

THE αg -KERNEL AND WEAKLY ULTRA αg -SEPARATION AXIOMS VIA αg -OPEN SETS

النواة لمجموعات αg و بديهية الفصل فوق الضعيف بين تلك المجموعات باستخدام مجموعات αg المفتوحة

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Abstract:

The aim of this paper to introduce the concept of αg -kernel also we introduced the concept of the weakly ultra αg -separation of two sets in a topological space using αg -open sets. The αg -closure is defined in terms of this weakly ultra αg -separation. We also investigate some properties of weak separation axioms like αg - R_i -spaces, $i = 0, 1$ and αg - T_i -spaces, $i = 0, 1, 2$.

Keywords: αg -open sets, αg -kernel, αg -closure, weakly ultra αg -separation, αg - R_i -spaces, $i = 0, 1$ and αg - T_i -spaces, $i = 0, 1, 2$.

الخلاصة :

يهدف البحث لدراسة مفهوم النواة لمجموعات αg و بديهية الفصل فوق الضعيف بين تلك المجموعات باستخدام مجموعات αg المفتوحة في الفضاء التوبولوجي. ومن ثم استخدمنا هذين المفهومين لتعريف الانغلاق لهذه المجموعات ودراسة بعض خواص بديهيات الفصل αg بين مجموعات αg مثل αg - R_i , $i = 0, 1$ و αg - T_i , $i = 0, 1, 2$ والعلاقة بينهم.

1. Introduction:

In 1970, Levine [2],[3], introduced the concept of generalized closed sets as a generalization of closed sets in topological spaces. This concept was found to be useful and many results in general topology were improved. In 2014, V. Senthilkumar, R. Krishnakumar and Y. Palaniappan [5], introduced αg -closed set. In this paper, we introduced some properties of αg -separation axioms by using some definition of new concept via αg -open sets. Throughout this paper, the closure and the interior of A are denoted by $cl(A)$ and $int(A)$, respectively.

2. Preliminaries:

Before entering to our work, we recall the following definitions, which are useful in the sequel.

Definition 2.1:[4] A subset A of a topological space X is called a α -open set if $A \subseteq int(cl(int(A)))$ and a α -closed set if $cl(int(cl(A))) \subseteq A$.

Definition 2.2:[2] A subset A of a topological space X is called a generalized closed set (briefly g -closed set) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

Definition 2.3:[1] A subset A of a topological space X is called α generalized closed set (briefly αg -closed set) if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in X .

Definition 2.4:[5] A subset A of a topological space X is called $\hat{\alpha}$ generalized closed set (briefly $\hat{\alpha}g$ -closed set) if $int(cl(int(A))) \subseteq U$ whenever $A \subseteq U$ and U is open in X . The complement of $\hat{\alpha}g$ -closed set in X is $\hat{\alpha}g$ -open in X .

Definition 2.5:[6] The intersection of all $\hat{\alpha}g$ -closed sets in X containing A is called $\hat{\alpha}$ generalized closure of A and is denoted by $\hat{\alpha}g-cl(A)$.

3. $\hat{\alpha}g$ -Kernel and $\hat{\alpha}g$ - R_i -Spaces, $i = 0, 1$:

Definition 3.1: The intersection of all $\hat{\alpha}g$ -open subset of X containing A is called the $\hat{\alpha}g$ -kernel of A (briefly $\hat{\alpha}g-ker(A)$), this means $\hat{\alpha}g-ker(A) = \bigcap \{G \in \hat{\alpha}g-O(X) : A \subseteq G\}$, where $O(X)$ is an open sets in X

Example 3.2: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$
 $\hat{\alpha}g$ -open sets = $\{\emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, d\}, \{a, c\}, \{a, b, d\}, \{a, c, d\}, X\}$
 $\hat{\alpha}g-ker\{a\} = \{a\} \cap \{a, b\} \cap \{a, d\} \cap \{a, c\} \cap \{a, b, d\} \cap \{a, c, d\} = \{a\}$
 $\hat{\alpha}g-ker\{b\} = \{a, b\} \cap \{a, b, d\} = \{a, b\}$.

Definition 3.3: In a space (X, τ) , a set A is said to be weakly ultra $\hat{\alpha}g$ -separated from B if there exists an $\hat{\alpha}g$ -open set G such that $G \cap B = \emptyset$ or $A \cap \hat{\alpha}g-cl\{B\} = \emptyset$.

By the definition 3.2, we have the following for $x, y \in X$ of a topological space,

- (i) $\hat{\alpha}g-cl\{x\} = \{y : \{y\} \text{ is not weakly ultra } \hat{\alpha}g\text{-separated from } \{x\}\}$
- (ii) $\hat{\alpha}g-ker\{x\} = \{y : \{x\} \text{ is not weakly ultra } \hat{\alpha}g\text{-separated from } \{y\}\}$.

Example 3.4: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c, d\}, X\}$
 $\hat{\alpha}g$ -open sets = $\{\emptyset, \{a\}, \{c\}, \{d\}, \{a, b\}, \{c, d\}, \{a, d\}, \{a, c\}, \{a, b, d\}, \{a, c, d\}, X\}$
 $\{a\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{b\}$, but $\{b\}$ is not weakly ultra $\hat{\alpha}g$ -separated from $\{a\}$.

Theorem 3.5: Let (X, τ) be a topological space then $x \in \hat{\alpha}g-cl\{y\}$ iff $y \in \hat{\alpha}g-ker\{x\}$ for each $x \neq y \in X$.

Proof: Let (X, τ) be a topological space. And let $x \in \hat{\alpha}g-cl\{y\}$, then for each U is an $\hat{\alpha}g$ -open set such that $x \in U$ implies $y \in U$ this means $y \in \hat{\alpha}g-ker\{x\}$. Let $y \in \hat{\alpha}g-ker\{x\}$, then for each U is an $\hat{\alpha}g$ -open set such that $x \in U$ implies $y \in U$ this means $U \cap \{y\} \neq \emptyset$. Hence $x \in \hat{\alpha}g-cl\{y\}$.

Definition 3.6: A topological space (X, τ) is called an $\hat{\alpha}g$ - R_0 -space if for each $x \in X$ and U $\hat{\alpha}g$ -open set containing x , then $\hat{\alpha}g-cl\{x\} \subseteq U$.

Example 3.7: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$
 $\hat{\alpha}g$ -open sets = $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Definition 3.8: A topological space (X, τ) is called an $\hat{\alpha}g$ - R_1 -space if for each two distinct point x, y of X with $\hat{\alpha}g-cl\{x\} \neq \hat{\alpha}g-cl\{y\}$, there exist disjoint $\hat{\alpha}g$ -open sets U, V such that $\hat{\alpha}g-cl\{x\} \subseteq U$ and $\hat{\alpha}g-cl\{y\} \subseteq V$.

Example 3.9: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$
 $\hat{\alpha}g$ -open sets = $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$.

Theorem 3.10: Let (X, τ) be a topological space. Then (X, τ) is $\hat{\alpha}g$ - R_0 -space if and only if, $\hat{\alpha}g-cl\{x\} = \hat{\alpha}g-ker\{x\}$, for each $x \in X$.

Proof: Let (X, τ) be an $\hat{\alpha}g$ - R_0 -space, and let y be another point such that $y \notin \hat{\alpha}g-cl\{x\}$, implies $y \in (\hat{\alpha}g-cl\{x\})^c$ is an $\hat{\alpha}g$ -open set. By assumption (X, τ) be an $\hat{\alpha}g$ - R_0 -space, then $\hat{\alpha}g-cl\{y\} \subseteq (\hat{\alpha}g-cl\{x\})^c$ [By Definition 3.6], there for $x \notin \hat{\alpha}g-cl\{y\}$ and $y \notin \hat{\alpha}g-ker\{x\}$ [By Theorem 3.5]. So we get $\hat{\alpha}g-ker\{x\} \subseteq \hat{\alpha}g-cl\{x\}$. Also since (X, τ) be an $\hat{\alpha}g$ - R_0 -space, then $\hat{\alpha}g-cl\{x\} \subseteq U$ for each U $\hat{\alpha}g$ -open set

containing x [By Definition 3.6], implies $\hat{\alpha}g\text{-cl}\{x\} \subseteq \bigcap \{U : x \in U\}$. So we get $\hat{\alpha}g\text{-cl}\{x\} \subseteq \hat{\alpha}g\text{-ker}\{x\}$ [By Definition 3.1]. Thus $\hat{\alpha}g\text{-cl}\{x\} = \hat{\alpha}g\text{-ker}\{x\}$.

Conversely, let $\hat{\alpha}g\text{-cl}\{x\} = \hat{\alpha}g\text{-ker}\{x\}$, for each $\hat{\alpha}g$ -open set U and $x \in U$, then $\hat{\alpha}g\text{-ker}\{x\} = \hat{\alpha}g\text{-cl}\{x\} \subseteq U$ [By Definition 3.1]. Hence by Definition 3.6, (X, τ) is an $\hat{\alpha}g\text{-R}_0$ -space.

Theorem 3.11: A topological space (X, τ) is an $\hat{\alpha}g\text{-R}_0$ -space if and only if for each F $\hat{\alpha}g$ -closed set and $x \in F$ then $\hat{\alpha}g\text{-ker}\{x\} \subseteq F$.

Proof: Let for each F $\hat{\alpha}g$ -closed set and $x \in F$ then $\hat{\alpha}g\text{-ker}\{x\} \subseteq F$ and let U be an $\hat{\alpha}g$ -open set, $x \in U$ then for each $y \notin U$ implies $y \in U^c$ is an $\hat{\alpha}g$ -closed set implies $\hat{\alpha}g\text{-ker}\{y\} \subseteq U^c$ [By assumption]. Therefore $x \notin \hat{\alpha}g\text{-ker}\{y\}$ implies $y \notin \hat{\alpha}g\text{-cl}\{x\}$ [By corollary 3.5]. So $\hat{\alpha}g\text{-cl}\{x\} \subseteq U$. Thus (X, τ) is an $\hat{\alpha}g\text{-R}_0$ -space.

Conversely, let a topological space (X, τ) be a $\hat{\alpha}g\text{-R}_0$ -space and F be $\hat{\alpha}g$ -closed set and $x \in F$. Then for each $y \notin F$ implies $y \in F^c$ is $\hat{\alpha}g$ -open set, then $\hat{\alpha}g\text{-cl}\{y\} \subseteq F^c$ [since (X, τ) is $\hat{\alpha}g\text{-R}_0$ -space], so $\hat{\alpha}g\text{-ker}\{x\} = \hat{\alpha}g\text{-cl}\{x\}$. Thus $\hat{\alpha}g\text{-ker}\{x\} \subseteq F$.

Corollary 3.12: A topological space (X, τ) is an $\hat{\alpha}g\text{-R}_0$ -space if and only if for each U $\hat{\alpha}g$ -open set and $x \in U$ then $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{x\}) \subseteq U$.

Proof: Clearly.

Theorem 3.13: Every $\hat{\alpha}g\text{-R}_1$ -space is an $\hat{\alpha}g\text{-R}_0$ -space.

Proof: Let (X, τ) be an $\hat{\alpha}g\text{-R}_1$ -space and let U be an $\hat{\alpha}g$ -open set, $x \in U$, then for each $y \notin U$ implies $y \in U^c$ is an $\hat{\alpha}g$ -closed set and $\hat{\alpha}g\text{-cl}\{y\} \subseteq U^c$ implies $\hat{\alpha}g\text{-cl}\{x\} \neq \hat{\alpha}g\text{-cl}\{y\}$. Hence by definition 3.8, $\hat{\alpha}g\text{-cl}\{x\} \subseteq U$. Thus (X, τ) is an $\hat{\alpha}g\text{-R}_0$ -space.

Theorem 3.14: A topological space (X, τ) is an $\hat{\alpha}g\text{-R}_1$ -space if and only if for each $x \neq y \in X$ with $\hat{\alpha}g\text{-ker}\{x\} \neq \hat{\alpha}g\text{-ker}\{y\}$ then there exist $\hat{\alpha}g$ -closed sets F_1, F_2 such that $\hat{\alpha}g\text{-ker}\{x\} \subseteq F_1, \hat{\alpha}g\text{-ker}\{x\} \cap F_2 = \emptyset$ and $\hat{\alpha}g\text{-ker}\{y\} \subseteq F_2, \hat{\alpha}g\text{-ker}\{y\} \cap F_1 = \emptyset$ and $F_1 \cup F_2 = X$.

Proof: Let a topological space (X, τ) be an $\hat{\alpha}g\text{-R}_1$ -space. Then for each $x \neq y \in X$ with $\hat{\alpha}g\text{-ker}\{x\} \neq \hat{\alpha}g\text{-ker}\{y\}$. Since every $\hat{\alpha}g\text{-R}_1$ -space is an $\hat{\alpha}g\text{-R}_0$ -space [by theorem 3.13], and by theorem 3.10, $\hat{\alpha}g\text{-cl}\{x\} \neq \hat{\alpha}g\text{-cl}\{y\}$, then there exist $\hat{\alpha}g$ -open sets G_1, G_2 such that $\hat{\alpha}g\text{-cl}\{x\} \subseteq G_1$ and $\hat{\alpha}g\text{-cl}\{y\} \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ [Since (X, τ) is $\hat{\alpha}g\text{-R}_1$ -space], then G_1^c and G_2^c are $\hat{\alpha}g$ -closed sets such that $G_1^c \cup G_2^c = X$. Put $F_1 = G_1^c$ and $F_2 = G_2^c$. Thus, $x \in G_1 \subseteq F_2$ and $y \in G_2 \subseteq F_1$ so that $\hat{\alpha}g\text{-ker}\{x\} \subseteq G_1 \subseteq F_2$ and $\hat{\alpha}g\text{-ker}\{y\} \subseteq G_2 \subseteq F_1$.

Conversely, let for each $x \neq y \in X$ with $\hat{\alpha}g\text{-ker}\{x\} \neq \hat{\alpha}g\text{-ker}\{y\}$, there exist $\hat{\alpha}g$ -closed sets F_1, F_2 such that $\hat{\alpha}g\text{-ker}\{x\} \subseteq F_1, \hat{\alpha}g\text{-ker}\{x\} \cap F_2 = \emptyset$ and $\hat{\alpha}g\text{-ker}\{y\} \subseteq F_2, \hat{\alpha}g\text{-ker}\{y\} \cap F_1 = \emptyset$ and $F_1 \cup F_2 = X$, then F_1^c and F_2^c are $\hat{\alpha}g$ -open sets such that $F_1^c \cap F_2^c = \emptyset$. Put $F_1^c = G_2$ and $F_2^c = G_1$. Thus $\hat{\alpha}g\text{-ker}\{x\} \subseteq G_1$ and $\hat{\alpha}g\text{-ker}\{y\} \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$, so that $x \in G_1$ and $y \in G_2$ implies $x \notin \hat{\alpha}g\text{-cl}\{y\}$ and $y \notin \hat{\alpha}g\text{-cl}\{x\}$, then $\hat{\alpha}g\text{-cl}\{x\} \subseteq G_1$ and $\hat{\alpha}g\text{-cl}\{y\} \subseteq G_2$. Thus, (X, τ) is an $\hat{\alpha}g\text{-R}_1$ -space.

Corollary 3.15: A topological space (X, τ) is an $\hat{\alpha}g\text{-R}_1$ -space if and only if for each $x \neq y \in X$ with $\hat{\alpha}g\text{-cl}\{x\} \neq \hat{\alpha}g\text{-cl}\{y\}$ there exist disjoint $\hat{\alpha}g$ -open sets U, V such that $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{x\}) \subseteq U$ and $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{y\}) \subseteq V$.

Proof: Let (X, τ) be an $\hat{\alpha}g\text{-R}_1$ -space and let $x \neq y \in X$ with $\hat{\alpha}g\text{-cl}\{x\} \neq \hat{\alpha}g\text{-cl}\{y\}$, then there exist disjoint $\hat{\alpha}g$ -open sets U, V such that $\hat{\alpha}g\text{-cl}\{x\} \subseteq U$ and $\hat{\alpha}g\text{-cl}\{y\} \subseteq V$. Also (X, τ) is $\hat{\alpha}g\text{-R}_0$ -space [by Theorem 3.13] implies for each $x \in X$, then $\hat{\alpha}g\text{-cl}\{x\} = \hat{\alpha}g\text{-ker}\{x\}$ [By Theorem 3.10], but $\hat{\alpha}g\text{-cl}\{x\} = \hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-cl}\{x\}) = \hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{x\})$. Thus $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{x\}) \subseteq U$ and $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{y\}) \subseteq V$.

Conversely, let for each $x \neq y \in X$ with $\hat{\alpha}g\text{-cl}\{x\} \neq \hat{\alpha}g\text{-cl}\{y\}$ there exist disjoint $\hat{\alpha}g$ -open sets U, V such that $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{x\}) \subseteq U$ and $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{y\}) \subseteq V$. Since $\{x\} \subseteq \hat{\alpha}g\text{-ker}\{x\}$ then $\hat{\alpha}g\text{-cl}\{x\} \subseteq \hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{x\})$ for each $x \in X$. So we get $\hat{\alpha}g\text{-cl}\{x\} \subseteq U$ and $\hat{\alpha}g\text{-cl}\{y\} \subseteq V$. Thus, (X, τ) be an $\hat{\alpha}g\text{-R}_1$ -space.

4. $\hat{\alpha}g$ - T_i -Spaces, $i = 0, 1, 2$:

Definition 4.1: Let (X, τ) be a topological space. Then X is called $\hat{\alpha}g$ - T_0 -space iff for each pair of distinct points in X , there exists a $\hat{\alpha}g$ -open set in X containing one and not the other.

Example 4.2: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$
 $\hat{\alpha}g$ -open sets = $\{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$.

Definition 4.3: Let (X, τ) be a topological space. Then X is called $\hat{\alpha}g$ - T_1 -space iff for each pair of distinct points x and y of X , there exists $\hat{\alpha}g$ -open sets G, H containing x and y respectively such that $y \notin G$ and $x \notin H$.

Example 4.4: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, X\}$
 $\hat{\alpha}g$ -open sets = $\{\emptyset, \{b\}, \{a, b\}, \{a, c\}, X\}$.

Definition 4.5: Let (X, τ) be a topological space. Then X is called $\hat{\alpha}g$ - T_2 -space iff for each pair of distinct points x and y of X , there exist disjoint $\hat{\alpha}g$ -open sets G, H in X such that $x \in G$ and $y \in H$.

Example 4.6: Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$
 $\hat{\alpha}g$ -open sets = $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$.

Theorem 4.7: A topological space (X, τ) is an $\hat{\alpha}g$ - T_0 -space if and only if either $y \notin \hat{\alpha}g\text{-ker}\{x\}$ or $x \notin \hat{\alpha}g\text{-ker}\{y\}$, for each $x \neq y \in X$.

Proof: Let (X, τ) be an $\hat{\alpha}g$ - T_0 -space then for each $x \neq y \in X$, there exists an $\hat{\alpha}g$ -open set G such that either $x \in G, y \notin G$ or $x \notin G, y \in G$. Thus either $x \in G, y \notin G$ implies $y \notin \hat{\alpha}g\text{-ker}\{x\}$ or $x \notin G, y \in G$ implies $x \notin \hat{\alpha}g\text{-ker}\{y\}$.

Conversely, let either $y \notin \hat{\alpha}g\text{-ker}\{x\}$ or $x \notin \hat{\alpha}g\text{-ker}\{y\}$, for each $x \neq y \in X$. Then there exists an $\hat{\alpha}g$ -open set G such that either $x \in G, y \notin G$ or $x \notin G, y \in G$. Thus (X, τ) is a $\hat{\alpha}g$ - T_0 -space.

Theorem 4.8: A topological space (X, τ) is an $\hat{\alpha}g$ - T_0 -space if and only if either $\hat{\alpha}g\text{-ker}\{x\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$ or $\hat{\alpha}g\text{-ker}\{y\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{x\}$ for each $x \neq y \in X$.

Proof: Let (X, τ) be an $\hat{\alpha}g$ - T_0 -space then for each $x \neq y \in X$, there exists an $\hat{\alpha}g$ -open set G such that $x \in G, y \notin G$ or $x \notin G, y \in G$. Now if $x \in G, y \notin G$ implies $\hat{\alpha}g\text{-ker}\{x\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$. Or if $x \notin G, y \in G$ implies $\hat{\alpha}g\text{-ker}\{y\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{x\}$.

Conversely, let either $\hat{\alpha}g\text{-ker}\{x\}$ be weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$ or $\hat{\alpha}g\text{-ker}\{y\}$ be weakly ultra $\hat{\alpha}g$ -separated from $\{x\}$. Then there exists an $\hat{\alpha}g$ -open set G such that $\hat{\alpha}g\text{-ker}\{x\} \subseteq G$ and $y \notin G$ or $\hat{\alpha}g\text{-ker}\{y\} \subseteq G, x \notin G$ implies $x \in G, y \notin G$ or $x \notin G, y \in G$. Thus, (X, τ) is a $\hat{\alpha}g$ - T_0 -space.

Theorem 4.9: A topological space (X, τ) is an $\hat{\alpha}g$ - T_1 -space if and only if for each $x \neq y \in X$, $\hat{\alpha}g\text{-ker}\{x\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$ and $\hat{\alpha}g\text{-ker}\{y\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{x\}$.

Proof: Let (X, τ) be an $\hat{\alpha}g$ - T_1 -space then for each $x \neq y \in X$, there exists an $\hat{\alpha}g$ -open sets U, V such that $x \in U, y \notin U$ and $x \notin V, y \in V$. Implies $\hat{\alpha}g\text{-ker}\{x\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$ and $\hat{\alpha}g\text{-ker}\{y\}$ is weakly ultra $\hat{\alpha}g$ -separated from $\{x\}$.

Conversely, let $\hat{\alpha}g\text{-ker}\{x\}$ be weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$ and $\hat{\alpha}g\text{-ker}\{y\}$ be weakly ultra $\hat{\alpha}g$ -separated from $\{x\}$. Then there exists an $\hat{\alpha}g$ -open sets U, V such that $\hat{\alpha}g\text{-ker}\{x\} \subseteq U, y \notin U$ and $\hat{\alpha}g\text{-ker}\{y\} \subseteq V, x \notin V$ implies $x \in U, y \notin U$ and $x \notin V, y \in V$. Thus, (X, τ) is a $\hat{\alpha}g$ - T_1 -space.

Theorem 4.10: A topological space (X, τ) is an $\hat{\alpha}g$ - T_1 -space if and only if for each $x \in X$, $\hat{\alpha}g$ - $\ker\{x\} = \{x\}$.

Proof: Let (X, τ) be an $\hat{\alpha}g$ - T_1 -space and let $\hat{\alpha}g$ - $\ker\{x\} \neq \{x\}$. Then $\hat{\alpha}g$ - $\ker\{x\}$ contains another point distinct from x say y . So $y \in \hat{\alpha}g$ - $\ker\{x\}$ implies $\hat{\alpha}g$ - $\ker\{x\}$ is not weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$. Hence by Theorem 4.9, (X, τ) is not an $\hat{\alpha}g$ - T_1 -space this is a contradiction. Thus $\hat{\alpha}g$ - $\ker\{x\} = \{x\}$.

Conversely, let $\hat{\alpha}g$ - $\ker\{x\} = \{x\}$, for each $x \in X$ and let (X, τ) be not an $\hat{\alpha}g$ - T_1 -space. Then by theorem 4.9, $\hat{\alpha}g$ - $\ker\{x\}$ is not weakly ultra $\hat{\alpha}g$ -separated from $\{y\}$, this means that for every $\hat{\alpha}g$ -open set G contains $\hat{\alpha}g$ - $\ker\{x\}$ then $y \in G$ implies $y \in \bigcap \{G \in \hat{\alpha}g$ - $O(X) : x \in G\}$ implies $y \in \hat{\alpha}g$ - $\ker\{x\}$, this is a contradiction. Thus, (X, τ) is a $\hat{\alpha}g$ - T_1 -space.

Theorem 4.11: A topological space (X, τ) is an $\hat{\alpha}g$ - T_1 -space if and only if for each $x \neq y \in X$, $y \notin \hat{\alpha}g$ - $\ker\{x\}$ and $x \notin \hat{\alpha}g$ - $\ker\{y\}$.

Proof: Let (X, τ) be a $\hat{\alpha}g$ - T_1 -space then for each $x \neq y \in X$, there exists an $\hat{\alpha}g$ -open sets U, V such that $x \in U, y \notin U$ or $y \in V, x \notin V$. Implies $y \notin \hat{\alpha}g$ - $\ker\{x\}$ and $x \notin \hat{\alpha}g$ - $\ker\{y\}$.

Conversely, let $y \notin \hat{\alpha}g$ - $\ker\{x\}$ and $x \notin \hat{\alpha}g$ - $\ker\{y\}$, for each $x \neq y \in X$. Then there exists an $\hat{\alpha}g$ -open sets U, V such that $x \in U, y \notin U$ and $y \in V, x \notin V$. Thus (X, τ) is a $\hat{\alpha}g$ - T_1 -space.

Theorem 4.12: A topological space (X, τ) is an $\hat{\alpha}g$ - T_1 -space if and only if for each $x \neq y \in X$ implies $\hat{\alpha}g$ - $\ker\{x\} \cap \hat{\alpha}g$ - $\ker\{y\} = \emptyset$.

Proof: Let a topological space (X, τ) be $\hat{\alpha}g$ - T_1 -space. Then $\hat{\alpha}g$ - $\ker\{x\} = \{x\}$ and $\hat{\alpha}g$ - $\ker\{y\} = \{y\}$ [By Theorem 4.10]. Thus $\hat{\alpha}g$ - $\ker\{x\} \cap \hat{\alpha}g$ - $\ker\{y\} = \emptyset$.

Conversely, let for each $x \neq y \in X$ implies $\hat{\alpha}g$ - $\ker\{x\} \cap \hat{\alpha}g$ - $\ker\{y\} = \emptyset$ and let (X, τ) be not $\hat{\alpha}g$ - T_1 -space then for each $x \neq y \in X$ implies $y \in \hat{\alpha}g$ - $\ker\{x\}$ or $x \in \hat{\alpha}g$ - $\ker\{y\}$ [by theorem 4.10], then $\hat{\alpha}g$ - $\ker\{x\} \cap \hat{\alpha}g$ - $\ker\{y\} \neq \emptyset$, this is a contradiction. Thus, (X, τ) is a $\hat{\alpha}g$ - T_1 -space.

Corollary 4.13: Every $\hat{\alpha}g$ - T_2 -space is $\hat{\alpha}g$ - T_1 -space and every $\hat{\alpha}g$ - T_1 -space is an $\hat{\alpha}g$ - T_0 -space.

Proof: Clearly.

Theorem 4.14: A topological space (X, τ) is an $\hat{\alpha}g$ - T_1 -space if and only if (X, τ) is $\hat{\alpha}g$ - T_0 -space and $\hat{\alpha}g$ - R_0 -space.

Proof: Let (X, τ) be $\hat{\alpha}g$ - T_1 -space and let $x \in U$ be and $\hat{\alpha}g$ -open set, then for each $x \neq y \in X$, $\hat{\alpha}g$ - $\ker\{x\} \cap \hat{\alpha}g$ - $\ker\{y\} = \emptyset$ [by theorem 4.12] implies $x \notin \hat{\alpha}g$ - $\ker\{y\}$ and $y \notin \hat{\alpha}g$ - $\text{cl}\{x\}$ this means $\hat{\alpha}g$ - $\text{cl}\{x\} = \{x\}$, hence $\hat{\alpha}g$ - $\text{cl}\{x\} \subseteq U$. Thus, (X, τ) is a $\hat{\alpha}g$ - R_0 -space.

Conversely, let (X, τ) be $\hat{\alpha}g$ - T_0 -space and $\hat{\alpha}g$ - R_0 -space, then for each $x \neq y \in X$ there exists an $\hat{\alpha}g$ -open set U such that $x \in U, y \notin U$ or $x \notin U, y \in U$. Say $x \in U, y \notin U$ since (X, τ) is a $\hat{\alpha}g$ - R_0 -space, then $\hat{\alpha}g$ - $\text{cl}\{x\} \subseteq U$, this means there exists an $\hat{\alpha}g$ -open set V such that $y \in V, x \notin V$. Thus, (X, τ) is a $\hat{\alpha}g$ - T_1 -space.

Theorem 4.15: A topological space (X, τ) is an $\hat{\alpha}g$ - T_2 -space if and only if

- (1) (X, τ) is an $\hat{\alpha}g$ - T_0 -space and $\hat{\alpha}g$ - R_1 -space.
- (2) (X, τ) is an $\hat{\alpha}g$ - T_1 -space and $\hat{\alpha}g$ - R_1 -space.

Proof (1): Let (X, τ) be an $\hat{\alpha}g$ - T_2 -space then it is an $\hat{\alpha}g$ - T_0 -space. Now since (X, τ) be an $\hat{\alpha}g$ - T_2 -space then for each $x \neq y \in X$, there exist disjoint $\hat{\alpha}g$ -open sets U, V such that $x \in U$ and $y \in V$ implies $x \notin \hat{\alpha}g$ - $\text{cl}\{y\}$ and $y \notin \hat{\alpha}g$ - $\text{cl}\{x\}$, therefore $\hat{\alpha}g$ - $\text{cl}\{x\} = \{x\} \subseteq U$ and $\hat{\alpha}g$ - $\text{cl}\{y\} = \{y\} \subseteq V$. Thus, (X, τ) is a $\hat{\alpha}g$ - R_1 -space.

Conversely, let (X, τ) be an $\hat{\alpha}g$ - T_0 -space and $\hat{\alpha}g$ - R_1 -space, then for each $x \neq y \in X$, there exists $\hat{\alpha}g$ -open set U such that $x \in U, y \notin U$ or $y \in U, x \notin U$, implies $\hat{\alpha}g$ - $\text{cl}\{x\} \neq \hat{\alpha}g$ - $\text{cl}\{y\}$, since (X, τ) is an $\hat{\alpha}g$ - R_1 -space [By assumption], then there exist disjoint $\hat{\alpha}g$ -open sets G, H such that $x \in G$ and $y \in H$ [Def. 3.8]. Thus, (X, τ) is a $\hat{\alpha}g$ - T_2 -space.

Proof (2): By the same way of part (1) an $\hat{\alpha}g$ - T_2 -space is an $\hat{\alpha}g$ - T_1 -space and $\hat{\alpha}g$ - R_1 -space.

Conversely, let (X, τ) be $\hat{\alpha}g$ - T_1 -space and $\hat{\alpha}g$ - R_1 -space, then for each $x \neq y \in X$, there exists $\hat{\alpha}g$ -open sets U, V such that $x \in U, y \notin U$ and $y \in V, x \notin V$, implies $\hat{\alpha}g\text{-cl}\{x\} \neq \hat{\alpha}g\text{-cl}\{y\}$, since (X, τ) is an $\hat{\alpha}g$ - R_1 -space, then there exist disjoint $\hat{\alpha}g$ -open sets G, H such that $x \in G$ and $y \in H$. Thus, (X, τ) is a $\hat{\alpha}g$ - T_2 -space.

Corollary 4.16: A topological $\hat{\alpha}g$ - T_0 -space is an $\hat{\alpha}g$ - T_2 -space if and only if for each $x \neq y \in X$ with $\hat{\alpha}g\text{-ker}\{x\} \neq \hat{\alpha}g\text{-ker}\{y\}$ then there exist $\hat{\alpha}g$ -closed sets F_1, F_2 such that $\hat{\alpha}g\text{-ker}\{x\} \subseteq F_1, \hat{\alpha}g\text{-ker}\{x\} \cap F_2 = \emptyset$ and $\hat{\alpha}g\text{-ker}\{y\} \subseteq F_2, \hat{\alpha}g\text{-ker}\{y\} \cap F_1 = \emptyset$ and $F_1 \cup F_2 = X$.

Proof: By Theorem 3.14 and Theorem 4.15.

Corollary 4.17: A topological $\hat{\alpha}g$ - T_1 -space is an $\hat{\alpha}g$ - T_2 -space if and only if one of the following conditions holds:

(1) For each $x \neq y \in X$ with $\hat{\alpha}g\text{-cl}\{x\} \neq \hat{\alpha}g\text{-cl}\{y\}$ then there exist an $\hat{\alpha}g$ -open sets U, V such that $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{x\}) \subseteq U$ and $\hat{\alpha}g\text{-cl}(\hat{\alpha}g\text{-ker}\{y\}) \subseteq V$.

(2) For each $x \neq y \in X$ with $\hat{\alpha}g\text{-ker}\{x\} \neq \hat{\alpha}g\text{-ker}\{y\}$ then there exist $\hat{\alpha}g$ -closed sets F_1, F_2 such that $\hat{\alpha}g\text{-ker}\{x\} \subseteq F_1, \hat{\alpha}g\text{-ker}\{x\} \cap F_2 = \emptyset$ and $\hat{\alpha}g\text{-ker}\{y\} \subseteq F_2, \hat{\alpha}g\text{-ker}\{y\} \cap F_1 = \emptyset$ and $F_1 \cup F_2 = X$.

Proof (1): By Corollary 3.15 and Theorem 4.15.

Proof (2): By Theorem 3.14 and Theorem 4.13.

Theorem 4.18: A topological $\hat{\alpha}g$ - R_1 -space is an $\hat{\alpha}g$ - T_2 -space if and only if one of the following conditions holds:

(1) For each $x \in X, \hat{\alpha}g\text{-ker}\{x\} = \{x\}$.

(2) For each $x \neq y \in X, \hat{\alpha}g\text{-ker}\{x\} \neq \hat{\alpha}g\text{-ker}\{y\}$ implies $\hat{\alpha}g\text{-ker}\{x\} \cap \hat{\alpha}g\text{-ker}\{y\} = \emptyset$.

(3) For each $x \neq y \in X, \text{either } x \notin \hat{\alpha}g\text{-ker}\{y\} \text{ or } y \notin \hat{\alpha}g\text{-ker}\{x\}$.

(4) For each $x \neq y \in X$ then $x \notin \hat{\alpha}g\text{-ker}\{y\}$ and $y \notin \hat{\alpha}g\text{-ker}\{x\}$.

Proof (1): Let (X, τ) be an $\hat{\alpha}g$ - T_2 -space. Then (X, τ) is a $\hat{\alpha}g$ - T_1 -space and $\hat{\alpha}g$ - R_1 -space [By theorem 4.15]. Hence by Theorem 4.10, $\hat{\alpha}g\text{-ker}\{x\} = \{x\}$ for each $x \in X$.

Conversely, let for each $x \in X, \hat{\alpha}g\text{-ker}\{x\} = \{x\}$, then by Theorem 4.10, (X, τ) is a $\hat{\alpha}g$ - T_1 -space. Also (X, τ) is an $\hat{\alpha}g$ - R_1 -space by assumption. Hence by Theorem 4.15, (X, τ) is an $\hat{\alpha}g$ - T_2 -space.

Proof (2): Let (X, τ) be an $\hat{\alpha}g$ - T_2 -space. Then (X, τ) is an $\hat{\alpha}g$ - T_1 -space [By theorem 4.3]. Hence by theorem 4.12, $\hat{\alpha}g\text{-ker}\{x\} \cap \hat{\alpha}g\text{-ker}\{y\} = \emptyset$ for each $x \neq y \in X$.

Conversely, assume that for each $x \neq y \in X, \hat{\alpha}g\text{-ker}\{x\} \neq \hat{\alpha}g\text{-ker}\{y\}$ implies $\hat{\alpha}g\text{-ker}\{x\} \cap \hat{\alpha}g\text{-ker}\{y\} = \emptyset$, so by Theorem 4.12, (X, τ) is an $\hat{\alpha}g$ - T_1 -space, also (X, τ) is an $\hat{\alpha}g$ - R_1 -space by assumption. Hence by Theorem 4.15, (X, τ) is an $\hat{\alpha}g$ - T_2 -space.

Proof (3): Let (X, τ) be an $\hat{\alpha}g$ - T_2 -space. Then (X, τ) is an $\hat{\alpha}g$ - T_0 -space [By Theorem 4.13]. Hence by theorem 4.3, either $x \notin \hat{\alpha}g\text{-ker}\{y\}$ or $y \notin \hat{\alpha}g\text{-ker}\{x\}$ for each $x \neq y \in X$.

Conversely, assume that for each $x \neq y \in X, \text{either } x \notin \hat{\alpha}g\text{-ker}\{y\} \text{ or } y \notin \hat{\alpha}g\text{-ker}\{x\}$ for each $x \neq y \in X$, so by theorem 4.3, (X, τ) is an $\hat{\alpha}g$ - T_0 -space also (X, τ) is an $\hat{\alpha}g$ - R_1 -space by assumption. Thus, (X, τ) is an $\hat{\alpha}g$ - T_2 -space [By Theorem 4.15].

Proof (4): Let (X, τ) be an $\hat{\alpha}g$ - T_2 -space. Then (X, τ) is an $\hat{\alpha}g$ - T_1 -space and an $\hat{\alpha}g$ - R_1 -space [By theorem 4.15]. Hence by Theorem 4.11, $x \notin \hat{\alpha}g\text{-ker}\{y\}$ and $y \notin \hat{\alpha}g\text{-ker}\{x\}$.

Conversely, let for each $x \neq y \in X$ then $x \notin \hat{\alpha}g\text{-ker}\{y\}$ and $y \notin \hat{\alpha}g\text{-ker}\{x\}$. Then by Theorem 4.11, (X, τ) is an $\hat{\alpha}g$ - T_1 -space. Also (X, τ) is an $\hat{\alpha}g$ - R_1 -space by assumption. Hence by Theorem 4.15, (X, τ) is an $\hat{\alpha}g$ - T_2 -space.

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