$* -$ Embded 0f $* -$ Semigroup into $* -$ Ring

انغمار * - شبه الزمرة في *- الحلقة

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Abstract

The purpose of this paper is to study the embeddability of the $* -$ semigroup $(S,.,*)$ into $* -$ ring , and prove that embeddability referred to, is possible if the ring $(Z[S],*)$ resulted from the semigroup S and the integer ring $(Z,+,.)$ is proper involution.

الخلاصة:

الغرض من هذا البحث هو دراسة انغمار *- شبه الزمرة في * - الحلقة واثبات ان هذا الانغمار ممكن اذا كانت الحلقة $(Z[S],*)$ انعكاسية فعلية

1 Introduction

In this paper one of the important of ring is the class of $* -$semigroup ring with involution is studied .

In Section 2 all important definitions of $* -$semigroup, n-proper maximal, involution on a ring, proper involution on $* -$ring and $* -$semigroup ring are discussed.

In Section 3, we proved some propositions, remarks, and examples of the embeddable $* -$semigroup into $* -$ring.

At last, we proved that if $(S,.,*)$ is $* -$semigroup finite commutative, maybe we can be $* -$embedded into a ring involution, or we cannot be $* -$embedded into any ring with involution.

2 $* -$Semigroup Ring

Definition 2.1,[1]. Let $(S,.)$ be any semigroup. A mab $*: (S,.) \to (S,.)$ is called an involution of $(S,.)$ if the conditions are holds $(a^*)^* = a \ \ and \ \ (ab)^* = b^* a^*$ for all a,b in S, $(S,.,*)$ is called $* -$semigroup with involution $*$.

Definition 2.2,[1]. $(S,.,*)$ is called n-proper if whenever $s_1 s_1{}^* = s_1 s_2{}^*, s_2 s_2{}^* = s_2 s_3{}^*, \dots, s_n s_n{}^* = s_n s_n{}^*$ implies $s_1 = s_2 = \cdots = s_n \ \forall \ s_1, s_2, \dots, s_n \in S$.

Definition 2.3,[1]. Let $(S,.,*)$ be a $* -$semigroup and $= \{s_1, s_2, \dots, s_n\} \subset S$. An element $s_k \in A$ is called maximal in A if the following two conditions are holds

1) $s_k s_k{}^* = s_k s_i{}^* \ \forall \ i = 1,2, \dots, n$ impliesed $s_k = s_i$.

2) $s_k s_k{}^* = s_{ij}{}^* (i \neq k \neq j)$ implies $s_k{}^* s_i = s_k{}^* s_j$.

Definition 2.4,[1]. Let $(R,+,.)$be a ring, an involution on this ring is a map $*: (R,+,.) \to (R,+,.)$ such that for all A, B, and C the following conditions are holds:

1) $(A + B) = A^* + B^*$

2) $(AB)^* = B^* A^*$

3) $(A^*)^* = A$

Definition 2.3,[2]. An involution $*$ on a ring R is called a proper involution if for every $A \in R$ such that $AA^* = 0$ implies $A = 0$. In this case $(R,*)$ is called a P^* −ring.

Example 2.6,[3]. $(C,*)$ is a P^* −ring where C is the complex field and $*$ is the conjugate operator.
Definition 2.7,[3].Let $(S,.,*)$ be a $*$ −semigroup and let R be a $*$ − ring. We say that $(S,.,*)$ is $*$ −embedded in a $*$ −ring R if there is injective map $f: (S,.,*) \to (R,+,*)$ such that $f(a.b) = f(a).f(b)$ and $f(a^*) = \left(f(a)\right)^* \ \forall\, a,b \in S$.

Definition 2.8,[4]. Let $(S,.,*)$ be a $*$ −semigroup and let R be a $*$ −ring
We define $R[S] = \{\sum_{i=1}^{N} a_i g_i : a_i \in R, g_i \in S, n \in Z^+ \}$
Where $\sum_{i=1}^{N} a_i g_i$ is just a formal symbol.

Thus, $\sum_{i=1}^{N} a_i g_i = \sum_{i=1}^{N} b_i g_i \Leftrightarrow a_i = b_i \ \forall\, i = 1,2,\dots,n$.
Also, we define additive "+" and multiplication "." On $R[S]$ as follows.
$\sum_{i=1}^{N} a_i g_i + \sum_{i=1}^{N} b_i g_i = \sum_{i=1}^{N} (a_i + b_i) g_i$ and
$\sum_{i=1}^{N} a_i g_i \, . \sum_{i=1}^{N} b_i g_i = \sum_{i=1}^{N} c_i g_i$ where $c_i = \sum_{i=1}^{N} a_i b_i$.
The sum being taken over all pairs (a_i, b_i) such that $g_i \,.\, g_j = g_k$.
Define $*$ on $R[S]$ as $(\sum_{i=1}^{N} a_i g_i)^* = \sum_{i=1}^{N} a_i^* g_i^{\ *} \ \forall a_i \in R, g_i \in S$

Proposition 2.9. $(R[S],+,.,*)$ is a $*$ −ring.
Proof. Clearly $(R[S],+)$is an abelian group and $(R[S],.)$ is semigroup under multiplication and
$\sum_{i=1}^{N} a_i g_i \ (\sum_{i=1}^{N} b_i g_i + \sum_{i=1}^{N} c_i g_i) =$
$(\sum_{i=1}^{N} a_i g_i \, . \sum_{i=1}^{N} b_i g_i) + (\sum_{i=1}^{N} a_i g_i \, . \sum_{i=1}^{N} b_i g_i)$
Thus, the left distributive law holds. Similarly the right distributive law holds.
Thus, $(R[S],+,.,*)$ is $*$ −ring.
$(R[S],+,.,*)$ is called $*$ −semigroup ring of $(R,+,.,*)$ over $(S,.,*)$

3 Embeddability into $(R,+,.,*)$

Proposition 3.1. If $(R[S],+,.,*)$ $*$ −embeds $(S,.,*)$ through a $*$ −embeding $f, w: S \to Z[S]$ is the inclution map and $g: Z[S] \to R$ is defined by
$g(\sum_{i=1}^{N} m_i s_i) = \sum_{i=1}^{N} m_i \, f(s_i) \ \ \forall m_i \in Z, s \in S$. Then the following diagram is commute

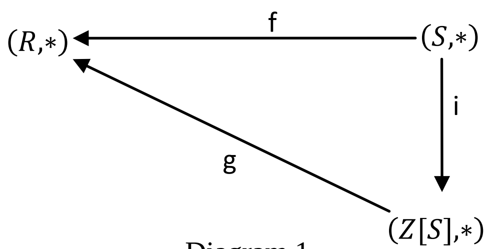

Diagram 1

Proof. Clearly g is a $*$ −homomorphism and $g \circ i = f$. If $(Z[S],*)$is a proper $*$ −ring , then $(S,*)$ is $*$ −embeddable in P* −ring $(Z[S],*)$. It turns out that if S ia an inverse semigroup, then $(Z[S],*)$is a proper $*$ −ring and $(S,*)$ is $*$ −embeddable in $(Z[S],*)$.
Proposition 3.2. Let $(S,*)$ be a proper $*$ −semigroup, and let $A_1 \neq A_2 , A_1, A_2 \in Z[S]$ such that $A_1 {A_1}^* = A_2 {A_2}^* \in Z[S]$ and $C \in Z[S]$ such that $CC^* = m_1 A_1 + m_2 A_2$.
If $s_1 - s_2$ is a linear combination of $A_1, A_2, and\ C$ in $Z[S]$, then there is no P* −ring which $*$ $-embedding$ $(S,*)$.

Proof. Let $(R,*^{\backslash})$ be a P^* −ring, $*-embedding$ $(S,*)$ consider diagram(1) , since $g(A_1A^*{}_1) = g(A_2A^*{}_2) = 0$ in R .
$g(A_1).g(A_1)^* = g(A_2).g(A_2)^* = 0$ and hence $g(A_1) = g(A_2) = 0$.
Now $g(CC^*) = 0 = g(C).g(C)^*$ and since $(R,*)$ is P^* −ring, then $g(D) = f(s_1).f(s_2) = 0$ which implies that $f(s_1) = f(s_2)$ and so f is not injective. Thus, $(S,*)$ in not $*-embeddable$.

Proposition 3.3. Let $(S,.,*)$ be a $*-$ semigroup with n-proper maximal involution. Then $*$ is an $n-$proper involution.
Proof. Suppose that $(S,.,*)$ is $n-$proper involution, then
$s_1s_1{}^* = s_1s_2{}^*, s_2s_2{}^* = s_2s_3{}^*, \dots, s_ns_n{}^* = s_ns_n{}^*$ $\forall s_1, s_2, \dots, s_n \in S$. Since $(S,.,*)$ has a maximal, then there exist a maximal element $s_1 \in \{s_1, s_2, \dots, s_n\}$. From the fact
$s_is_i{}^* = s_is_{i+1}{}^*(mod\ n)$ implies $s_i = s_{i+1}(mod\ n)$, it follows that $s_1 = s_2 = \cdots = s_n$.

Proposition 3.4.[2]. Let $(R,*)$ be a ring with a 1-formally complex involution, then the involution is $*$ 2 –proper. Moreover if the involution $*$ is $n-$formally complex, then it is n-proper.

Corollary 3.5. Let $(R,*)$ be a ring with a formally complex involution, then the involution $*$ is $n-$proper.
Proof. Follows directly from proposition 3.4.
Remark 3.6. If $(S,.,*)$ is a finite commutative, then
1) We can be $*-embedded$ into a ring with involution, as shown in Example 3.7.
2) We cannot be $*-$ embedded into any ring with involution, as shown Example 3.8.

Example 3.7. Let $S \subset Z^2$be such that $S = \{s_1 = (1,1), s_2 = (0,1), s_3 = (-1,-1), s_1 = (0,-1)\}$. It is clear that S is semigroup under pointwise multiplication
$(a,b).(c,d) = (ab,cd)$
Define a map $*: (S,.) \to (S,.)$ by $(a,b)^* = (ab,b)$. This satisfies all conditions of involution.
It's clear that $*$ is proper (since if $s_1s_1{}^* = s_1s_2{}^* = s_2s_2{}^*, then\ s_1 = s_2$) and S is commutative.
Let $(Z[S],*)$ be $*-$semigroup ring of S over the integer where $*$ is the involution induced from S in $Z[S]$.
$X = \sum_{i=1}^{4} x_i\, s_i$ such that $XX^* = 0$, let
$f_1 = x_1^2 + x_3^2 = 0, f_2 = 2x_1x_2 + x_2^2 + 2x_3x_4 + x_4^2 = 0, f_3 = 2x_1x_3 = 0,$
$f_4 = 2x_1x_4 + 2x_3x_2 + 2x_2x_4 = 0$. It is clear that in Z if $x_1^2 + x_2^2 = 0$ then $x_1 = x_2 = 0$.
By substituting in f_2 we get $x_3 = x_4 = 0$, thus the solution of XX^* in $(Z[S],*)$ is trivial in this case. Then $(S,*)$ is $*-$ embeddable.

Example 3.8. Let $S \subset Z^3$be such that $S = \{s_1 = (-1,1,1), s_2 = (-1,1,1), s_3 = (-1,-1,1), s_1 = (1,-1,1)\}$.
Clearly S is semigroup under pointwise multiplication $(a_1,b_1,c_1).(a_2,b_2,c_2) = (a_1a_2,b_1b_2,c_1c_2)$.
Define a map $*: (S,.) \to (S,.)$ by
$(a,b,c)^* = (a,ab,c)$, this map satisfies all conditions of involution and it is proper and commutative. $X = \sum_{i=1}^{4} x_i\, s_i$ such that $XX^* = 0$, this implies
$f_1 = x_1^2 + 2x_2x_3 + x_4^2 = 0, f_2 = x_1x_3 + x_1x_2 + x_3x_4 + x_2x_4 = 0,$
$f_3 = x_2^2 + 2x_1x_4 + x_3^2 = 0,$
$f_4 = x_1^2 + 2x_1x_4 + x_3^2 = 0$
$f_1 + f_3 = (x_1 + x_4)^2 + (x_2 + x_3)^2$, $f_2 = (x_1 + x_4)(x_2 + x_3) = 0$, thus $x_1 = -x_4$, $x_2 = -x_3$, $x_2 = \pm x_1$

$X = t_1s_1 + t_1s_2 - t_1s_3 - t_1s_4$ or $X = t_1s_1 - t_1s_2 + t_1s_3 - t_1s_4 \in Z$.
Thus, the solution of the equation $XX^* = 0$ is non-trivial in this case
$A = s_1 + s_2 - s_3 - s_4$, let $B = s_1 - s_2 + s_3 - s_4$ and $C = 2s_1 + 3s_2 - 3s_3 - 2s_4$
$AA^* = 0, BB^* = 0, CC^* = -5A - 5B$
$g(AA^*) = g(BB^*) = 0$. Since g is $*-$homomorphism then $g(A) = g(B) = O_R$(since R a proper 8-ring). Hence $g(C)g(C)^* = g(CC^*) = g(-5A - 5B)$
$g(CC^*) = O_R$, $g(C) = O_R$. Let $D = C - 2A$, then $D = s_2 - s_3$
By proposition 3.1. $(S,.,*)$ is not $*-$ embeddable.

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