Journal University of Kerbala , Vol. 15 No.4 Scientific . 2017

Fuzzy translations and fuzzy multiplications of QS-algebras

التحويلات الضبابية والضرب الضبابي الى حبر-QS

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Abstract.
Fuzzy translations, (normalized, maximal) fuzzy extensions and fuzzy multiplications of fuzzy QS-subalgebras of QS-algebras are discussed relations among fuzzy translations, fuzzy extensions and fuzzy multiplications of fuzzy QS-ideals are investigated.
2010 Mathematics Subject Classification: 06F35, 03G25, 08A72.

الملخص :
في هذا البحث نعطي تعريف التحويلات الضبابية و التوسعات الضبابية (الاعتيادية ، الاعظمية) والضرب الضبابي الى الجبر الجزئي من النوع QS وناقشنا العلاقات بينها .وعرفنا التحويلات الضبابية والتوسعات الضبابية والضرب الضبابي على المثاليات الضبابية في الجبر من النوع QSوحققنا علاقات بينها .

1. Introduction

Several authors ([7],[3],[8]) have introduced of BCK-algebras as a generalization of the concept of set-theoretic difference and propositional calculus and studied some important properties. The concept of a fuzzy set, was introduced by L.A. Zadeh [10]. In ([5], [6],[9]) , they applied the concept of fuzzy set to BCK/BCI-algebras and gave some of its properties. A.T. Hameed and et.al. [2] introduced the notions fuzzy QS-subalgebras , fuzzy QS-ideals of QS-algebras and investigated relations among them . In this paper, we discuss fuzzy translation, (normalized, maximal) fuzzy S-extension and fuzzy multiplication of fuzzy QS-subalgebras in QS-algebra. We discuss fuzzy translation, fuzzy extension and fuzzy multiplication of fuzzy QS-ideals in QS-algebra.

2. Preliminaries

Now, we introduced the concept of algebraic structure of QS-algebra and we give some results and theorems of it .

Definition 2.1([1]). Let $(X; *,0)$ be an algebra of type (2,0) with a single binary operation ($*$). X is called a **QS-algebra** if it satisfies the following identities: for any $x, y, z \in X$,

(QS_1) : $(z*y) * (z*x) = x*y$,

(QS_2) : $x * 0 = x$,

(QS_3) : $x * x = 0$,

(QS_4) : $(x * y) * z = (x* z)*y$.

In X we can define z\sorder relation ($\leq$) by : $x \leq y$ if and only if, $x* y = 0$.

Lemma 2.2 ([1]). In any QS-algebra X, the following properties hold: for all $x, y, z \in X$;

a) $x* y = 0$ and $y*x = 0$ imply $x = y$,

b) $y * [(y*z) *z]= 0$,

c) $x \leq y$ implies that $y *z \leq x *z$,

d) $x \leq y$ implies that $z*x \leq z*y$,

e) $x \leq y$ and $y \leq z$ imply $x \leq z$,

f) $x*y \leq z$ implies that $z*y\leq x$.

Definition 2.3([1]). Let X be a QS-algebra and let S be a nonempty subset of X. S is called a **QS-subalgebra** of X if $x *y \in S$ whenever $x , y \in S$.

Definition 2.4([1]). A nonempty subset I of a QS-algebra X is called a **QS-ideal** of X if it satisfies: for $x , y, z \in X$,

(IQS_1) $(0 \in I)$,
(IQS_2) $(z*y) \in I$ and $(x*y) \in I$ imply $(z*x) \in I$.

Definition 2.5([10]). Let X be a nonempty set, a fuzzy subset μ in X is a function $\mu: X \to [0,1]$.

Proposition 2.6([1]). Every QS-ideal of QS-algebra X is a QS-subalgebra of X**.**

Definition 2.7([2]). Let X be a QS-algebra, a fuzzy subset μ in X is called a **fuzzy QS-subalgebra** of X if for all x , y $\in$ X , $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$.

Definition 2.8([2]). Let X be a QS-algebra, a fuzzy subset μ in X is called a **fuzzy QS-ideal** of X if it satisfies the following conditions: , for all x , y, z $\in$ X ,
(FQS_1) $\mu(0) \geq \mu(x)$,
(FQS_2) $\mu(z*x) \geq \min\{\mu(z*y), \mu(x*y)\}$.

Proposition 2.9([2]). Every fuzzy QS-ideal of QS-algebra X is a fuzzy QS-subalgebra of X**.**

3. Fuzzy translations and fuzzy multiplications of fuzzy QS-subalgebras .

We study the relations among fuzzy translation,(normalized, maximal) fuzzy S-extension and fuzzy multiplication of QS-subalgebras of QS-algebra X as([3],[4],[7]).

In what follows let $(X; *,0)$ denote a QS-algebra, and for any fuzzy set μ of X, we denote $T = 1 - \sup\{\mu(x) \mid x \in X\}$ unless otherwise specified.

Definition 3.1([3]). Let X be a nonempty set and μ be a fuzzy subset of X and let $\alpha \in [0,T]$. A mapping $\mu_\alpha^T : X \to [0,1]$ is called a **fuzzy translation subset** of μ if it satisfies:

$\mu_\alpha^T(x) = \mu(x) + \alpha$, for all $x \in X$.

Theorem 3.2. Let X be a QS-algebra and μ be a fuzzy QS-subalgebra of X and $\alpha \in [0,T]$. Then the fuzzy translation subset μ_α^T of μ is a fuzzy QS-subalgebra of X .

Proof. Assume μ be a fuzzy QS-subalgebra of X and $\alpha \in [0,T]$, let x, y $\in$ X. Then
$\mu_\alpha^T(x*y) = \mu(x*y)+\alpha \geq \min\{\mu(x), \mu(y)\}+\alpha = \min\{\mu(x)+\alpha, \mu(y)+\alpha\} = \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$.

Hence μ_α^T is a fuzzy translation QS-subalgebra of X. ⌂

Theorem 3.3. Let X be a QS-algebra and μ be a fuzzy subset of X such that the fuzzy translation subset μ_α^T of μ is a fuzzy QS-subalgebra of X for some $\alpha \in [0,T]$. Then μ is a fuzzy QS-subalgebra of X .

Proof. Assume μ_α^T be a fuzzy translation QS-subalgebra of X for some $\alpha \in [0,T]$. Let x, y $\in$ X, then $\mu(x*y)+\alpha = \mu_\alpha^T(x*y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\} = \min\{\mu(x)+\alpha, \mu(y)+\alpha\}$
$= \min\{\mu(x), \mu(y)\}+\alpha$ and so $\mu(x*y) \geq \min\{\mu(x), \mu(y)\}$.
Hence μ is a fuzzy QS-subalgebra of X .⌂

Definition 3.4([3]). Let μ_1 and μ_2 be fuzzy subsets of a set X. If $\mu_1(x) \leq \mu_2(x)$ for all $x \in X$, then we say that μ_2 is **a fuzzy extension of** μ_1.

Definition 3.5. Let X be a QS-algebra, μ_1 and μ_2 be fuzzy subsets of X. Then μ_2 is called **a fuzzy S-extension** of μ_1 if the following assertions are valid:
(S_i) μ_2 is a fuzzy extension of μ_1.
(S_{ii}) If μ_1 is a fuzzy QS-subalgebra of X, then μ_2 is a fuzzy QS-subalgebra of X.

By means of the definition of fuzzy translation, we know that $\mu_\alpha^T(x) \geq \mu(x)$ for all $x \in X$. Hence we have the following proposition.

Proposition 3.6. Let μ be a fuzzy QS-subalgebra of a QS-algebra X and $\alpha \in [0,T]$. Then the fuzzy translation subset μ_α^T of μ is a fuzzy S-extension of μ.

Proof. Straightforward.⌂

In general, the converse of Proposition (3.6) is not true as seen in the following example.

Example 3.7. Consider a QS-algebra X = {0, a, b, c} with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a fuzzy QS-subalgebra μ of X by:

X	0	a	b	c
μ	0.9	0.6	0.7	0.6
μ_α^T	0.91	0.61	0.71	0.61

Then μ_α^T is a fuzzy QS-subalgebra of X. which is a fuzzy S-extension of μ for all $\alpha \in [0,T]$. But μ is not a fuzzy S-extension of μ_α^T .

Proposition 3.8. The intersection of fuzzy S-extensions of a fuzzy subset μ of X is a fuzzy S-extension of μ.

Proof: Let $\{ \mu_i \mid i \in \Lambda.\}$ be a family of fuzzy QS-subalgebras of QS-algebra X , then for any x, y $\in$ X , i $\in\Lambda$,

$$(\bigcap_{i\in\wedge} \mu_i)(x * y) = \inf(\mu_i(x * y)) \geq \inf(\min\{\mu_i(x), \mu_i(y)\})$$

$$= \min\{\inf(\mu_i(x)), \inf(\mu_i(y)\}$$

$$= \min\{((\bigcap_{i\in\wedge} \mu_i)(x)), (\bigcap_{i\in\wedge} \mu_i(y)\}$$. This completes the proof . ⌂

Clearly, the union of fuzzy S-extensions of a fuzzy subset μ of X , is not a fuzzy S-extension of μ as seen in the following example.

Example 3.9. Consider a QS-algebra X = {0, a, b, c} with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	b	b
b	b	c	0	a
c	c	b	a	0

Define a fuzzy subset μ of X by:

X	0	a	b	c
μ	0.8	0.5	0.7	0.5

Then μ is a fuzzy QS-subalgebra of X. Let ν and δ be fuzzy subsets of X given by

X	0	a	b	c
ν	0.8	0.6	0.7	0.7
δ	0.9	0.5	0.8	0.5

Then ν and δ are fuzzy S-extensions of μ. But the union $\nu \cup \delta$ is not a fuzzy S-extension of μ since $(\nu \cup \delta)(c * b) = 0.6 < 0.7 = \min\{(\nu \cup \delta)(c), (\nu \cup \delta)(b)\}$.

Definition 3.10. For a fuzzy subset μ of a QS-algebra X, $\alpha \in [0,T]$ and $t \in [0, 1]$ with $t \geq \alpha$, let $U_\alpha(\mu; t) := \{x \in X \mid \mu(x) \geq t - \alpha\}$.

If μ is a fuzzy QS-subalgebra of X, then it is clear that $U_\alpha(\mu; t)$ is a QS-subalgebra of X, for all $t \in Im(\mu)$ with $t \geq \alpha$. But if we do not give a condition that μ is a fuzzy QS-subalgebra of X, then $U_\alpha(\mu; t)$ is not a QS-subalgebra of X as seen in the following example.

Example 3.11. Let X = {0, a, b, c} be a QS-algebra which is given in Example (3.9). Define a fuzzy subset λ of X by

X	0	a	b	c
λ	0.7	0.6	0.4	0.3

Then λ is not a fuzzy QS-subalgebra of X since
$\lambda(a * b) = \lambda(c) = 0.3 < 0.4 = \min\{\lambda(a), \lambda(b)\}$. For $\alpha = 0.1$ and $t = 0.5$, we obtain
$U_\alpha(\lambda; t) = \{0, a, b\}$ which is not a QS-subalgebra of X since $a * b = c \notin U_\alpha(\lambda; t)$.

Proposition 3.12. Let μ be a fuzzy subset of a QS-algebra X and $\alpha \in [0,T]$. Then the fuzzy subset translation μ_α^T of μ is a fuzzy QS-subalgebra of X if and only if,
$U_\alpha(\mu; t)$ is a QS-subalgebra of X, for all $t \in Im(\mu)$ with $t \geq \alpha$.

Proof. Necessity is clear. To prove the sufficiency, assume that there exist $x, y \in X$,
$\gamma \in [0, 1]$ with $\gamma \geq \alpha$ such that $\mu_\alpha^T(x * y) < \gamma \leq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$. Then $\mu(x) \geq \gamma - \alpha$ and $\mu(y) \geq \gamma - \alpha$, but $\mu(x * y) < \gamma - \alpha$. This shows that $x, y \in U_\alpha(\mu; \gamma)$ and $x * y \notin U_\alpha(\mu; \gamma)$. This is a contradiction, and so $\mu_\alpha^T(x * y) \geq \min\{\mu_\alpha^T(x), \mu_\alpha^T(y)\}$, for all $x, y \in X$. Hence
μ_α^T is a fuzzy translation QS-subalgebra of X. ⌂

Proposition 3.13. Let μ be a fuzzy QS-subalgebra of QS-algebra X and $\alpha, \lambda \in [0,T]$. If $\alpha \geq \lambda$, then the fuzzy translation QS-subalgebra μ_α^T of μ is a fuzzy S-extension of the fuzzy translation QS-subalgebra μ_λ^T of μ.

Proof. For every $x \in X$ and $\alpha, \lambda \in [0,T]$ and $\alpha \geq \lambda$, we have $\mu_\alpha^T(x) = \mu(x) + \alpha \geq \mu(x) + \lambda = \mu_\lambda^T(x)$, then $\mu_\alpha^T(x) \geq \mu_\lambda^T(x)$ therefor $\mu_\alpha^T(x)$ is a fuzzy extension of $\mu_\lambda^T(x)$, since μ is a fuzzy QS-subalgebra of X then μ_α^T of μ is a fuzzy QS-subalgebra (by Theorem (3.2)),hence μ_α^T of μ is a fuzzy S-extension of the fuzzy translation QS-subalgebra μ_λ^T of μ. ⌂

Proposition 3.14. Let μ be a fuzzy QS-subalgebra of a QS-algebra X and $\lambda \in [0,T]$. For every fuzzy S-extension ν of the fuzzy translation QS-subalgebra μ_λ^T of μ, there exists $\alpha \in [0,T]$ such that $\alpha \geq \lambda$ and ν is a fuzzy S-extension of the fuzzy translation QS-subalgebra μ_α^T of μ.

Proof. Since μ is afuzzy QS-subalgebra of a QS-algebra X and $\lambda \in [0,T]$, the fuzzy translation subset μ_λ^T of μ is a fuzzy QS-subalgebra of X. If ν is a fuzzy S-extension of μ_λ^T, then there exists $\alpha \in [0,T]$ such that $\alpha \geq \lambda$ and $\nu(x) \geq \mu_\alpha^T(x)$, for all $x \in X$, hence
ν is a fuzzy S-extension of the fuzzy translation QS-subalgebra μ_α^T of μ.⌂

The following example illustrates Proposition (3.14).

Example 3.15. Consider a QS-algebra X = {0, a, b, c} with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	b	b
b	b	c	0	a
c	c	b	a	0

Define a fuzzy subset μ of X by:

X	0	a	b	c
μ	0.7	0.5	0.4	0.4

Then μ is a fuzzy QS-subalgebra of X and T=0.3 . If we take $\lambda = 0.2$, then the fuzzy QS-subalgebra translation μ_λ^T of μ is given by :

X	0	a	b	c
μ_λ^T	0.9	0.7	0.6	0.6

Let ν be a fuzzy subset of X defined by:

X	0	a	b	c
ν	0.94	0.76	0.64	0.64

Then ν is clearly a fuzzy QS-subalgebra of X which is fuzzy extension of μ_λ^T and hence ν is a fuzzy S-extension of fuzzy translation subset μ_λ^T of μ. Take $\alpha = 0.23$, then $\alpha = 0.23 > 0.2 = \lambda$, and the fuzzy translation QS-subalgebra μ_α^T of μ is given as follows:

X	0	a	b	c
μ_α^T	0.93	0.73	0.63	0.63

Note that $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in X$, and hence ν is a fuzzy S-extension of the fuzzy translation QS-subalgebra μ_α^T of μ.

Definition 3.16. A fuzzy S-extension ν of a fuzzy QS-subalgebra μ in a QS-algebra X is said to be **normalized** if there exists $x_0 \in X$ such that $\nu(x_0) = 1$. Let μ be a fuzzy QS-subalgebra of X. A fuzzy subset ν of X is called a **maximal fuzzy S-extension** of μ if it satisfies:

(M_i) ν is a fuzzy S-extension of μ,

(M_{ii}) there does not exist another fuzzy QS-subalgebras of a QS-algebra X which is a fuzzy extension of ν.

Example 3.17. let X = {0, a, b, c} with the following Cayley table:

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

$(X; *, 0)$ is a QS-algebra. Let μ and ν be fuzzy subsets of X which are defined by

$\mu(x) = \frac{1}{5}$ and $\nu(x) = 1$ for all $x \in X$. Clearly μ and ν are fuzzy QS-subalgebras of X. It is easy to verify that ν is a maximal fuzzy S-extension of μ.

Proposition 3.18. If a fuzzy subset ν of a QS-algebra X is a normalized fuzzy S-extension of a fuzzy QS-subalgebra μ of X, then $\nu(0) = 1$.

Proof. It is clear because $\nu(0) \geq \nu(x)$ for all $x \in X$. ⌂

Proposition 3.19. Let μ be a fuzzy QS-subalgebra of a QS-algebra X. Then every maximal fuzzy S-extension of μ is normalized.

Proof. This follows from the definitions of the maximal fuzzy S-extensions and normalized.⌂

Definition 3.20. Let μ be a fuzzy subset of a QS-algebra X and $\beta \in (0, 1]$. A fuzzy multiplication of μ, denoted by μ_β^M is defined to be a mapping $\mu_\beta^M : X \to [0, 1]$ define by $\mu_\beta^M(x) = \beta.\mu(x)$, for all $x \in X$.

Proposition 3.21. If μ is a fuzzy QS-subalgebra of a QS-algebra X, then the fuzzy multiplication subset of μ is a fuzzy QS-subalgebra of X for all $\beta \in [0, 1]$.

Proof. Assume μ is a fuzzy QS-algebra of a QS-algebra X , Then for all $x, y \in X$ and $\beta \in [0, 1]$, $\mu_\beta^M(x*y)=\beta.\mu(x*y) \geq \beta.\min\{\mu(x),\mu(y)\}=\{\beta\ \mu(x),\beta\ \mu(y)\}= \{\mu_\beta^M(x), \mu_\beta^M(y)\}$. Hence μ_β^M is a fuzzy QS-subalgebra of X for all $\beta \in (0, 1]$. ⌂

Proposition 3.22. For any fuzzy subset μ of QS-algebra X, the following are equivalent:

(A) μ is a fuzzy QS-subalgebra of X.

(B) For all $\beta \in (0, 1]$, μ_β^M is a fuzzy multiplication QS-subalgebra of X.

Proof. Necessity follows from Proposition (3.21). Let $\beta \in (0, 1]$ be such that μ_β^M is a fuzzy multiplication QS-subalgebra of X. Then for all $x, y \in X$, $\beta.\mu(x * y) = \mu_\beta^M(x * y) \geq \min\{\mu_\beta^M(x), \mu_\beta^M(y)\} = \min\{\beta.\mu(x), \beta.\mu(y)\} = \beta.\min\{\mu(x), \mu(y)\}$ and so $\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$. Hence μ is a fuzzy QS-subalgebra of X. ⌂

Proposition 3.23. Let μ be a fuzzy subset of QS-algebra X, $\alpha \in [0,T]$ and $\beta \in (0, 1]$. Then every fuzzy translation QS-subalgebra μ_α^T of μ is a fuzzy S-extension of the fuzzy multiplication QS-subalgebra μ_β^M of μ.

Proof. For every $x \in X$, we have $\mu_\alpha^T(x) = \mu(x)+\alpha \geq \mu(x) \geq \beta.\mu(x) = \mu_\beta^M(x)$, and so μ_α^T is a fuzzy extension of μ_β^M. Assume that μ_β^M is a fuzzy multiplication QS - subalgebra of X. Then μ is a fuzzy QS-subalgebra of X by Proposition (3.22). It follows from Theorem (3.2) that μ_α^T is a fuzzy QS-subalgebra of X for all $\alpha \in [0,T]$. Hence every fuzzy translation QS-subalgebra μ_α^T is a fuzzy S-extension of the fuzzy multiplication QS-subalgebra μ_β^M . ⌂

4. Fuzzy translations and fuzzy multiplications of fuzzy QS-ideals .

We study the relations among fuzzy translation, fuzzy extension and fuzzy multiplication of QS-ideals of QS-algebra X.

Theorem 4.1. Let μ be a fuzzy QS-ideal of a QS-algebra X , then the fuzzy translation subset μ_α^T of μ is a fuzzy QS-ideal of X, for all $\alpha \in [0,T]$.

Proof. Assume μ be a fuzzy QS-ideal of X and let $\alpha \in [0,T]$. For all $x, y, z \in X$ and $\mu(0) \geq \mu(x)$. Then $\mu_\alpha^T(0) = \mu(0)+\alpha \geq \mu(x)+\alpha = \mu_\alpha^T(x)$. and $\mu_\alpha^T(z * x) = \mu(z * x)+\alpha \geq \min\{\mu(z * y), \mu(x * y)\}+\alpha = \min\{\mu(z * y)+\alpha, \mu(x * y)+\alpha\}= \min\{\mu_\alpha^T(z * y), \mu_\alpha^T(x * y)\}$. Hence μ_α^T is a fuzzy translation QS-ideal of X .⌂

Theorem 4.2. Let μ be a fuzzy subset of QS-algebra X such that the fuzzy translation subset μ_α^T of μ is a fuzzy QS-ideal of X for some $\alpha \in [0,T]$. Then μ is a fuzzy QS-ideal of X.

Proof. Assume μ_α^T is a fuzzy translation QS-ideal of X for some $\alpha \in [0,T]$. Let $x, y, z \in X$, we have $\mu(0)+\alpha = \mu_\alpha^T(0) \geq \mu_\alpha^T(x) = \mu(x)+\alpha$. So $\mu(0) \geq \mu(x)$ and

$\mu(z*x)+\alpha = \mu_\alpha^T(z*x) \geq \min\{\mu_\alpha^T(z*y), \mu_\alpha^T(x*y)\} = \min\{\mu(z*y)+\alpha, \mu(x*y)+\alpha\}$
$= \min\{\mu(z*y), \mu(x*y)\}+ \alpha$ and so $\mu(z*x) \geq \min\{\mu(z*y), \mu(x*y)\}$. Hence μ is a fuzzy QS-ideal of X .⌂

Definition 4.3. Let μ_1 and μ_2 be fuzzy subsets of a QS-algebra X. Then μ_2 is called a **fuzzy extension QS-ideal** of μ_1 if the following assertions are valid:

(I_i) μ_2 is a fuzzy extension of μ_1.

(I_{ii}) If μ_1 is a fuzzy QS-ideal of X, then μ_2 is a fuzzy QS-ideal of X.

Proposition 4.4. Let μ be a fuzzy QS-ideal of X and let $\alpha, \gamma \in [0,T]$. If $\alpha \geq \gamma$, then the fuzzy translation subset μ_α^T of μ is a fuzzy extension QS-ideal of the fuzzy translation QS-ideal μ_γ^T of μ.

Proof. Let μ be a fuzzy QS-ideal of X and let $\alpha, \gamma \in [0,T]$ and $\alpha \geq \gamma$,then $\mu(x)+\alpha \geq \mu(y)+\gamma$, $\mu_\alpha^T \geq \mu_\gamma^T$, hence μ_α^T of μ is a fuzzy extension of μ_γ^T of μ and by Theorem(4.1) then the fuzzy translation subset μ_α^T of μ is a fuzzy QS-ideal of X, hence μ_α^T of μ is a fuzzy extension QS-ideal of the fuzzy translation QS-ideal μ_γ^T of μ. ⌂

Proposition 4.5. Let μ be a fuzzy QS-ideal of a QS-algebra X and $\gamma \in [0,T]$. For every fuzzy extension QS-ideal ν of the fuzzy translation QS-ideal μ_γ^T of μ, there exists $\alpha \in [0,T]$ such that $\alpha \geq \gamma$ and ν is a fuzzy extension QS-ideal of the fuzzy translation QS-ideal μ_α^T of μ.

Proof. For every fuzzy QS-ideal μ of X and $\gamma \in [0,T]$, the fuzzy translation subset μ_γ^T of μ is a fuzzy translation QS-ideal of X. If ν is a fuzzy extension QS-ideal of μ_γ^T, then there exists $\alpha \in [0,T]$ such that $\alpha \geq \gamma$ and $\nu(x) \geq \mu_\alpha^T(x)$ for all $x \in X$. and by Theorem(4.1) then the fuzzy translation subset μ_α^T of μ is a fuzzy QS-ideal of X, hence μ_α^T of μ is a fuzzy extension QS-ideal of the fuzzy translation QS-ideal μ_γ^T of μ. ⌂

The following example illustrates Proposition (4.5).

Example 4.6. Consider a QS-algebra X = {0, a, b, c} with the following Cayley table:

*	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a fuzzy subset μ of X by:

X	0	a	b	c
μ	0.8	0.5	0.6	0.5

Then μ is a fuzzy QS-ideal of X and T = 0.2 . If we take $\gamma = 0.12$, then the fuzzy translation QS-ideal μ_γ^T of μ is given by :

X	0	a	b	c
μ_γ^T	0.92	0.62	0.72	0.62

Let ν be a fuzzy subset of X defined by:

X	0	a	b	c
ν	0.98	0.67	0.76	0.67

Then ν is clearly a fuzzy extension QS-ideal of the fuzzy translation QS-ideal μ_{γ}^{T} of μ . for all α ∈[0,T]. Take α = 0.17, then α = 0.17 > 0.12 = γ , and the fuzzy translation QS-ideal μ_{α}^{T} of μ is given as follows:

X	0	a	b	c
μ_{α}^{T}	0.97	0.67	0.77	0.67

Note that $\nu(x) \geq \mu_{\alpha}^{T}(x)$, for all x ∈ X, and hence ν is a fuzzy extension QS-ideal of the fuzzy translation QS-ideal μ_{α}^{T} of μ.

Proposition 4.7. Let μ be a fuzzy QS-ideal of a QS-algebra X and α∈ [0,T]. Then the fuzzy translation subset μ_{α}^{T} of μ is a fuzzy extension QS-ideal of μ.

Proof. Since $\mu(x)+\alpha= \mu_{\alpha}^{T}(x)\geq \mu(x)$,for all x∈X ,and α∈ [0,T] and by Theorem(4.1)then the fuzzy translation subset μ_{α}^{T} of μ is a fuzzy extension QS-ideal of μ . ⌂

A fuzzy extension QS-ideal of a fuzzy QS-ideal μ may not be represented as a fuzzy translation QS-ideal μ_{α}^{T} of μ , that is , the converse of Proposition (4.7) is not true in general , as shown by the following example.

Example 4.8. Consider a QS-algebra X = {0, a, b, c} with the following Cayley table:

∗	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	0

Define a fuzzy subset μ of X by:

X	0	a	b	c
μ	0.8	0.5	0.3	0.3

Then μ is a fuzzy QS-ideal of X . Let ν be a fuzzy subset of X defined by:

X	0	a	b	c
ν	0.82	0.56	0.35	0.35

Then ν is a fuzzy extension QS-ideal of μ . But ν is not the fuzzy QS-ideal which is fuzzy translation QS-ideal μ_{α}^{T} of μ for all α ∈ [0,T].

Proposition 4.9. The intersection of any set of fuzzy extension QS-ideals of QS-algebra X is also fuzzy extension QS-ideal of X.

Proof: Let {μ_i| i ∈ Λ} be a family of fuzzy extension QS-ideals of QS-algebra X, then for any x, y, z ∈ X , i ∈Λ,

$(\bigcap_{i\in\wedge}(\mu)_i)(0) = \inf((\mu)_i(0)) \geq \inf((\mu)_i(x)) = (\bigcap_{i\in\wedge}(\mu_{\alpha}^{T})_i(x)$ and $(\bigcap_{i\in\wedge}(\mu)_i)(z*x) = \inf((\mu)_i)(z*x)) \geq \inf(\min\{\mu_i(z*y), \mu_i(x*y)\}) =$

min $\{(\cap_{i\in\Lambda}\mu_i)(z*y)\ ,\ (\cap_{i\in\Lambda}\mu_i)(x*y)\ \}$ = min $\{(\bigcap_{i\in\wedge}(\mu)_i)(z*y)\ ,\ (\bigcap_{i\in\wedge}(\mu)_i)(x*y)\ \}$. ⌂ Clearly, the union of fuzzy extensions QS-ideals of a fuzzy subset of QS-algebraX is not a fuzzy extension of μ as seen in the following example.

Example 4.10. Consider a QS-algebra X = {0, 1, 2, 3} with the following Cayley table:

$*$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Define a fuzzy subset μ of X by:

X	0	1	2	3
μ	0.8	0.5	0.3	0.3

then μ is a fuzzy QS-ideal of X. Let ν and δ be fuzzy subsets of X given by

X	0	1	2	3
ν	0.8	0.6	0.8	0.6
δ	0.9	0.6	0.6	0.7

Then ν and δ are fuzzy extensions QS-ideals of μ. But the union $\nu\cup\delta$ is not a fuzzy extension QS-ideals of μ since $(\nu\cup\delta)(3*2) = 0.6 < 0.7 = \min\{(\nu\cup\delta)(3), (\nu\cup\delta)(2)\}$.

Theorem 4.11. The intersection of any set of fuzzy translations QS-ideals of QS-algebra X is also fuzzy translations QS-ideals of X.

Proof: Let $\{(\mu_\alpha^T)_i \mid i\in\Lambda\}$ be a family of fuzzy translations QS-ideals of QS-algebra X , then for any x, y, z ∈ X , i ∈Λ,

1) The intersection of any set of fuzzy QS-ideals of QS-algebra X is also fuzzy QS-ideals of X, (by Proposition(3.10) from [1]) .

2) since $(\bigcap_{i\in\wedge}(\mu_\alpha^T)_i)(x) = \inf\{(\mu_\alpha^T)_i(x): i\in\Lambda\} = \inf\{\mu_i(x)+\alpha : i\in\Lambda\} = \inf\{\mu_i(x) : i\in\Lambda\}+\alpha$

$= \bigcap_{i\in\wedge}(\mu)_i(x)+\alpha$ hence intersection of any set of fuzzy translations of QS-algebra X is also fuzzy translations of X. ⌂

Theorem 4.12. Let $\alpha\in[0,T]$, μ_α^T be the fuzzy translation subset of μ. Then the following are equivalent:

(1) μ_α^T is a fuzzy translation QS-ideal of X.

(2) $\forall t\in Im(\mu)$, $t>\alpha \Rightarrow U_\alpha(\mu; t)$ is QS-ideal of X.

Proof. Assume that μ_α^T is a fuzzy translation QS-ideal of X and let $t\in Im(\mu)$ be such that $t>\alpha$. Since $\mu_\alpha^T(0)\geq\mu_\alpha^T(x)$ for all $x\in X$, we have $\mu(0)+\alpha=\mu_\alpha^T(0)\geq\mu_\alpha^T(x)=\mu(x)+\alpha$ that mean $\mu(0)\geq\mu(x)$, for all x∈ X. Let $x\in U_\alpha(\mu; t)$, then $\mu(x) > t-\alpha$ and

$\mu(0)\geq\mu(x)$ imply $\mu(0)\geq\mu(x)\geq t-\alpha$. Hence $0\in U_\alpha(\mu; t)$.

Let x, y, z ∈ X be such that $(z*y)\in U_\alpha(\mu; t)$ and $(x*y)\in U_\alpha(\mu; t)$. Then $\mu(z*y)\geq t-\alpha$ and $\mu(x*y)\geq t-\alpha$, i.e., $\mu_\alpha^T(z*y)=\mu(z*y)+\alpha\geq t$ and $\mu_\alpha^T(x*y)=\mu(x*y)+\alpha\geq t$. Since μ_α^T is a fuzzy translation QS-ideal of X, it follows that $\mu(z*x)+\alpha=\mu_\alpha^T(z*x)\geq\min\{\mu_\alpha^T(z*y), \mu_\alpha^T(x*y)\}\geq t$, that is, $\mu(z*x)\geq t-\alpha$ so that $(z*x)\in U_\alpha(\mu; t)$. Therefore $U_\alpha(\mu; t)$ is QS-ideal of X.

Conversely, suppose that U_α (μ; t) is QS-ideal of X for every t ∈ Im(μ) with t > α. If there exists x ∈ X such that $\mu_\alpha^T(0) < \lambda \le \mu_\alpha^T(x)$, then $\mu(x) \ge \lambda - \alpha$ but $\mu(0) < \lambda - \alpha$. This shows that x∈ U_α (μ; t) and $0 \notin U_\alpha$ (μ; t). This is a contradiction, and so $\mu_\alpha^T(0) \ge \mu_\alpha^T(x)$ for all x ∈ X.

Now assume that there exist x, y, z ∈ X such that $\mu_\alpha^T(z*x) < \gamma \le \min\{\mu_\alpha^T(z*y), \mu_\alpha^T(x*y)\}$. Then $\mu(z*y) \ge \gamma - \alpha$ and $\mu(x*y) \ge \gamma - \alpha$, but $\mu(z*x) < \gamma - \alpha$. Hence $(z*y) \in U_\alpha(\mu;\gamma)$ and $(x*y) \in U_\alpha(\mu;\gamma)$, but $(z*x) \notin U_\alpha(\mu;\gamma)$. This is a contradiction, and therefore $\mu_\alpha^T(z*x) \ge \min\{\mu_\alpha^T(z*y), \mu_\alpha^T(x*y)\}$, for all x,y, z ∈ X. Hence μ_α^T is a fuzzy translation QS-ideal of X. ⌂

In Theorem (4.11(2)), if $t \le \alpha$, then U_α (μ; t) = X.

Proposition 4.13. Let μ be a fuzzy QS-ideal of a QS-algebra X and let α ∈ [0,T], then the fuzzy translation subset μ_α^T of μ is a fuzzy QS-subalgebra of X.

Proof: Since μ be a fuzzy QS-ideal of a QS-algebra X , then by Proposition (2.9) μ be a fuzzy QS-subalgebra of a QS-algebraX and let α ∈ [0,T], then by Proposition (3.2), the fuzzy translation subset μ_α^T of μ is a fuzzy translation QS-subalgebra of X. ⌂

In general, the converse of the Proposition (4.13) is not true .

Proposition 4.14. For any fuzzy subset μ of X, the following are equivalent:

(i) μ is a fuzzy QS-ideal of X.

(ii) For all β ∈ (0, 1], μ_β^M is a fuzzy multiplication QS-ideal of X.

Proof. Let μ be a fuzzy QS-ideal of a QS-algebra X ,and x, y, z ∈ X, For all β ∈ (0, 1] then μ(0) ≥ μ(x) , β.μ(0) ≥ β.μ(x),hence $\mu_\beta^M(0) \ge \mu_\beta^M(x)$ and $\mu(z*x) \ge \min\{\mu(z*y), \mu(y*x)\}$ = β.μ(z∗x)≥ = β. min{μ(z∗y), μ(y∗x)} = min{β.μ(z∗y), β.μ(y∗x)} and so $\mu_\beta^M(x) \ge \min\{\mu_\beta^M(z*y), \mu_\beta^M(y*x)\}$ for all x, y, z ∈ X.Hence μ_β^M is a fuzzy QS-ideal of X. Let β ∈ (0, 1] be such that μ_β^M is a fuzzy multiplication QS-ideal of X.Then for all x, y, z∈ X, β.μ(0) ≥ β.μ(x) , then μ(0) ≥ μ(x) and β.μ(z∗x) = $\mu_\beta^M(z*x)$ ≥ $\min\{\mu_\beta^M(z*y), \mu_\beta^M(x*y)\}$ = min{β.μ(z∗y), β.μ(x∗y)} = β. min{μ(z∗y), μ(x∗y)} and so μ(z∗x) ≥ min{μ(z∗y), μ(x∗y)} for all x, y, z ∈ X. Hence μ is a fuzzy QS-ideal of X. ⌂

Proposition 4.15. Let μ be a fuzzy subset of a QS-algebraX, α ∈ [0,T] and λ ∈ (0, 1]. Then every fuzzy translation subset μ_α^T of μ is a fuzzy extension QS-ideal of the fuzzy multiplication QS-ideal μ_β^M of μ

Proof. For every x ∈ X, we have $\mu_\alpha^T(x) = \mu(x)+\alpha \ge \mu(x) \ge \beta.\mu(x) = \mu_\beta^M(x)$, and so μ_α^T is a fuzzy extension QS-ideal of μ_β^M. Assume that μ_β^M is a fuzzy QS-ideal of X. Then μ is a fuzzy QS-ideal of X by Proposition (4.14). It follows from Theorem (3.2) that μ_α^T is a fuzzy QS-ideal of X for all α ∈ [0,T]. Hence every fuzzy translation subset μ_α^T is a fuzzy extension QS-ideal of the fuzzy multiplication QS-ideal μ_β^M . ⌂

The following example illustrates Proposition (4.15).

Example 4.16. Consider a QS-algebraX = {0, 1, 2, 3} with the Example (4.10). Define a fuzzy subset μ of X by:

X	0	1	2	3
μ	0.8	0.5	0.3	0.3

Then μ is a fuzzy QS-ideal of X. If we take λ = 0.1, then the fuzzy subset multiplication $\mu_{0.1}^{M}$ of μ is given by:

X	0	1	2	3
$\mu_{0.1}^{M}$	0.08	0.05	0.03	0.03
$\mu_{0.3}^{M}$	0.24	0.15	0.09	0.09

Clearly $\mu_{0.3}^{M}$ is a fuzzy multiplication QS-ideal of X. Also, for any $\alpha \in [0, 0.2]$, the fuzzy translation μ_{α}^{T} of μ is given by:

X	0	1	2	3
μ_{α}^{T}	0.8+α	0.5+α	0.3+α	0.3+α

Then μ_{α}^{T} is a fuzzy extension QS-ideal of $\mu_{0.3}^{M}$ and μ_{α}^{T} is always a fuzzy translation QS-ideal of X for all $\alpha \in [0, 0.2]$. Hence μ_{α}^{T} is a fuzzy extension QS-ideal multiplication of $\mu_{0.3}^{M}$ for all $\alpha \in [0, 0.2]$.

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