
0 & \xrightarrow{\partial^{\bar{A}}_{\ell+1}} & \bar{\mathrm{A}}_\ell(A) & \xrightarrow{\partial^{\bar{A}}_{\ell}} & \bar{\mathrm{A}}_{\ell-1}(A) & \xrightarrow{\partial^{\bar{A}}_{\ell-1}} \cdots \xrightarrow{\partial^{\bar{A}}_{1}} & \bar{\mathrm{A}}_0(A) & \xrightarrow{\partial^{\bar{A}}_{0}} & 0
\end{array}$$

□

Definition (3.15):

Let A be a hypersolvable r-arrangement with HP $\Pi=(\Pi_1,\cdots,\Pi_\ell)$. The following diagram of acyclic chain complexes gives a definition of a connection between the structures of the Orlik-Solomon algebra and the quadratic Orlik-Solomon algebra for the hypersolvable class of arrangements by using the partition complex:

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{\partial^{A}_{r+1}} & A_r(A) & \xrightarrow{\partial^{A}_{r}} & A_{r-1}(A) & \xrightarrow{\partial^{A}_{r-1}} \cdots \xrightarrow{\partial^{A}_{1}} & A_0(A) & \xrightarrow{\partial^{A}_{0}} & 0 \\
 & & \tau_r \downarrow & & \tau_{r-1}\downarrow & & \tau_0 \downarrow & & \\
\cdots & \xrightarrow{\partial^{\Pi}_{r+1}} & (\Pi)_r & \xrightarrow{\partial^{\Pi}_{r}} & (\Pi)_{r-1} & \xrightarrow{\partial^{\Pi}_{r-1}} \cdots \xrightarrow{\partial^{\Pi}_{1}} & (\Pi)_0 & \xrightarrow{\partial^{\Pi}_{0}} & 0 \\
 & & \bar{\kappa}_r \downarrow & & \bar{\kappa}_{r-1}\downarrow & & \bar{\kappa}_0 \downarrow & & \\
\cdots & \xrightarrow{\partial^{\bar{A}}_{r+1}} & \bar{A}_r(A) & \xrightarrow{\partial^{\bar{A}}_{r}} & \bar{A}_{r-1}(A) & \xrightarrow{\partial^{\bar{A}}_{r-1}} \cdots \xrightarrow{\partial^{\bar{A}}_{1}} & \bar{A}_0(A) & \xrightarrow{\partial^{\bar{A}}_{0}} & 0 \\
 & & \bar{\tau}_r \downarrow & & \bar{\tau}_{r-1}\downarrow & & \bar{\tau}_0 \downarrow & & \\
\cdots & \xrightarrow{\partial^{\Pi}_{r+1}} & (\Pi)_r & \xrightarrow{\partial^{\Pi}_{r}} & (\Pi)_{r-1} & \xrightarrow{\partial^{\Pi}_{r-1}} \cdots \xrightarrow{\partial^{\Pi}_{1}} & (\Pi)_0 & \xrightarrow{\partial^{\Pi}_{0}} & 0 \\
 & & \kappa_r \downarrow & & \kappa_r \downarrow & & \kappa_0 \downarrow & & \\
0 & \xrightarrow{\partial^{A}_{r+1}} & A_r(A) & \xrightarrow{\partial^{A}_{r}} & A_{r-1}(A) & \xrightarrow{\partial^{A}_{r-1}} \cdots \xrightarrow{\partial^{A}_{1}} & A_0(A) & \xrightarrow{\partial^{A}_{0}} & 0
\end{array}$$

Define the K-linear maps of graded K-modules $\tau'=\bar{\kappa}\circ\tau : A^*(A)\to\bar{A}^*(A)$ and $\kappa'=\kappa\circ\bar{\tau} : \bar{A}^*(A)\to A^*(A)$.

Observe that τ' is mono. and κ' is epi. in general.

Theorem (3.16):

Let A be a hypersolvable r-arrangement with HP $\Pi=(\Pi_1,\cdots,\Pi_\ell)$. Then, A is fiber-type arrangement if, and only if, the K-linear map $\kappa' : \bar{A}^*(A)\to A^*(A)$ is an isomorphism.

Proof: This is a direct result of construction (3.10).

□

(4) APPLICATIONS OF THE HYPERSOLVABLE PARITION COMPLEX

Ali in [1] studied the properties of the hypersolvable NBC bases of a hypersolvable arrangement. Our aim in this section is to reproduce these properties into the structure the Orlik-Solomon algebra for the hypersolvable class of arrangements and we compare it with the structure of the quadratic Orlik-Solomon algebra by using the partition complex analogue:

Theorem (4.1):

Let A be a hypersolvable r-arrangement with HP $\Pi=(\Pi_1,\cdots,\Pi_\ell)$ and exponent vector $d=(d_1,\ldots,d_\ell)$. Then $A_0(A)\approx K$, $A_1(A)\approx E_1$ and;

$$A_2(A)\approx(\Pi)_2=\bigoplus_{S\in NBC_2(A)} Kp_S \approx \bigoplus_{\substack{i_1=1}}^{\ell-1}(\bigoplus_{\substack{j_1=1\\ H^{i_1}_{j_1}\in\Pi_{i_1}}}^{d_{i_1}}(\bigoplus_{i_2=i_1+1}^{\ell}(\bigoplus_{\substack{j_2=1\\ H^{i_2}_{j_2}\in\Pi_{i_2}}}^{d_{i_2}} Kp_{\{H^{i_1}_{j_1},H^{i_2}_{j_2}\}})))) .$$

Proof: from the definition of the Orlik-Solomon algebra $\mathrm{A}_0(A) \approx \mathrm{K}$, $\mathrm{A}_1(A) \approx E_1$ and by applying a result of Ali and Al-Ta'ai states that "*of a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$, every sub arrangement contains two hyperplanes from two different blocks (2-section of Π) form a 2-NBC base of A*", we deduce that $NBC_2(A) = \mathbf{S}_2$. Thus;

$$\mathrm{A}_2(A) \approx (\Pi)_2 = \bigoplus_{S \in NBC_2(A)} \mathrm{K}p_S \approx \bigoplus_{i_1=1}^{\ell-1}(\bigoplus_{\substack{j_1=1 \\ H_{j_1}^{i_1} \in \Pi_{i_1}}}^{d_{i_1}}(\bigoplus_{i_2=i_1+1}^{\ell}(\bigoplus_{\substack{j_2=1 \\ H_{j_2}^{i_2} \in \Pi_{i_2}}}^{d_{i_2}} \mathrm{K}p_{\{H_{j_1}^{i_1}, H_{j_2}^{i_2}\}})))) . \quad \square$$

Corollary (4.2):

Let A be a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$. Then $\mathrm{A}_0(A) \approx \overline{\mathrm{A}}_0(A)$, $\mathrm{A}_1(A) \approx \overline{\mathrm{A}}_1(A)$ and $\mathrm{A}_2(A) \approx \overline{\mathrm{A}}_2(A)$.

Proof: By applying theorem (4.1), our aim is held. $\square$

Corollary (4.3):

Let A be a hypersolvable r-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ and exponent vector $d = (d_1, \ldots, d_\ell)$ such that $rk(A) = r \geq 2$. Then, $b_0 = rk(\mathrm{A}_0(A)) = 1$, $b_1 = rk(\mathrm{A}_1(A)) = |A|$ and $b_2 = rk(\mathrm{A}_2(A)) = \sum_{i_1=1}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}$.

Proof: It is clear that $b_0 = 1$ and $b_1 = |A|$. Since $b_2 = |NBC_2(A)|$ = the number of all 2-NBC base of A and from theorem (2.1) such number is equal to the number of all 2-section of Π. Therefore, $b_2 = \sum_{i_1=1}^{\ell-1} \sum_{i_2=i_1+1}^{\ell} d_{i_1} d_{i_2}$.
$\square$

Corollary (4.4):

Let A be an essential central 2-arrangement. Then;

$$\mathrm{A}(A) \approx \mathrm{K} \oplus E_1 \oplus (\Pi)_2 \approx \overline{\mathrm{A}}(A) \text{ and}$$

$$p(\mathrm{A}(A), t) = 1 + |A|t + (|A| - 1)t^2 .$$

Proof: Deduce that, each essential central 2-arrangement $A = \{H_1, \ldots, H_n\}$ is a hypersolvable arrangement with Hp $\Pi = (\Pi_1, \Pi_2) = (\{H_1\}, \{H_2, \ldots, H_\ell\})$ and exponent vector $d = (d_1, d_2) = (1, |A| - 1)$. Then by applying theorem (4.1) we have $\mathrm{A}(A) \approx \mathrm{K} \oplus E_1 \oplus (\Pi)_2$. Thus, $p(\mathrm{A}(A), t) = 1 + |A|t + (|A| - 1)t^2$.
$\square$

Theorem (4.5):

Let A be a hypersolvable 3-arrangement with Hp $\Pi = (\Pi_1, \ldots, \Pi_\ell)$ and exponent vector $d = (d_1, \ldots, d_\ell)$ such that $\ell > 3$, (i.e. A is not fiber-type arrangement). Then, $\mathrm{A}(A) \approx \mathrm{K} \oplus E_1 \oplus (\Pi)_2 \oplus (\Pi)'_3$, where $(\Pi)'_3$ is the free K-submodule of $(\Pi)_3$ with basis $\{p_S \mid S \in \mathbf{S}_3 \textit{ and } S \cap \Pi_1 \neq \phi\}$.

Proof: by applying theorem (4.1), we need only to show that $A_3(A) \approx (\Pi)'_3$, i.e. we want to prove every 3-section $S = \{H_{i_1}, H_{i_2}, H_{i_3}\} \in \mathbf{S_3}$ of Π forms a 3-NBC base of A if and only if $H_{i_1} = H_1$, where the hyperplane $H_1 \in \Pi_1$ is the minimal hyperplane of A via the hypersolvable ordering.

In light of remark (1.2), every 3-section $S = \{H_{i_1}, H_{i_2}, H_{i_3}\}$ of Π is independent. Therefore it is either a 3-NBC base of A or a 3-broken circuit of A. *I*f $S = \{H_1, H_{i_2}, H_{i_3}\}$, deduce that S is a 3-NBC base of A. On the other hand, if $S = \{H_{i_1}, H_{i_2}, H_{i_3}\} \in \mathbf{S_3}$ and $H_{i_1} \neq H_1$, then S is a 3-broken circuit of A, since $H_1 \trianglelefteq H_{i_k}$, $1 \le k \le 3$ and $\{H_1\} \bigcup S$ is a 3-circuit of A. In particular, $NBC_3(A) = \{S \subseteq A \mid S \in \mathbf{S}_3 \textit{ and } S \bigcap \Pi_1 \neq \varphi\}$.

Now recall definition (2.16) and theorem (2.18), the restriction of the homogeneous K-linear map $\tau : A^*(A) \to (\Pi)^*$ of graded free K-module $K \oplus E_1 \oplus (\Pi)_2 \oplus (\Pi)'_3$ splits into a chain isomorphism of chain complexes as the following commutative diagram:

$$\begin{array}{ccccccccc}
A_3(A) & \xrightarrow{\partial_3^A} & A_2(A) & \xrightarrow{\partial_2^A} & A_1(A) & \xrightarrow{\partial_1^A} & A_0(A) & \xrightarrow{\partial_0^A} & 0 \\
\tau'_3 \downarrow & & \tau_2 \downarrow & & \tau_1 \downarrow & & \tau_0 \downarrow & & \\
(\Pi)'_3 & \xrightarrow{\partial_3^\Pi} & (\Pi)_2 & \xrightarrow{\partial_2^\Pi} & (\Pi)_1 & \xrightarrow{\partial_1^\Pi} & (\Pi)_0 & \xrightarrow{\partial_0^\Pi} & 0
\end{array}$$

Where $\tau'_3 = \tau_{3/(\Pi)'_3} : A_3(A) \xrightarrow{\sim} (\Pi)'_3$ is the restriction of $\tau_3 : A_3(A) \to (\Pi)_3$ on $(\Pi)'_3$ which is an isomorphism. Thus, $A(A) = \bigoplus_{j=1}^{3} A_j(A) \approx K \oplus E_1 \oplus (\Pi)_2 \oplus (\Pi)'_3$. □

Corollary (4.6):

Assume we have the conclusion of theorem (4.5). Then;

$$A_3(A) \approx (\Pi)'_3 = \bigoplus_{S \in NBC_3(A)} Kp_S \approx \bigoplus_{\substack{i_1=2 \\ H^{i_1}_{j_1} \in \Pi_{i_1}}}^{\ell-1} (\bigoplus_{j_1=1}^{d_{i_1}} (\bigoplus_{\substack{i_2=i_1+1 \\ H^{i_2}_{j_2} \in \Pi_{i_2}}}^{\ell} (\bigoplus_{j_2=1}^{d_{i_2}} Kp_{\{H_1, H^{i_1}_{j_1}, H^{i_2}_{j_2}\}}))));$$

where the hyperplane $H_1 \in \Pi_1$ is the minimal hyperplane of A with the hypersolvable ordering.

***Proof*:** In a straight line of the theorem (4.5). □

Definition (4.7): [6]

An r-arrangement A with $|A| = \ell > r$ is called a ℓ-*generic* if for every subarrangement $B \subseteq A$ with $|B| = r$ are linearly independent. Observe that, if A is a ℓ-generic r-arrangement, then all the subarrangements $B \subseteq A$ with $|B| \le r$ are linearly independent.

Remark (4.8):

Deduce that if A is a ℓ-generic r-arrangement, then A is a hypersolvable arrangement with exponents $d_i = 1$, $1 \le i \le \ell$ (i.e. $d = (1, \ldots, 1)$), where the converse need not to be true in general. Ra'ad in [7] explained that by an example. He used the quantity

$m(A)=\min\{|B| \mid B\subseteq A \text{ and } |B|>rk(B)\}$ to classified such r-arrangements into two classes as "A is a ℓ-generic r-arrangement if, and only if, $m(A)=r+1$".

Notice that, $m(A)$ is the minimal cardinality of subarrangement $B\subseteq A$ such that B is linearly dependent. Thus, $m(A)\geq 3$ in general and if $m(A)>3$, then there are no collinear relations among any three hyperplanes of A and such arrangement is hypersolvable with exponent vector $d=(1,\dots,1)$ which cannot be ℓ-generic if $m(A)\leq r$.

Theorem (4.9):

Let A be a ℓ-generic r-arrangement and $\Pi=(\Pi_1,\dots,\Pi_\ell)$ be the Hp of A with exponent vector $d=(1,\dots,1)$. Then $\mathrm{A}(A)=\mathrm{K}\oplus E_1\oplus_{k=2}^{r-1}(\Pi)_k\oplus(\Pi)'_r$, where for $2\leq k\leq r-1$;

$$(\Pi)_k=\bigoplus_{S\in NBC_k(A)}\mathrm{K}p_S\approx\bigoplus_{\substack{i_1=1\\ H_{i_1}\in\Pi_{i_1}}}^{\ell-k+1}\Big(\bigoplus_{\substack{i_2=i_1+1\\ H_{i_2}\in\Pi_{i_2}}}^{\ell-k+2}\Big(\cdots\Big(\bigoplus_{\substack{i_k=i_{k-1}+1\\ H_{i_k}\in\Pi_{i_k}}}^{\ell}\mathrm{K}p_{\{H_{i_1},H_{i_2},\dots,H_{i_k}\}}\Big)\cdots\Big)\Big) \text{ and}$$

$$(\Pi)'_r=\bigoplus_{S\in NBC_r(A)}\mathrm{K}p_S\approx\bigoplus_{\substack{i_1=2\\ H_{i_1}\in\Pi_{i_1}}}^{\ell-r+2}\Big(\bigoplus_{\substack{i_2=i_1+1\\ H_{i_2}\in\Pi_{i_2}}}^{\ell-r+3}\Big(\cdots\Big(\bigoplus_{\substack{i_{r-1}=i_{r-2}+1\\ H_{i_{r-1}}\in\Pi_{i_{r-1}}}}^{\ell}\mathrm{K}p_{\{H_1,H_{i_1},\dots,H_{i_{r-1}}\}}\Big)\cdots\Big)\Big),$$

is a free K-submodule of $(\Pi)_r$ with basis $\{p_S \mid S\in\mathbf{S}_r \text{ and } S\cap\Pi_1=\{H_1\}\}$, where the hyperplane $H_1\in\Pi_1$ is the minimal hyperplane of A with the hypersolvable ordering.

Proof: Ali and Al-Ta'ai in [1] proved that for ℓ-generic r-arrangement A, if $2\leq k\leq r-1$ every subarrangement which contains k-hyperplanes from k-different blocks form a k-NBC base of A and every sub arrangement which contains r-hyperplanes from r-different blocks form a r-NBC base of A if, and only if, it contains the minimal hyperplane of A with the hypersolvable ordering $H_1\in\Pi_1$. That is $NBC_k(A)=\mathbf{S}_k$ for $2\leq k\leq r-1$ and;

$$NBC_r(A)=\{S\subseteq A \mid S\in\mathbf{S}_r \text{ and } S\cap\Pi_1=\{H_1\}\}.\ \square$$

Recall the definition of $p(A)=\sup\{i \mid b_k(A)=\bar{b}_k(A),\ \forall k\leq i\}$, where for $0\leq k\leq r$ $b_k(A)$ and $\bar{b}_k(A)=\sum_{i_k>\cdots>i_1=1}^{\ell}d_{i_1}\dots d_{i_k}=|\mathbf{S}_k|$ are the j^{th}-Betti numbers of the Poincaré polynomial of $\mathrm{A}^*(A)$ and $\overline{\mathrm{A}}^*(A)$ respectively. Then we state the following result:

Theorem (4.10):

Let A be a hypersolvable r-arrangement with HP $\Pi=(\Pi_1,\cdots,\Pi_\ell)$ and exponent vector $d=(d_1,\dots,d_\ell)$ such that $r<\ell$. Then for $2\leq k\leq p(A)$;

$$\mathrm{A}_k(A)\approx(\Pi)_k=\bigoplus_{S\in\mathbf{S}_k}\mathrm{K}p_S\approx\bigoplus_{\substack{i_1=1\\ H^{i_1}_{j_1}\in\Pi_{i_1}}}^{\ell-k+1}\Big(\bigoplus_{j_1=1}^{d_{i_1}}\Big(\cdots\Big(\bigoplus_{\substack{i_k=i_{k-1}+1\\ H^{i_k}_{j_k}\in\Pi_{i_k}}}^{\ell}\Big(\bigoplus_{j_k=1}^{d_{i_k}}\mathrm{K}p_{\{H^{i_1}_{j_1},\dots,H^{i_k}_{j_k}\}}\Big)\Big)\cdots\Big)\Big)\approx\overline{\mathrm{A}}_k(A)$$

and for $p(A)+1 \le k \le r$;

$$\mathrm{A}_k(A) \approx (\Pi)'_k = \bigoplus_{S \in NBC_k(A)} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H^{i_1}_{j_1} \in \Pi_{i_1}}}^{\ell-k+1} (\bigoplus_{j_1=1}^{d_{i_1}} (\cdots (\bigoplus_{\substack{i_k = i_{k-1}+1 \\ H^{i_k}_{j_k} \in \Pi_{i_k} \\ \{H^{i_1}_{j_1},\ldots,H^{i_k}_{j_k}\} \in NBC_k(A)}}^{\ell} (\bigoplus_{j_k=1}^{d_{i_k}} \mathrm{K}p_{\{H^{i_1}_{j_1},\ldots,H^{i_k}_{j_k}\}}))\cdots)) \not\approx \overline{\mathrm{A}}_k(A);$$

where $\overline{\mathrm{A}}_k(A) \approx (\Pi)_k = \bigoplus_{S \in \mathbf{S}_k} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H^{i_1}_{j_1} \in \Pi_{i_1}}}^{\ell-k+1} (\bigoplus_{j_1=1}^{d_{i_1}} (\cdots (\bigoplus_{\substack{i_k = i_{k-1}+1 \\ H^{i_k}_{j_k} \in \Pi_{i_k}}}^{\ell} (\bigoplus_{j_k=1}^{d_{i_k}} \mathrm{K}p_{\{H^{i_1}_{j_1},\ldots,H^{i_k}_{j_k}\}}))\cdots))$.

Proof: ***For*** $2 \le k \le p(A)$: We have that $b_k(A) = \overline{b}_k(A)$. That is $rk(\mathrm{A}_k(A)) = |NBC_k(A)| = rk(\overline{\mathrm{A}}_k(A)) = |\mathbf{S}_k| = rk((\Pi)_k)$, i.e. every k-section of A forms a k-NBC base of A. Thus $\mathrm{A}_k(A) \approx (\Pi)_k \approx \overline{\mathrm{A}}_k(A)$ as graded free K-modules.

For $p(A)+1 \le k \le r$: We have $b_k(A) = |NBC_k(A)| \ne \overline{b}_k(A) = |\mathbf{S}_k|$ and since the number of all k-NBC bases of A cannot exceed the number of all k-sections of A. Therefore, $rk(\mathrm{A}_k(A)) = |NBC_k(A)| < rk(\overline{\mathrm{A}}_k(A)) = |\mathbf{S}_k| = rk((\Pi)_k)$. Thus the Orlik-Solomon Algebra can be embedded as a free K -submodule of the quadratic Orlik-Solomon Algebra by the hypersolvable partition module as an application of section two and three as;

$$\mathrm{A}_k(A) \approx (\Pi)'_k = \bigoplus_{S \in NBC_k(A)} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H^{i_1}_{j_1} \in \Pi_{i_1}}}^{\ell-k+1} (\bigoplus_{j_1=1}^{d_{i_1}} (\cdots (\bigoplus_{\substack{i_k = i_{k-1}+1 \\ H^{i_k}_{j_k} \in \Pi_{i_k} \\ \{H^{i_1}_{j_1},\ldots,H^{i_k}_{j_k}\} \in NBC_k(A)}}^{\ell} (\bigoplus_{j_k=1}^{d_{i_k}} \mathrm{K}p_{\{H^{i_1}_{j_1},\ldots,H^{i_k}_{j_k}\}}))\cdots)),$$

where $(\Pi)'_k$ is the free K-submodule of ;

$$(\Pi)_k = \bigoplus_{S \in \mathbf{S}_k} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H^{i_1}_{j_1} \in \Pi_{i_1}}}^{\ell-k+1} (\bigoplus_{j_1=1}^{d_{i_1}} (\cdots (\bigoplus_{\substack{i_k = i_{k-1}+1 \\ H^{i_k}_{j_k} \in \Pi_{i_k}}}^{\ell} (\bigoplus_{j_k=1}^{d_{i_k}} \mathrm{K}p_{\{H^{i_1}_{j_1},\ldots,H^{i_k}_{j_k}\}}))\cdots)) \approx \overline{\mathrm{A}}_k(A),$$

with basis $NBC_k(A) \subseteq \mathbf{S}_k$. □

Corollary (4.11):

Let A be a hypersolvable r-arrangement with HP $\Pi = (\Pi_1, \cdots, \Pi_\ell)$ and exponent vector $d = (d_1, \ldots, d_\ell)$ such that $r < \ell$. Then:

i- $p(A) \ge 2$ in general.

ii- If A is a 3-arrangement, then $p(A) = 2$.

iii- If A is a ℓ-generic r-arrangement, then $p(A) = r-1$.

Proof: (i), (ii) and (iii) form direct results for Theorem (4.1), Theorem (4.5) and Theorem (4.9) respectively.

□

Example (4.12):

Let A be a central complex 4-arrangement defined by

$Q(A) = xyz(x+y+z)w$, with this ordering of the hyperplanes of A, write $A = \{H_1, H_2, H_3, H_4, H_5\}$ where;

$$H_1 = \{(x,y,z,w) \in C^4 \mid x = 0\},\ H_2 = \{(x,y,z,w) \in C^4 \mid y = 0\},$$

$$H_3 = \{(x,y,z,w) \in C^4 \mid z = 0\},\ H_4 = \{(x,y,z,w) \in C^4 \mid x+y+z = 0\} \text{ and}$$

$$H_5 = \{(x,y,z,w) \in C^4 \mid w = 0\}.$$

In fact there are no collinear (rank two) relations among any three hyperplanes of A. Therefore A has a hypersolvable partition $\Pi = (\{H_1\},\{H_2\},\{H_3\},\{H_4\},\{H_5\})$ with exponent vector is $d = (1,1,1,1,1)$. Note that the subarrangement $\{H_1, H_2, H_3, H_4\}$ of A forms a unique linearly dependent relation among four different blocks of Π which implies A is not fiber type since Π is not nice and A is not generic since $m(A) = 4 \neq 5$. From theorem (4.1) $\mathrm{A}_0(A) = \overline{\mathrm{A}}_0(A) = \mathrm{K}$;

$$\mathrm{A}_1(A) = \overline{\mathrm{A}}_1(A) = E_1 \approx \bigoplus_{k=1}^{5} \mathrm{K}p_{H_k} \text{ and}$$

$$\mathrm{A}_2(A) \approx (\Pi)_2 = \bigoplus_{S \in \mathbf{S}_2} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H_{i_1}\in\Pi_{i_1}}}^{4} \Big(\bigoplus_{\substack{i_2=i_1+1 \\ H_{i_2}\in\Pi_{i_2}}}^{5} \mathrm{K}p_{\{H_{i_1}, H_{i_2}\}} \Big) \approx \overline{\mathrm{A}}_2(A) \cdot$$

Deduce that the subarrangement $\{H_2, H_3, H_4\}$ is a 3-section where it is not a 3-NBC base indeed there is a hyperplane $H_1 \in A$ with for each $2 \leq j \leq 4$, $H_1 \trianglelefteq H_j$ and $rk(H_1, H_2, H_3, H_4) = 3$. Thus $\{H_2, H_3, H_4\}$ is a 3-broken circuit of A. That is $b_3(A) = 9 = |NBC_3(A)| < \bar{b}_3(A) = 10 = |\mathbf{S}_3|$ and $p(A) = 2$. By applying theorem (4.10);

$$\mathrm{A}_3(A) \approx (\Pi)'_3 = \bigoplus_{S \in \mathbf{S}_3 \setminus \{H_2, H_3, H_4\}} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H_{i_1}\in\Pi_{i_1}}}^{3} \Big(\bigoplus_{\substack{i_2=i_1+1 \\ H_{i_2}\in\Pi_{i_2}}}^{4} \Big(\bigoplus_{\substack{i_3=i_2+1 \\ H_{i_3}\in\Pi_{i_3} \\ \{H_{i_1},H_{i_2},H_{i_3}\}\neq\{H_2,H_3,H_4\}}}^{5} \mathrm{K}p_{\{H_{i_1}, H_{i_2}, H_{i_3}\}} \Big)\Big), \text{ where;}$$

$$\overline{\mathrm{A}}_3(A) \approx (\Pi)_3 = \bigoplus_{S \in \mathbf{S}_3} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H_{i_1}\in\Pi_{i_1}}}^{3} \Big(\bigoplus_{\substack{i_2=i_1+1 \\ H_{i_2}\in\Pi_{i_2}}}^{4} \Big(\bigoplus_{\substack{i_3=i_2+1 \\ H_{i_3}\in\Pi_{i_3}}}^{5} \mathrm{K}p_{\{H_{i_1}, H_{i_2}, H_{i_3}\}} \Big)\Big).$$

Observe that $NBC_4(A) = \mathbf{S}_4 \setminus \{\{H_1, H_2, H_3, H_4\}, \{H_2, H_3, H_4, H_5\}\}$, indeed $\{H_1, H_2, H_3, H_4\}$ is a circuit and $\{H_2, H_3, H_4, H_5\}$ is a 4-broken circuit of A. Thus $b_4(A) = 3 = |NBC_4(A)| < \bar{b}_4(A) = 5 = |\mathbf{S}_4|$ and by applying the theorem (4.10) we have;

$$\mathrm{A}_4(A) \approx (\Pi)'_4 = \bigoplus_{S \in NBC_4(A)} \mathrm{K}p_S = \mathrm{K}p_{\{H_1,H_2,H_3,H_5\}} \oplus \mathrm{K}p_{\{H_1,H_2,H_4,H_5\}} \oplus \mathrm{K}p_{\{H_1,H_3,H_4,H_5\}},$$

$$\text{where } \overline{\mathrm{A}}_4(A) \approx (\Pi)_4 = \bigoplus_{S \in \mathbf{S}_4} \mathrm{K}p_S \approx \bigoplus_{\substack{i_1=1 \\ H_{i_1}\in\Pi_{i_1}}}^{2} \Big(\cdots \Big(\bigoplus_{\substack{i_4=i_3+1 \\ H_{i_4}\in\Pi_{i_4}}}^{5} \mathrm{K}p_{\{H_{i_1}, H_{i_2}, H_{i_3}, H_{i_4}\}} \Big)\Big).$$

References:

[1] H. M. Ali, *A Topological Interpretation for Vanishing of Higher Homotopy Groups of a Hypersolvable Arrangement***.** A Ph.D. Thesis, College of Science - University of Basrah, (2007).

[2] V. I. Arnold, *The Cohomology Ring of The Colored Braid Group*. Mat. Zametki, 5, 227, (1969).

[3] E. Brieskorn, *Sur les Groupes de Tresses*. Séminaire Bourbaki 1971/72. Lecture Notes in Math. 317, Springer, 21, (1973).

[4] M. Falk, H**.** Terao, *βNBC-Bases for Cohomology of Local Systems on Hyperplane Complements*, available at arxiv:alg-geom / 9412009 v3, (24-May-1995).

[5] M. Jambu, S. Papadima , *A Generalization of Fiber type Arrangements and a new Deformation Method* . Topology, 3, 1135, (1998).

[6] M. Jambu, S. Papadima, *Deformation of Hypersolvable Arrangements*. Topology and it is Applications, 103, (2002).

[7] R. S. Mahdi, *Product of Hypersolvable Arrangements*, A Ph.D. Thesis, College of Science - University of Basrah, (2006).

[8] P. Orlik, L. Solomon, *Combinatorics and Topology of complements of hyperplanes*, Invent. Math. 56, 167, (1980).

[9] P. Orlik, H. Terao, *Arrangements of Hyperplanes*, Grundlhern Math. Wiss., vol. 300, Springer-Verlag Berlin, (1992).

[10] Kelly J. Pearson, *Cohomology of OS Algebras for Quadratic Arrangements*, lecturas Matemáticas, vol. 22, 103, (2001).

[11] G. C. Rota, *On the Foundations of Combinatorial Theory I. Theory of the Möbius Functions*, Z. Wahrsheinlichkeitstheorie 2, 340, (1964).

[12] B. E. Sagan, *A Generalization of Rota's NBC Theorem*, AMS subject classification, (1991).

[13] R. P. Stanley, *Supersolvable Lattices*. Algebra Universalis 2, 197, (1972).

المستخلص:

الهدف من هذا البحث بناء جبر أورلاك–سولومون لترتيبةr –قابلة للحل فوقيا"باستخدام بنية التجزئة القابلة للحل فوقيا" و باستخدام الترتيب القابل للحل فوقيا" الذي يحافظ على البنية القابلة للحل فوقيا", عرفنا قاعدة الNBC لجبر أورلاك–سولومون, ثم غمرناه في موديول التجزئة كموديول جزئي حر. و استخدمنا هذا الترتيب ايضا"في أثبات ان ترتيبة قابلة للحل فوقيا"تكون ثنائية اذا و فقط اذا كانت قابلة للحل كليا". و كتعميم لنتيجة جامبو و باباديما اعطينا لجبر أورلاك–سولومون الثنائي بنية تكافئ موديول التجزئة, وباستخدام هذا التكافؤ قمنا بدراسة الفروقات بين جبر أورلاك–سولومون و جبر أورلاك–سولومون الثنائي لترتيبة قابلة للحل فوقيا" كموديولات حرة مع العديد من التطبيقات.