Classification of Zero Divisor Graphs of a Commutative Ring With Degree Equal 7 and 8

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ABSTRACT

In 2005 J. T Wang investigated the zero divisor graphs of degrees 5 and 6. In this paper, we consider the zero divisor graphs of a commutative rings of degrees 7 and 8.

Keywords: Zero-divisor, Ring, Zero-divisor graph.

تصنيف بيانات قواسم الصفر للحلقات الإبدالية ذات الدرجات 7 و 8

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الملخص

في عام 2005 درس Wang بيانات قواسم الصفر للحلقات الإبدالية من الدرجة 5 و6. في هذا البحث درسنا بيانات قواسم الصفر للحلقات الإبدالية من الدرجتين 7 و8.

الكلمات المفتاحية: قواسم الصفر، حلقة، بيان قواسم الصفر.

1. Introduction

The concept of zero divisor graph of a commutative ring was introduced by Beck in [3]. He let all the elements of the ring be vertices of the graph. In [1] Anderson and Livingston introduced and studied the zero divisor graph whose vertices are the non-zero zero divisors.

Throughout this paper, all rings are assumed to be commutative rings with identity, and Z(R) be the set of zero divisors. We associate a simple graph Γ(R) to a ring R with vertices $Z(R)^*= Z(R)-\{0\}$, the set of all non-zero zero divisors of R. For all distinct $x,y\in Z(R)^*$, the vertices x and y are adjacent if and only if xy=0. (R,m) and |S| will stand respectively for the local ring with maximal ideal m and cardinal numbers of a set S.

In [1] Anderson and Livingston proved that for any commutative ring R Γ(R) is connected.

In 2005 J. T Wang [5] investigated the zero divisor graphs of degrees 5 and 6. In this paper, we extend this results to consider the zero divisor graphs of commutative rings of degrees 7 and 8.

The main result when $|Z(R)^*|=7$ is given in Theorem 2.7,while when $|Z(R)^*|=8$ the main result is given in Theorem 3.4. We also extend Wang's result concerning local rings (Theorem 2.2)

2. Rings with $|Z(R)^*|=7$

It is known that if R is a ring then Γ(R) is connected. In this section, we find all possible graphs of Γ(R) with Γ(R)=7.

Recall that if R is finite ring, then every element of R is either a unit or a zero divisor [2]. In [5] Wang proved the following result.

Lemma 2.1 :

Let (R_1,m_1) and (R_2,m_2) are local rings, then $|Z(R_1xR_2)^*|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1||m_2|-1$. ■

Now, we shall prove the following theorem which extends Wang's result.

Theorem 2.2 :

If (R_1,m_1), (R_2,m_2) and (R_3,m_3) are finite local rings, then $|Z(R_1xR_2xR_3)^*|=|R_1|x|R_2|x|m_3|+|Z(R_1xR_2)|x(|R_3|-|m_3|)-1$ where $|Z(R_1xR_2)|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1|x|m_2|$.

Proof :

By Lemma 2.1 $|Z(R_1xR_2)^*| = |R_1|x|m_2| + |R_2|x|m_1| - |m_1|x|m_2| - 1$, therefore $|Z(R_1xR_2)|= |Z(R_1xR_2)^*|+1=|R_1|x|m_2|+|R_2|x|m_1|-|m_1|x|m_2|$. Let $R_{(1)(2)}=R_1xR_2$, then $|R_{(1)(2)}|=|R_1|x|R_2|$ and $|Z(R_{(1)(2)})|=Z(R_1xR_2)$.

For any non-zero-divisor (a,b) in $R_{(1)(2)}xR_3$, we have the following cases:

1- If a is non-zero divisor of $R_{(1)(2)}$, then a must be a unit element. If b is a zero divisor of R_3, then there are $(|R_{(1)(2)}|-|Z(R_{(1)(2)})|)x|m_3|$ elements of this type.
2- If a is a non-zero zero divisor of $R_{(1)(2)}$ and b any element in R_3, then there are $(|Z(R_{(1)(2)})|-1)x|R_3|$ elements of this type.
3- If a=0, and b is a non-zero element in R_3, then there are $1x(|R_3|-1)$.

Now, we sum up these three types of elements; there are as follows:
$(|R_{(1)(2)}|-|Z(R_{(1)(2)})|)x|m_3|+(|Z(R_{(1)(2)})|-1)x|R_3|+1x(|R_3|-1)=$
$|R_{(1)(2)}|x|m_3|-|Z(R_{(1)(2)})|x|m_3|+|Z(R_{(1)(2)})|x|R_3|-|R_3|+|R_3|-1=$
$|R_1|x|R_2|x|m_3|+|Z(R_1xR_2)|(|R_3|-|m_3|)-1$ where
$Z(R_1xR_2)|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1|x|m_2|$. ■

As a direct consequence to Theorem 2.2, we obtain the following:

Corollary 2.3 :

If R_1, R_2 and R_3 are finite fields, then
$|Z(R_1xR_2xR_3)^*|= |R_1||R_2|+|R_1||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3|$. ■

Corollary 2.4 :

If R finite and $R\cong R_1xR_2xR_3$, then $|Z(R)^*|\geq 13$ for some local rings R_i but not field.

Proof :

Suppose that R_3 is local which is not a field, then clearly $|R_3|\geq 4$ and $|m_3|\geq 2$ and since $|R_1|,|R_2|\geq 2$ and $|m_1|,|m_2|\geq 1$, then $Z(R_1xR_2)\geq 3$, therefore $|Z(R)^*|\geq 2.2.2+3(4-2)-1=13$. ■

Next, we prove two fundamental lemmas

Lemma 2.5 :

If R is a ring with $|Z(R)|^*=7$, then is either R local ring or R is isomorphic to a product of two local rings.

Proof:

Since $|Z(R)^*|=7$, then R is finite and hence $R\cong R_1xR_2x\ldots xR_n$ where R_i, $i=1,2,\ldots,n$ are local rings. If $n\geq 4$, then by [5,Lemma 4.7], $|Z(R)^*|\geq 14$ this is a contradiction.

Now, consider n=3, if R_i local, but not field for some $1\leq i\leq 3$, then by Corollary 2.4, $|Z(R)^*|\geq 13$ which is a contradiction. Hence R_i are fields for all $1\leq i\leq 3$. Applying Corollary 2.3 $|Z(R_1xR_2xR_3)^*| = |R_1||R_2|+|R_1||R_3|+ |R_2||R_3|-|R_1|-|R_2|-|R_3|=7$. If $|R_1|=|R_2|=2$

then $|R_3|=7/3$ which is also a contradiction. Finally, if $|R_i|\geq 3$ for some i, then by [5,Lemma 4.5], $|Z(R)^*|\geq 9$ which is also a contradiction. Therefore, n=1 or 2 ■

Lemma 2.6 :

Let R be a ring which is not local and $|Z(R)|^*=7$, then $R\cong Z_4xZ_3$ or $Z_2[X]/(X^2)x\ Z_3$ or Z_2xZ_7 or F_4xZ_5.

Proof:

Suppose that R is a ring which is not local, then by Lemma 2.5 $R\cong R_1xR_2$. If R_1 and R_2 are local, but not a field, then by [5, Corollary 4.4], $|Z(R)^*|\geq 11$ which is a contradiction. If R_1 local, but not a field, R_2 field, then we have $|Z(R)^*|=|R_1|x|m_2|+|R_2|x|m_1|-|m_1|x|m_2|-1=7$, this yields to $|R_1|+|m_1|(|R_2|-1)-8=0$ …(1)

Now, if $|m_1|=p$ where p is prime number, then by [5, Lemma 4.2], $|R_1|=|m_1|^*=p^2$, so from equation (1) we have $p^2+kp-8=0$ …(2), where $k=|R_2|-1$ this implies that $p=\frac{-k\mp\sqrt{k^2+32}}{2}$, so the only solution for p to be prime is k=2, and hence p=2, and this implies $|R_1|=4$ and $|R_2|=3$. Then, by [4,pp.687] $R_1\cong Z_4$ or $Z_2[X]/(X^2)$ and $R_2\cong Z_3$. Hence, $R\cong Z_4xZ_3$ or $Z_2[X]/(X^2)xZ_3$. Now if R_1 and R_2 are fields, then $|Z(R)^*|=|R_1|+|R_2|-2=7$, this yields to $|R_1|+|R_2|=9$. Therefore, $|R_1|=2$, $|R_2|=7$ or $|R_1|=4$, $|R_2|=5$. Thus, $R\cong Z_2xZ_7$ or F_4xZ_5. ■

Now, we shall prove the main result of this section.

Theorem 2.7 :

Let R be a ring which is not local and $|Z(R)|^*=7$, then the following graph can be realized as $\Gamma(R)$

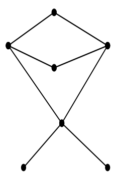

Figure (1)

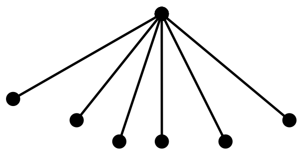

Figure (2)

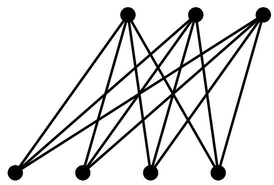

Figure (3),

Proof:

By Lemma 2.6, $R\cong Z_4xZ_3$ or $Z_2[X]/(X^2)x\ Z_3$ or Z_2xZ_7 or F_4xZ_5. In Figure (1), can be realized as $\Gamma(Z_4xZ_3)$or $\Gamma(Z_2[X]/(X^2)xZ_3)$, Figure (2) can be realized as $\Gamma(Z_2xZ_7)$ and Figure (3) can be realized as $\Gamma(F_4xZ_5)$. ■

3. Rings with $|Z(R)^*|=8$

The main aim of this section is to find all possible zero divisor graphs of 8 vertices and rings which correspond to them.

We shall start this section with following lemmas which play a central role in the sequel.

Lemma 3.1 :

Let R be a ring with $|Z(R)|^*=8$, then R is local or R is isomorphic to a product of two local rings.

Proof:

Since $|Z(R)^*|=8$, then R is finite and hence, $R\cong R_1xR_2x\ldots xR_n$ where R_i, i=1,2,…,n are local rings.

If $n\geq 4$, then by [5, Lemma 4.7], $|Z(R)^*|\geq 14$; this is a contradiction.

Now, consider n=3, if R_i local but not field for some $1 \leq i \leq 3$, then by Corollary 2.4, $|Z(R)^*| \geq 13$ which is a contradiction. So R_i is a field for all $1 \leq i \leq 3$. Then, by Corollary 2.3
$|Z(R_1xR_2xR_3)^*| = |R_1||R_2|+|R_1||R_3|+|R_2||R_3|-|R_1|-|R_2|-|R_3| = 8$. If $|R_1|=|R_2|=2$ then and $|R_3|=8/3$ which is a contradiction. If $|R_i| \geq 3$ for some i, then by [5, Lemma 4.5], $|Z(R)^*| \geq 9$ which is a contradiction. Therefore, n=1 or 2. ■

Lemma 3.2 :

Let R be a ring which is not local and $|Z(R)|^*=8$, then $R \cong F_1xF_2$, where F_1 and F_2 are fields

Proof:

Since R not local, then by Lemma 3.1 $R \cong R_1xR_2$, where R_1, R_2 are local rings. If R_1 and R_2 local, but not field, then by [5, Corollary 4.4], $|Z(R)^*| \geq 11$ which is a contradiction.

If R_1 field and R_2 local not field, then $|m_1|=1$. if $|m_2|=p$ is prime number, then by [5,Lemma 4.8], $|R_2|= p^2$ and applied [5,Lemma 4.2], we have $p^2+kp-9=0$ where $k=|R_2|-1$, so that $p = \frac{-k+\sqrt{k^2+36}}{2}$ ……(3) , since p is prime, then we have a contradiction. If $|m_1|$ not prime then $|m_1| \geq 4$ and since $|R_2| \geq 2$, then $|R_1|=9-|m_1|(|R_2|-1) \leq 9-4(2-1)=5$ which is a contradiction. Therefore, R_1 and R_2 are fields. Hence, $R \cong F_1xF_2$, where F_1 and F_2 are fields. ■

Lemma 3.3 :

Let R be a ring which is not local and $|Z(R)|^*=8$, then $R \cong Z_2xF_8$ or Z_3xZ_7 or Z_5xZ_5.

Proof:

By Lemma3.2 $R \cong F_1xF_2$, where F_1,F_2 are fields, we have $|F_1|+|F_2|-2 =8$ which implies that $|F_1|+|F_2|=10$, so that $|F_1|=2$, $|F_2|=8$ or $|F_1|=3$, $|F_2|=7$ or $|F_1|=5$, $|F_2|=5$. Therefore, $R \cong Z_2xF_8$ or Z_3xZ_7 or Z_5xZ_5. ■

Now, we are in a position to give the main result of this section

Theorem 3.4 :

Let R be a ring which is not local and $|Z(R)^*| =8$, then the following graph can be realized as $\Gamma(R)$.

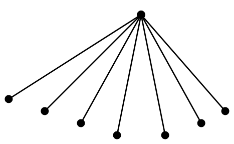

Figure (1)

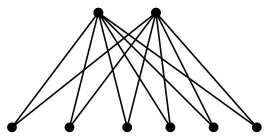

Figure (2)

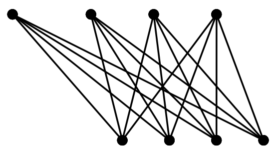

Figure (3),

Proof:

By Lemma 3.3, then $R \cong Z_2xF_8$ or Z_3xZ_7 or Z_5xZ_5. In Figure (1), can be realized as $\Gamma(Z_2xF_8)$. Figure (2) , can be realized as $\Gamma(Z_3xZ_7)$. Figure (3) , can be realized as $\Gamma(Z_5xZ_5)$. ■

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