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On Some Properties of Functions on Convex Galaxies

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ABSTRACT

In this paper, we define and study extensively a new type of external sets in R, we call it "convex galaxies". We show that these convex external sets may be classified in some definite types. More precisely, we obtain the following :

(1) Let $G \subset R$ be a convex galaxy which is symmetric with respect to zero, then

(i) G is an α- galaxy (0) if and only if there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n\in \underline{N}}$ with $G = \bigcup_{n\in \underline{N}} [-a_n, a_n]$ such that $a_0 = \alpha$ and $\frac{a_{n+1}}{a_n} = c$, for all $n \in \underline{N}$, where, c is some limited real number such that $c > 1$.

(ii) G is a non-linear galaxy if and only if there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n\in \underline{N}}$ with $G = \bigcup_{n\in \underline{N}} [-a_n, a_n]$ such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \underline{N}$.

(2) Let $G \subset R$ be a convex galaxy which is symmetric with respect to zero, then

(i) G is an α- galaxy (0) iff there exists a real internal strictly increasing C^∞- function f, such that $f(G) = G$, and $\frac{f'(t)}{f(t)} = c$ for all limited $t \geq 1$, where c is a positive real number.

(ii) G is a non-linear galaxy if and only if there exists a real internal strictly increasing C^∞- function f, such that $f(G) = G$ and $\frac{f'(t)}{f(t)}$ is positive unlimited, for all appreciable $t \geq 1$.

Keywords: Convex, Galaxy, External Sets.

حول بعض خواص الدوال في الكالكسيات المحدبة

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الملخص

في هذا البحث، تم تعريف نوع جديد من المجموعات الخارجية في R سميت بـ "الكالكسيات المحدبة" كما تم دراستها بشكل مستفيض. يمكن أن تصنف هذه المجموعات المحدبة الخارجية إلى بعض الأنواع المحددة. وعلى نحو أدق حصلنا على ما يلي:

(1) لتكن $G \subset R$ كالكسي محدبة متناظرة بالنسبة للصفر، عندها

(i) G α−كالكسي (0) إذا وفقط إذا وجدت متتابعة داخلية $\{a_n\}_{n\in\underline{N}}$ من الأعداد الحقيقية الموجبة متزايدة باضطراد و $G=\bigcup_{n\in\underline{N}}[-a_n,a_n]$ بحيث أن $a_0=\alpha$ و $\frac{a_{n+1}}{a_n}=c$ لكل $n\in\underline{N}$ ، حيث c هو عدد حقيقي محدد بحيث أن $c>1$.

(ii) G كالكسي غير خطية إذا وفقط إذا وجدت متتابعة داخلية $\{a_n\}_{n\in\underline{N}}$ من الأعداد الحقيقية الموجبة متزايدة باضطراد و $G=\bigcup_{n\in\underline{N}}[-a_n,a_n]$ بحيث أن $\frac{a_{n+1}}{a_n}$ غير محددة لكل $n\in\underline{N}$.

(2) لتكن $G\subset R$ كالكسي محدبة متناظرة بالنسبة للصفر ، عندها

(i) G α−كالكسي (0) إذا وفقط إذا وجدت دالة f حقيقية داخلية ومتزايدة باضطراد− C^{∞} ، بحيث أن $f(G)=G$ ، و $\frac{f'(t)}{f(t)}=c$ لكل $t\geq 1$ محددة، حيث c عدد حقيقي موجب.

(ii) G كالكسي غير خطية إذا وفقط إذا وجدت دالة f حقيقية داخلية ومتزايدة باضطراد− C^{∞} ، بحيث أن $f(G)=G$ ، و $\frac{f'(t)}{f(t)}$ موجبة غير محددة، لكل $t\geq 1$ ممكن تقديره.

الكلمات المفتاحية: محدبة، كالكسي، مجموعات خارجية.

1. Introduction

An application of this classification may be found in non-standard analysis approach. The study of slow-fast vector fields as shown by Diener F. [2]. For example, the notion width of jump may be defined in terms of a convex galaxy. Further, this classification can be used in approximations. Thus, the set of points on the real line, where two real functions f and g are infinitely close on R, that is the set $\{x\in R: f(x)\cong g(x)\}$ will often be a convex monad. [1]

For practice reasons we start with the study of convex galaxies which are symmetric with respect to zero.

The following definitions and notations are needed throughout this paper. See [4], [5], [6], [7], [8], [9] and [10].

Every concept concerning sets or elements defined in the classical mathematics is called **standard**.

Any set or formula which does not involve new predicates "standard, infinitesimals, limited, unlimited … etc" is called **internal,** otherwise it is called **external.**

A real number x is called **unlimited** if and only if $|x|>r$ for all positive standard real numbers r; otherwise it is called **limited**.

The set of all unlimited real numbers is denoted by $\overline{R}$, and the set of all limited real numbers is denoted by $\underline{R}$.

A real number x is called **infinitesimal** if $|x|<r$ for all positive standard real numbers r.

A real number x is called **appreciable**, if x is limited but not infinitesimal.

Two real numbers x and y are said to be **infinitely close** if and only if $x-y$ is infinitesimal and denoted by $x\cong y$.

The set of all limited real numbers is called **principal galaxy**, (denoted by $\mathbb{G}$).

For any real number a, the set of all real numbers x such that $x-a$ limited is called the **galaxy of *a*** (denoted by gal(a)).

Let $(\alpha \neq 0)$ α and $x \in R$,we define the *α-galaxy* (x) as follows:

α-galaxy(x)={ $y \in R : \frac{y-x}{\alpha}$ *is limited*} and denoted by *α-G*(x).

A subset G of R is a **convex galaxy** which is asymmetric with respect to zero **iff** there exists an internal strictly increasing sequence of strictly positive real numbers $\{t_n\}_{n\in N}$ such that $G = \bigcup_{n\in \underline{N}} [-t_n, t_n]$.

Theorem 1.1:(Extension Principle) [3]

Let X and Y be two standard sets, ${}^sX$ and ${}^sY$ be the subsets constitute of the standard elements X and Y, respectively. If we can associate with every $x \in {}^SX$ a unique $y = f(x) \in {}^SY$, then there exists a unique standard $y^* \in Y$ such that $\forall^{st} x \in {}^sX$, $y^* = f(x)$.

Theorem 1.2: (Principal of External Induction) [3]

If E is an internal or external property such that $E(0)$ is true and $E(n) \rightarrow E(n+1)$ true for all $n \in \underline{N}$. Then, $E(n)$ is true for all $n \in \underline{N}$.

Consider the following characterization of the galaxies:

We start firstly with some examples:

2. Examples: Let G and $\{a_n\}_{n\in N}$ be as mentioned previously, we may always assume that $a_0 = 1$

(1) Suppose that $\frac{a_{n+1}}{a_n} = 2$ for all $n \in \underline{N}$ **,** then G is an additive group. Because $x, y \in G$ and $|x|$, $|y|$ being less than a_n then

$|x+y| \leq |x| + |y| \leq 2a_n \leq a_{n+1}$. So $x+y \in G$.

Since, $a_0 = 1$, it follows that $G=\mathbb{G}$. If, on the contrary, we had $a_0 = \alpha$, then G will be *α-galaxy*(0).

(2) Suppose that $\frac{a_{n+1}}{a_n} = \omega$, for all $n \in \underline{N}$, where ω is a positive unlimited real number, then the galaxy G is again additive group. However, G is not an *α-galaxy*(0) because if $\alpha > 0$ be such that *α-galaxy* $(0) \subset G$, then there is $n \in \underline{N}$ such that $\alpha < a_n$ so $\alpha < a_{n+1}$ which implies that $a_{n+1} \in$*α-galaxy*(0). Hence, *α-galaxy*(0)$\subsetneq$ G.

(3) If $\frac{a_{n+1}}{a_n} = 1+\varepsilon$ for all $n \in \underline{N}$, where ε is a positive infinitesimal, then G is not additive group because $2 \notin G$.

A convex galaxy which is a group, but not an *α-galaxy* will be called non-linear. Informally, the *α-galaxy* is the set of all real numbers of order *α*, while a non-linear galaxy cannot be the set of real numbers of the order of one of its element.

We generalize the connection between the convex galaxy G and the ratio of consecutive terms $\frac{a_{n+1}}{a_n}$ of the sequence $\{a_n\}_{n\in N}$ such that $G = \bigcup_{n\in \underline{N}} [-a_n, a_n]$ of above examples. Consider the following theorem

Theorem 2.1 :

Let $G \subset R$ be a convex galaxy which is symmetric with respect to zero, then

(i) G is an *α-galaxy*(0) **iff** there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n\in N}$ with $G = \bigcup_{n\in \underline{N}} [-a_n, a_n]$ such that $a_0 = \alpha$ and $\frac{a_{n+1}}{a_n} = c$, for all $n \in \underline{N}$, where c is some limited real number such that $c \gneqq 1$.

(ii) G is a non-linear galaxy **iff** there exists an internal strictly increasing sequence of strictly positive real numbers $\{a_n\}_{n\in N}$ with $G = \bigcup_{n\in \underline{N}} [-a_n, a_n]$ such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \underline{N}$.

Proof:

(i) Let G be *α-galaxy*(0), then every sequence $\{\alpha^{k^n}\}_{n\in N}$, $k > 1$, clearly satisfies the relation.

Conversely let $\{a_n\}_{n\in N}$ be an internal strictly in an increasing sequence of strictly positive real numbers such that $\frac{a_{n+1}}{a_n} = c$, for all $n \in \underline{N}$, where c some limited real number such that $c \gneqq 1$, then it is clear that $\bigcup_{n\in \underline{N}} [-a_n, a_n]$ is the a_0 *-galaxy*(0).

(ii) Let G be a non-linear galaxy, let $\{k_n\}_{n\in \underline{N}}$ be strictly increasing sequence of strictly positive real numbers such that $G = \bigcup_{n\in \underline{N}} [-k_n, k_n]$. We define a subsequence $\{a_n\}_{n\in \underline{N}}$ of $\{k_n\}_{n\in \underline{N}}$ such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \underline{N}$, by the external induction. Put $a_0 = k_0$, suppose that a_n is defined to some k_n. Since G is a convex group, which contains a_n *-galaxy*(0). Because, G is non-linear a_n *-galaxy*(0) is strictly contained in G. So there is k in G-a_n *-galaxy*(0), by putting a_{n+1} the last k_n such that $k_n \in G - a_n$ *-galaxy*(0). Then $\frac{a_{n+1}}{a_n}$ is unlimited, by the principle extension there exists an internal extension $\{a_n\}_{n\in N}$ of $\{a_n\}_{n\in \underline{N}}$ which may be assume strictly increasing. This sequence has all the required properties.

Conversely let $\{a_n\}_{n\in \underline{N}}$ be an internal increasing sequence of strictly positive real numbers such that $\frac{a_{n+1}}{a_n}$ is unlimited for all $n \in \underline{N}$.

By putting $\bigcup_{n\in \underline{N}} [-a_n, a_n]$, we may prove in the same way as in example (2) that G is not *α-galaxy*(0), if $\alpha \le k_n$, for some $n \in \underline{N}$, then $\frac{k_{n+1}}{\alpha}$ is unlimited.

$\subsetneqq$

Hence, α-*galaxy*(x)$\subset G$ implies α-*galaxy*(0) G. ■

3. Convex Galaxies and Functions:

We now establish a relation between convex galaxies G and internal strictly functions C^{∞} and internal strictly increasing C^{∞} functions f, such that $f\left(G\right)=G$.

We use the following identity of these functions.

$$\frac{f\left(n+1\right)}{f\left(n\right)}=\exp\int_{n}^{n+1}(\frac{f'(t)}{f(t)})dt,\quad n\in N-\{0\}$$

Theorem 3.1:

Let $G\subset R$ be a convex galaxy which is symmetric with respect to zero. Then,

(i) G is an α-*galaxy*(0) **iff** there exists a real internal strictly C^{∞}-function f, such that $f\left(G\right)=G$, and $\frac{f'(t)}{f(t)}=c$ for all limited $t\geq 1$, where c is a positive real number.

(ii) G is a non-linear galaxy **iff** there exists a real internal strictly increasing C^{∞}-function f, such that $f\left(G\right)=G$ and $\frac{f'(t)}{f(t)}$ is positive unlimited, for all appreciable $t\geq 1$.

Proof :

(i) The part (i) follows from theorem(2.1) part (i).

(ii) Let G be non-linear and $\{a_n\}_{n\in N}$ be an internal increasing sequence of strictly positive real numbers such that $G=\bigcup_{n\in \underline{N}}\left[-a_n,a_n\right]$ and $\frac{a_{n+1}}{a_n}$ is unlimited for all $n\in \underline{N}$.

Now, we define the functions f_n on the [n, n+1] as follows :

$$f_n=\begin{cases} a_n\left(\frac{a_{n+1}}{a_n}\right)^{t-n} & if\ \ n\geq 1\\ \alpha t & if\ \ n=0\end{cases}$$

Then, f_n is internal strictly increasing and C^{∞} on [n, n+1], for all $n\in N$.

Furthermore $\frac{f_n'(t)}{f_n(t)}=\log\left(\frac{a_{n+1}}{a_n}\right)$, so $\frac{f_n'(t)}{f_n(t)}$ is unlimited for all limited $t\geq 1$, while $\bigcup_{n\in N} f_n$ is continuous, we can obtain a function f which conserves all these properties and is C^{∞} on the $[0,\infty)$ we may also expect that the odd function $f\cup g$ where $g:R^{-}\rightarrow R^{-}$ is defined by $g(t)=-f(-t)$ is C^{∞}. Then, $f\cup g$ is the required the function.

Conversely let f be a real internal strictly increasing odd C^{∞}-function such that $f\left(G\right)=G$, and $\frac{f'(t)}{f(t)}$ is positive unlimited for every limited $t\geq 1$, then we have for every $n\in \underline{N}$, $\left(n\geq 1\right)$, $\frac{f\left(n+1\right)}{f\left(n\right)}=\exp\int_{n}^{n+1}\frac{f'(t)}{f(t)}dt$, So that, $\frac{f\left(n+1\right)}{f\left(n\right)}$ is unlimited. Because $G=\bigcup_{n\in \underline{N}}\left[-f(n+1),f(n+1)\right]$, we deduct that G is not linear galaxy by theorem (2.1) part (ii)■.

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