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ON THE INVERSE OF PATTERN MATRICES WITH APPLICATION TO STATISICAL MODELS

Hiba Hani Abdullah
Department of Mathematics , College of Education for Women , Tikrit University , Tikrit , Iraq
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Corresponding Author:

Name: Hiba Hani Abdullah

E-mail: hiba83sef@yahoo.com

Tel:

Abstract

In this study the inverse of two patterned matrices has been investigated. First, for a Toeplitz-type matrix, it is proved that the exact number of independent cofactors is (n +2)/4 when n is even number and $(\boldsymbol{n}+\mathbf{1})^{2}/\mathbf{4}$ when n is an odd. Second, when the matrix is reduced to a Jacobi-type matrix B_n , two equivalent formulae for its determinant are obtained, one of which in terms of the eigen values. Moreover, it is proved that the independent cofactors $\boldsymbol{B_{ij}}$ of $\boldsymbol{B_n}$ are explicitly expressed as a product of the determinants of $\boldsymbol{B_{i-1}}$ and $\boldsymbol{B_{n-j}}$. So, the problem of finding the exact inverse of $\boldsymbol{B_n}$ is reduced to that one of finding the determinants of $\boldsymbol{B_i}$, i = 1, 2, …, n.

1- Introduction

One of the important problems involved in the analysis of such models is to find the exact inverse of these covariance matrices in explicit form which leads to the computation of determinants and other related characteristics such as their eigen values and spectral representation. Such computations are tedious especially when the order n of the matrix is large [2].

There is a large literature on inversion of covariance matrices (e.g. [3,2,4]). The problem has been approached either numerically to find fast algorithms or analytically to find explicit forms for the entries of the inverse. Naturally, analytical solution leads to numerical one.

Now, let A_n be an (n x n) symmetric, positive definite matrix. A_n is said to be a patterned matrix if its entries exhibit a structured form, for example the Toeplitz matrix, the Jacobi matrix, …. These patterned matrices are frequently encountered as covariance matrices of structured dependent errors or observations in statistical models or autoregressive and moving average time series models as well as in many other stochastic models [1].

The purpose of this work is divided to two parts. We first prove for a Toeplitz-type matrix that the number of independent cofactors is exactly $n(n+2)/4$ for n even and $(n+1)^2/4$ for n odd. This reduces the number of distinct cofactors to a little bit greater than the quarter of the total number n^2 of cofactors, which means that, practically, only these distinct entries of the adjoint matrix need to be calculated[5]. Further, these distinct elements have a certain arrangement along each diagonal on the upper half of the matrix. Second when the matrix is reduced to a Jacobi-type matrix B_n, two equivalent formulae for the determinant of B_n are given, one of them in terms of the eigen values of the matrix. Moreover, it is proved that the independent cofactors B_{ij} of B_n are exactly given by:

$$B_{ij} = (-1)^{j=i} b^{j-i} det(B_{i-1}) det(B_{n-j}),$$
$$i \leq j \leq n-i+1, i = 1,2, \dots, \tfrac{n+1}{2}$$

When n is odd or n/2 when n is even, and b is some entry of B_n.

So that the problem of finding the inverse of a Jacobi-matrix is reduced to that of finding the determinant of B_i, i = 1,2, …, n. [6]

2- The Adjoint of A Toeplitz-type matrix:

Suppose $A_n = [a_{ij}]$ is a Toeplitz matrix of order n having the form:

$a_{ij} = a_{i-j}, 1 \leq i,j \leq n.$ [4]

Let M_{ij} denote the submatrix of order n-1 obtained by deleting the ith row and the jth column of A_n, and let $A_{ij} = (-1)^{i+j} det(M_{ij})$ be the cofactor of a_{ij}. It is well-known that the inverse A_n^{-1} of A_n is given by:

$A_n^{-1}det(A_n) = [A_{ij}]^t$, where t denote the transpose of the matrix $= [A_{ij}]$, by symmetry of A_n.
This means that A_{ij} for all i > j are redundant. The following lemma proves that about the half of the remaining cofactors are redundant too.
In all what follows J_n denotes a reversing matrix of order n(secondary diagonal) , namely:

$$J_n = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 0 & \dots & 0 & 0 \end{bmatrix}$$

Lemma 2.1: [3]
Consider the matrix $A_n = [a_{(i-j)}]$. Then for all $1 \le i \le j \le n$,
$A_{ij} = A_{n-j+1,n-i+1}$.
Proof:
Let $\alpha_1, \alpha_2, \dots, \alpha_n$ denote the row vectors of A_n, and $\beta_1, \beta_2, \dots, \beta_n$ the column vectors. Then, by symmetry of A_n, $\alpha_i = \beta_{-i}^t$, and by the structured pattern of A_n, $\alpha_i = \alpha_{n-i+1}J_n$. It, thus follows that:
$\beta_i = (J_n\beta_{n-i+1})^t \dots$ (2.1)
and
$\beta_j = (\alpha_{n-j+1}J_n)^t \dots$ (2.2)
(2.1) and (2.2) imply immediately that,
$M_{ij} = J_{n-1}M_{n-j+1,n-i+1}^t J_{n-1}$,
from which,
$A_{ij} = A_{n-j+1,n-i+1}, \ for\ all\ 1 \le i \le j \le n.$
Theorem 2.1:
Consider the matrix $A_n = [a_{|i-j|}]$. If K denotes the number of independent cofactors of A_n, then :

$$K = \begin{cases} (n+1)^2/4, & n\ odd \\ n(n+2)/4, & n\ even \end{cases}$$

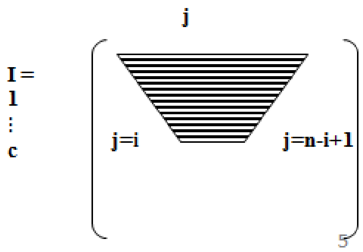
These independent cofactors are the elements A_{ij} with $i \le j \le n-i+1, i = 1,2,\dots,\frac{n+1}{2}$ when n is odd or $\frac{n}{2}$ when n is even. [6]
Proof:
Suppose n is odd. Put n = 2r + 1 , r = 1, 2, …. . It results from(lemma 2.1) that the independent cofactors are the (i, j) elements A_{ij} with $i \le j \le n - i + 1, i = 1,2,\dots,r+1$.
Thus

$$K = \sum_{i=1}^{r+1}\sum_{j=i}^{n-i+1} 1_{(i,j)} = \sum_{i=1}^{r+1}\sum_{s=0}^{n-2i+1} 1_{(i,s)}$$
$$= \sum_{i=1}^{r+1}(n-2i+2)$$
$$= (r+1)^2$$
$$= \frac{(n+1)^2}{4}$$

Now, let n be even, n = 2r with r a positive integer. Then the independent cofactors are those A_{ij} with i = 1,2, …, r, i≤ j ≤ n-i+1, so that

$$K = \sum_{i=1}^{r}\sum_{j=i}^{n-i+1} 1_{(i,j)} = \sum_{i=1}^{r}\sum_{s=0}^{n-2i+1} 1_{(i,s)}$$
$$r\,(r+1) = \frac{n(n+2)}{4}$$

Remark 2.1
The independent cofactors are exactly the entries of the adjoint matrix indicated by the hachured area



$c = \frac{n+1}{2}$ or $\frac{n}{2}$ according to n odd or even, respectively.[7]
3- The Jacobi-type matrix:[8]
In all this section we suppose that the matrix $A_n = [a_{|i-j|}]$ is now reduced to a Jacobi-type matrix where $a_{|i-j|}$=0 whenever $|i-j| > 1$. Precisely, we suppose a matrix $B_n = [b_{ij}]$ such that :

$$b_{ij} = \begin{cases} a, & i = j \\ b, & |i-j| = 1 \\ 0, & otherwise \end{cases} \dots (3.1)$$

3-1 The determinant of B_n:
Let $D_n = det(B_n)$. Then by expansion about the first column, it can be shown that D_n satisfies the difference equation of second order:
$D_n = aD_{n-1} - b^2D_{n-2}$, n = 2,3, …
with the two boundary conditions $D_0 = 1, D_1 = 0.$
The roots of the auxiliary equation $y^2 - ay + b^2 = 0$ are,
$y_{1,2} = \frac{a \pm \sqrt{a^2-4b^2}}{2}$ which with the boundary conditions give the solution.

$$D_n = \frac{1}{2^{n+1}\sqrt{a^2-4b^2}}\left[\left(a+\sqrt{a^2-4b^2}\right)^{n+1} - \left(a-\sqrt{a^2-4b^2}\right)^{n+1}\right], \text{ n≥0} \dots (3.2)$$

Expanding the binominals in (3.2), D_n reduced to :

$$D_n = 2^{-n}\sum_{s=0}^{n}\binom{n+1}{s+1}x^s a^{n-s}, with\ x = \sqrt{a^2-4b^2}$$
$$= \left(\frac{a}{2}\right)^n \sum_{s=0}^{\left[\frac{n}{2}\right]}\binom{n+1}{2s+1}a^{2s}\left(1-4\frac{b^2}{a^2}\right)^2,$$

$with \left[\frac{n}{2}\right] denotes\ the\ greatest\ integer\ \le \frac{n}{2}$
Expansion of the above binomial again yields.

$$D_n = \left(\frac{a}{2}\right)^n \sum_{s=0}^{\left[\frac{n}{2}\right]}\sum_{r=0}^{s}(-1)^r\binom{n+1}{2s+1}\binom{s}{r}\left(4\frac{b^2}{a^2}\right)^r$$
$$D_n = \left(\frac{a}{2}\right)^n \sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r\left(4\frac{b^2}{a^2}\right)^r \sum_{s=r}^{\left[\frac{n}{2}\right]}\binom{n+1}{2s+1}\binom{s}{r}$$

The last summation can be proved to be exactly $\binom{n-r}{r}2^{n-2r}$, which leads to the expression:

$$D_n = \sum_{r=0}^{\left[\frac{n}{2}\right]}(-1)^r\binom{n-r}{r}b^{2r}a^{n-2r},$$

n ≥ 0 , $\left[\frac{n}{2}\right]$ the greatest integer $\le \frac{n}{2}$ … (3.3)
3-2 The Eigen Values of B_n:
If λ is an Eigen value of B_n if λ satisfies the linear equation $B_nZ = \lambda Z$, with Z a nonzero column vector of dimension n, which is the Eigen vector corresponding to λ. To find the Eigen values of B_n we are motivated by the approach relating to this problem to the characteristic-value problem of a finite

homogeneous boundary difference system of equations (See [9]).
In theorem 3.1 below we prove that the n eigen values of B_n are exactly the ne eigen values of a system of ne difference equtions with two boundary conditions, and hence can be determined from the general solution of the system.

Theorem 3.1:
Let B_n be the Jacobi-type matrix given in (3.1). The n eigen values λ_m, m = 1,2, …, n of B_n are exactly the n eigen values of the different equation:

$$bZ_{m+1} + aZ_m + bZ_{m-1} = \lambda Z_m, m = 1,2, \dots, n$$

with,

$$Z_0 = 0, \quad Z_{n+1} = 0$$

Hence,

$$\lambda_m = a - 2b \cos\frac{m\pi}{n+1}, m = 1,2, \dots, n.$$

Proof:
Let Z = $(z_1, z_2, \dots, z_n)^r$. Write the equation $B_n Z = \lambda Z$ in the expansion form

$az_1 + bz_2 = \lambda z_1$
$bz_1 + az_2 + bz_3 = \lambda z_2$
……………………………………
$+ bz_{m-1} + az_m + bz_{m+1} = \lambda z_m$
……………………………………
$bz_{n-1} + az_n = \lambda z_n$

which is equivalent to the homogeneous system of different equations

$bz_{m+1} + az_m + bz_{m-1} = \lambda z_m$, m = 1,2, … , n.

with the two homogeneous boundary conditions $z_0 = 0$, $z_{m+1} = 0$.
For such a system, no nonzero solution exists unless λ takes on one of a set of eigen values $\lambda_1, \dots, \lambda_n$ which are exactly the required eigen values of B_n. In fact no nonzero solution to the above system exists unless $\left|\frac{a-\lambda}{2b}\right| < 1$ or equivalently unless λ = a - 2b cos θ. In this case the general solution to the system is $z_m = c_1 \cos m\theta$. The condition $z_0 = 0$ implies $c_1 = 0$, and the second condition $z_{m+1} = 0$ leads to $c_2 \sin(n+1)\theta = 0$, which unless θ takes a value for which $\sin(n+1)\theta = 0$, the only solution is $c_2 = 0$, in which case $z_m = 0$, m = 1,2, … , n.
However, if (n+1)θ = mπ , m = 1,2, … , c_2 is arbitrary and $z_m \neq 0$.
Thus $z_m \neq 0$ whenever $\theta = \frac{m\pi}{n+1}$, m = 1,2, …, n for in fact, all the other values of m lead either to the trival solution. when m=0, n+1, 2(n+1), …, or to solutions identical to those obtained: when m takes on one of the integers in the intervals (n+1, 2(n+1), 3(n+1)), … etc.
From all what preceeds , it follows that the required eigen values are :

$$\lambda_m = a - 2b \cos\frac{m\pi}{n+1}, \quad m = 1,2, \dots, n.$$

Corollary 3.1:
It can be easily shown that :

$$D_n = \prod_{m=1}^{n}\left[a - 2b \cos\frac{m\pi}{n+1}\right] \dots (3.4)$$

which is another expression of det (B_n).

3-3 The Inverse of B_n:
As proved in theorem 2.1, to find adj B_n it suffices to calculate the cofactors B_{ij}, j = i, i+1, …., n-i+1, i = 1, 2, …, $\frac{n+1}{2}$ for n odd or $\frac{n}{2}$ for n even. Observe that when deleting the ith row of B_n , for any fixed i , the obtained submatrix gives the following cofactors B_{ij} , j = i, i+1, …, n-i+1, where,

$B_{ij} = (-1)^{2i}$ det (B_{i-1}) det (B_{n-i}) , j = i , … (3.5)
$B_{ij} = (-1)^{i+j}$ det (C_{ij}) det (B_{n-j}), j = i+1, … , n-i+1

with C_{ij} a square matrix of order (j-1) satisfying the relation:

det (C_{ij}) = b det (C_{ij-1}), j = i+1, …, n-i+1. … (3.6)
det (C_{ij}) = det (B_{i-1}), j=i

(3.6) is clearly a first order different system of equations with boundary condition. It can be easily shown that:

$$det(C_{ij}) = b^{i-j} det(B_{j-1}), j = i, i+1, \dots, n-i+1 \quad \dots (3.7)$$

Varying i, (3.5) together with (3.7) imply, thus, that:

$$B_{ij} = (-1)^{i+j} b^{j-i} det(B_{i-1}) det(B_{n-j}), i \le j \le n - i - 1,$$

$i = 1, \dots, \frac{n+1}{2}$ or $\frac{n}{2}$ as n odd or even.
Clearly, this formula reflects the symmetry of cofactors proved before for the more general case by lemma 2.1. We can thus state the theorem:

Theorem 3.2:
For the matrix B_n given in (3.1), the independent cofactors B_{ij} are exactly:

$B_{ij} = (-1)^{i+j} b^{j-i} det(B_{i-1}) det(B_{n-j})$, j = i, j+1, …, n-i+1,
i = 1,2, …, $\frac{n+1}{2}$ or $\frac{n}{2}$ as n odd or even.

Hence, if $B_n^{-1} = [B^{ij}]$ denotes the inverse of B_n, then

$B^{ij} = (-1)^{i+j} b^{j-i} det(B_{i-1}) det(B_{n-j}) / det(B_n)$,
j = i, …, n-i+1, i = 1,2, …, $\frac{n+1}{2}$ or $\frac{n}{2}$ as n odd or even.

Remark 3.1:
1- It follows from theorem 3.2 that, to find B^{-1} it suffices to calculate the determinants of B_1, B_2, … B_n which can be calculated using either formula (3.3) or (3.4).
2- A statement similar to that of B^{ij} in the theorem but for the inverse of the covariance matrix of a first order moving average process has been observed before by Arato [7] and then used shaman [10].

4- Applications

Below are two examples of statistical models for which the involved covariance matrix is of the Toeplitz of Jacobi types studied in this work.

Example (1):
Suppose y_1, y_2, … , y_n is an observed time series generated by a stationary autoregressive process of order p given by:

$y_i = \theta_1 y_{i-1} + \theta_2 y_{i-2} + \cdots \theta_p y_{i-p} + e_i$, with (e_i) a white noise process, that is $E(e_i) = 0, \forall\, i$,

$$E(e_i e_j) = \begin{cases} 0\,, i \neq j \\ \sigma^2, i = j \end{cases}$$

This means that y_i have bounded means and variances, precisely for all i,

$E(y_i) = 0, E(y_i y_{i+k}) = \begin{cases} \sigma^2, k = 0 \\ a_k, k \neq 0 \end{cases}$

Put,

$Y_p = (y_{p+1} \dots \dots y_n)^t$

$$X_p = \begin{bmatrix} y_p & y_{p+1} & \cdots & y_1 \\ y_{p-1} & y_p & \cdots & y_2 \\ \vdots & \vdots & & \vdots \\ y_{n-1} & y_{n-2} & \cdots & y_{n-p} \end{bmatrix}$$

$\Theta_p = (\theta_1 \dots \dots \theta_p)^t$

Then, given y_1, ……., y_p, the least squares estimate of Θ_p is given by,

$\widehat{\Theta_p} = (X_p^t X_p^{-1})^{-1} X_p^t Y_p$, which, under the Gaussian assumption of the process, is consistent, asymptotically normally distributed, namely:

$n(\widehat{\Theta}_p - \Theta_p) \xrightarrow{D} N_p(0, \sigma^2 A_p^{-1})$, where,

A_p = [a_{i-j}] , which can be consistently estimated by $X_p^t X_p$.

A_p is obviously a matrix of the Eoeplitz-type studied in section 2.

It is well-known that [1] the asymptotic theory is not altered if $X_p^t X_p$ is replaced by the matrix $A_p^* = [a^*_{|i-j|}]$ with $a_k^* = \frac{1}{n}\sum_{r=1}^{n-k} y_r y_{r+k}$, k = 0,1, …, p-1, in which A_p^* is again of the same pattern as A_p. [11] ,[12].

Lemma 2.1 and theorem 2.1 are useful in calculating the inverse of A_p^* which is indispensable for making any inference concerning Θ_p.

Example (2):

We consider the stationary normal first order moving average stochastic process which is very common in time series analysis.

Here, if x_i, i =1, …, n, is an observed finite series, then

$x_i = e_i + \beta e_{i-1}$, with e_i a gaussian white noise, and $\beta, |\beta|\tau < 1$, is the parameter to be estimated.

Put X = (x_i, …., x_n)^t. Then, var X = B_n a matrix of the Jacobi type as given in section 3 , with a = $\sigma^2(1+\beta^2)$, b = $\sigma^2\beta$ and $\sigma^2 = var(e_i)\ \forall i$.

The log-likelihood function is thus:

$L(\beta, X) = \frac{1}{2} log 2\pi\sigma - \frac{1}{2} n X^t \beta_n^{-1} X_n$.

Clearly, the exact estimation of β is not an easy problem as long as the exact inverse of B_n is not available. Theorem 3.2 together with corollary 3.1 can be applied to obtain B_n^{-1}.

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معكوس المصفوفات النمطية مع تطبيقات على النماذج الإحصائية

هبه هاني عبدالله

قسم الرياضيات ، كلية التربية للبنات ، جامعة تكريت ، تكريت ، العراق

الملخص

قد بحثنا في البداية انعكاس نوعين من المصفوفات النمطية : النوع الاول هو المصفوفات من نوع Toeplitz وهذا النوع يثبت فيه العدد الدقيق او المضبوط من المعاملات المستقلة ويساوي (n +2)/4 عندما تكون nعدد وزجي و $(n+1)^2/4$ عندما تكون n عدد فردي. اما النوع الثاني هو النوع الذي يظهر عند تخفيض المصفوفة الى مصفوفة من نوع جاكوبي، وهناك صيغتين متكافئة في الحصول على المحددات. وقد برهنا في دراستنا على ان العامل المستقل B_{ij} من B_n عبر عنه ظاهريا بدلالة B_{i-1} و B_{n-j} لذلك قد انخفضت مشكلة ايجاد المعكوس بصورة دقيقة الى واحد من النتائج لمجموع العناصر الناتجة لـ B_i, i=1,…,n.