The n-Hosoya Polynomials of the Composite of Some Special Graphs

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ABSTRACT

It is not easy to find the n-Hosoya polynomial of the compound graphs constructed in the form $G_1 \boxtimes G_2$ for any two disjoint connected graphs G_1 and G_2. Therefore, in this paper, we obtain n-Hosoya polynomial of $G_1 \boxtimes G_2$ when G_1 is a complete graph and G_2 is a special graph such as a complete graph, a bipartite complete, a wheel, or a cycle. The n-Wiener index of each such composite graph is also obtained in this paper.

Keywords: *Composite graphs $G_1 \boxtimes G_2$, n-Hosoya polynomial ,n-Wiener index.*

متعددة حدود هوسويا – n لبيان مركب من بعض البيانات الخاصة

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الملخص

إن إيجاد صيغة بسيطة لمتعددة حدود هوسويا−n لبيان مركب من بيانين متصلين G_1 و G_2 ومنفصلين عن بعضهما بالنسبة إلى الرؤوس بالشكل $G_1 \boxtimes G_2$ صعب. ولأجل الحصول على متعددة حدود هوسويا−n لكثير من البيانات المركبة، فقد عالجنا هذه المشكلة عندما يكون البيان G_1 تاما والبيان G_2 تاما، أو ثنائي التجزئة تام، أو عجلة، أو دارة. ولقد أوجدنا دليل وينر−n لكل من البيانات المركبة المذكورة.

الكلمات المفتاحية: بيان مركب $G_1 \boxtimes G_2$ ، متعددة حدود هوسويا−n، دليل وينر−n .

1. Introduction:

We follow the terminology of [5,6,7,8].Let v be a vertex of a connected graph G, and S be an (n-1) subset of $V(G)$, $n \geq 2$, then the n-distance $d_n(v,S)$ is defined [3] by

$$d_n(v,S) = \min\{d(v,u) : u \in S\}. \qquad \dots(1.1)$$

The n-diameter of G is defined by

$$\mathrm{diam}_n G = \max\{d_n(v,S) : v \in V(G), |S| = n-1, S \subseteq V(G)\}. \qquad \dots(1.2)$$

The n-Wiener index of G is defined by

$$W_n(G)=\sum_{(v,S)} d_n(v,S)\,. \qquad \dots(1.3)$$

The n-Hosoya polynomial of a connected graph G of order p is defined by

$$H_n(G;x)=\sum_{k=0}^{\delta_n} C_n(G,k)x^k\,, \qquad \dots(1.4)$$

where $2\le n\le p$, δ_n is the n-diameter of G , and $C_n(G,k)$ is the number of order pairs $(v,S), v\in V(G), S\subseteq V(G), |S|=n-1,$ such that $d_n(v,S)=k$.

One can easily show that

$$C_n(G,0)=p\binom{p-1}{n-2}\,,\ C_n(G,1)=p\binom{p-1}{n-1}-\sum_{v\in V(G)}\binom{p-1-\deg v}{n-1}. \qquad \dots(1.5)$$

The n-Hosoya polynomial of a vertex v in G , denoted by $H_n(v,G;x)$, is defined [3] by

$$H_n(v,G;x)=\sum_{k\ge 0} C_n(v,G,k)x^k\,, \qquad \dots(1.6)$$

where $C_n(v,G,k)$ is the number of (n-1)-subsets of vertices S such that $d_n(v,S)=k$. It is clear that for each k , $0\le k\le \delta_n$,

$$C_n(G,k)=\sum_{v\in V(G)} C_n(v,G,k)\,, \qquad \dots(1.7)$$

and

$$H_n(G;x)=\sum_{v\in V(G)} H_n(v,G;x)\,, \qquad \dots(1.8)$$

Let T be a non-empty subset of vertices of G . We define

$$C_n(T,G,k)=\sum_{v\in T} C_n(v,G,k)\,. \qquad \dots(1.9)$$

We shall use this notation in our proofs.

Finally , if n=2 , then from (1.5) , we get

$C_2(G,0)=p=d(G,0)$, and $C_2(G,1)=p(p-1)-\sum_{v\in V(G)}(p-1-\deg v)=\sum_{v\in V(G)}\deg v=2q$,

then $d(G,1)=\frac{1}{2}C_2(G,1)=q$. Also, we notice that $d(G,k)=\frac{1}{2}C_2(G,k)\,,\quad k\ge 2$.

Hence $H(G;x)=\frac{1}{2}H_2(G;x)\,.$ …(1.10)

In [2], H. G. Ahmed gave the following result :

Lemma: Let v be any vertex of a connected graph G. If there are r vertices of distance $k\ge 1$ from v , and there are s vertices of distance more than k from v , then

$$C_n(v,G,k)=\binom{r+s}{n-1}-\binom{s}{n-1}. \qquad \dots(1.11)$$

In 2007, H.O. Abdulla [1] and A.S. Aziz [4] defined the composite graph $G_1 ⊠ G_2$ as follows:

Let G_1 and G_2 be disjoint connected graphs , and let $u_1u_2\in E(G_1)$ and $v_1v_2\in E(G_2)$, then the composite graph $G_1 ⊠ G_2$ is the graph constructed from G_1 and G_2 by adding the edges u_1v_1, u_1v_2, u_2v_1 , and u_2v_2. It is clear that $p(G_1 \boxtimes G_2)=p(G_1)+p(G_2)$ and $q(G_1 \boxtimes G_2)=q(G_1)+q(G_2)+4$.

In this paper , we obtain n-Hosoya polynomials , n-Wiener indices , Hosoya polynomials and Wiener indices of the composite of some special graphs .

2. The Composite Graph $K_\alpha \boxtimes K_\beta$:

Let K_α and K_β be complete graphs of orders α , $\alpha \ge 2$ and β , $\beta \ge 2$ respectively .The composite graph $K_\alpha \boxtimes K_\beta$ is depicted in Fig. 2.1. We assume, without loss of generality that $\alpha \ge \beta$.

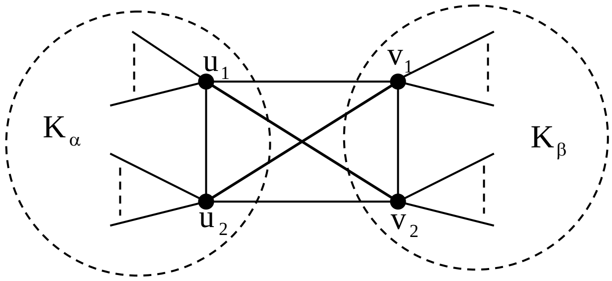

Fig.2.1. $K_\alpha \boxtimes K_\beta$

We notice that the n-diameter of $K_\alpha \boxtimes K_\beta$ is

$$\mathrm{diam}_n K_\alpha \boxtimes K_\beta = \begin{cases} 3, & \text{if } 2 \le n \le \alpha - 1 \\ 2 \text{ or } 1, & \text{if } n \ge \alpha \end{cases}$$

To simplify the notation, we denote $K_\alpha \boxtimes K_\beta$ by G .From ,Fig.2.1, we find that

$$C_n(G,3) = (\alpha-2)\binom{\beta-2}{n-1} + (\beta-2)\binom{\alpha-2}{n-1}. \qquad \ldots(2.1)$$

Since , $\mathrm{diam}_n K_\alpha \boxtimes K_\beta \le 3$, then from (1.5)

$$C_n(G,2) + C_n(G,3) = \sum_{v \in V(G)} \binom{p-1-\deg v}{n-1}.$$

Hence,

$$C_n(G,2) = (\alpha-2)\binom{\beta}{n-1} + (\beta-2)\binom{\alpha}{n-1} - (\alpha-4)\binom{\beta-2}{n-1} - (\beta-4)\binom{\alpha-2}{n-1}. \qquad \ldots(2.2)$$

From (1.5) we get

$$C_n(G,1) = p\binom{p-1}{n-1} - (\alpha-2)\binom{\beta}{n-1} - (\beta-2)\binom{\alpha}{n-1} - 2\binom{\beta-2}{n-1} - 2\binom{\alpha-2}{n-1}. \qquad \ldots(2.3)$$

From (2.1) , (2.2) , and (2.3) we have the next proposition:

Proposition 2.1: For $2 \le n \le p = \alpha + \beta$, $\alpha, \beta \ge 2$, then

$$H_n(G;x) = p\binom{p-1}{n-2} + \sum_{k=1}^{3} C_n(G,k)x^k,$$

where, $C_n(G,k)$, $1 \le k \le 3$, are given in (2.1), (2.2) , and (2.3).

And $W_n(G) = p\binom{p-1}{n-1} + \alpha\binom{\beta-2}{n-1} + \beta\binom{\alpha-2}{n-1} + (\alpha-2)\binom{\beta}{n-1} + (\beta-2)\binom{\alpha}{n-1}$. ■

From Proposition 2.1 and (1.10) , we get the next corollary.

Corollary 2.2: The Hosoya polynomial of the graph G of order $\alpha+\beta$, $\alpha,\beta \geq 2$, is given by :

$$H(G;x) = (\alpha+\beta) + \frac{1}{2}\left[\alpha(\alpha-1)+\beta(\beta-1)+8\right]x + 2\left[\alpha+\beta-4\right]x^2 + (\alpha-2)(\beta-2)x^3 .$$

And Wiener index of G is

$$W(G) = \frac{1}{2}\left[\alpha(\alpha-5)+\beta(\beta-5)+6\alpha\beta\right]. \qquad \blacksquare$$

3. The Composite Graph $K_\alpha \boxtimes K_{\beta,\gamma}$:

Let K_α be a complete graph of order $\alpha, \alpha \geq 2$ and $K_{\beta,\gamma}$ be a complete bipartite graph of order $\beta+\gamma$, $\beta,\gamma \geq 2$, then $K_\alpha \boxtimes K_{\beta,\gamma}$ is depicted in Fig. 3.1.

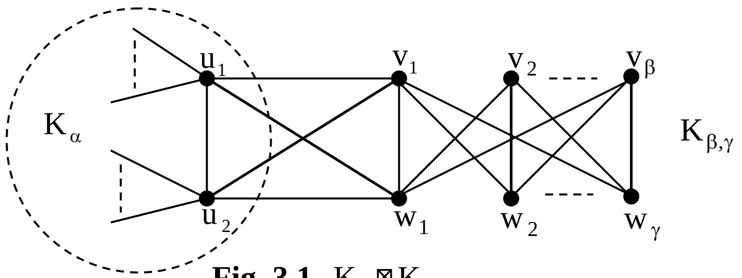

Fig. 3.1. $K_\alpha \boxtimes K_{\beta,\gamma}$

The order of $K_\alpha \boxtimes K_{\beta,\gamma}$ is $p = \alpha+\beta+\gamma$, the size is $q = \frac{1}{2}\{\alpha(\alpha-1)+2\gamma\beta+8\}$, and the diameter is 3 for $\alpha \geq 3$ and $\beta,\gamma \geq 2$. We denote $K_\alpha \boxtimes K_{\beta,\gamma}$ by G'.

The n-diameter of G' is given by

$$\mathrm{diam}_n G' = \begin{cases} 3, & \text{if } 2 \leq n \leq \max\{\alpha-1,\beta,\gamma\} \\ 2 \text{ or } 1, & \text{if } n > \max\{\alpha-1,\beta,\gamma\} \end{cases}.$$

In the next proposition , we obtain the n-Hosoya polynomial of G':

Proposition 3.1: For $2 \leq n \leq p = \alpha+\beta+\gamma$, $\alpha,\beta,\gamma \geq 2$, then

$$H_n(G';x) = p\binom{p-1}{n-2} + \sum_{k=1}^{3} C_n(G',k)x^k ,$$

where,

$$C_n(G',1) = p\binom{p-1}{n-1} - (\alpha-2)\binom{\beta+\gamma}{n-1} - (\beta-1)\binom{\alpha+\beta-1}{n-1} - (\gamma-1)\binom{\alpha+\gamma-1}{n-1}$$
$$-2\binom{\beta+\gamma-2}{n-1} - \binom{\alpha+\beta-3}{n-1} - \binom{\alpha+\gamma-3}{n-1}. \qquad \ldots(3.1.1)$$

$$C_n(G',2) = (\alpha-2)\binom{\beta+\gamma}{n-1} + (\beta-1)\binom{\alpha+\beta-1}{n-1} + (\gamma-1)\binom{\alpha+\gamma-1}{n-1} + \binom{\alpha+\beta-3}{n-1}$$
$$+\binom{\alpha+\gamma-3}{n-1} - (\alpha-4)\binom{\beta+\gamma-2}{n-1} - (\beta+\gamma-2)\binom{\alpha-2}{n-1}. \qquad \ldots(3.1.2)$$

$$C_n(G',3) = (\alpha-2)\binom{\beta+\gamma-2}{n-1} + (\beta+\gamma-2)\binom{\alpha-2}{n-1}. \qquad \ldots(3.1.3)$$

Proof: From (1.5) , we get (3.1.1) , and from Fig. 3.1 , we have

$$C_n(G',3) = (\alpha-2)\binom{\beta+\gamma-2}{n-1} + (\beta+\gamma-2)\binom{\alpha-2}{n-1}.$$

Since $\operatorname{diam}_n G' \le 3$, then,

$$C_n(G',2) + C_n(G',3) = \sum_{v\in V(G)} \binom{p-1-\deg v}{n-1} = (\alpha-2)\binom{\beta+\gamma}{n-1} + (\beta-1)\binom{\alpha+\beta-1}{n-1}$$
$$+ (\gamma-1)\binom{\alpha+\gamma-1}{n-1} + 2\binom{\beta+\gamma-2}{n-1} + \binom{\alpha+\beta-3}{n-1} + \binom{\alpha+\gamma-3}{n-1}$$

This completes the proof. ■

Corollary 3.2: : For $2 \le n \le p = \alpha+\beta+\gamma$, $\alpha \ge 2$ and $\beta,\gamma \ge 2$, we have

$$W_n(G') = p\binom{p-1}{n-1} + (\alpha-2)\binom{\beta+\gamma}{n-1} + (\beta-1)\binom{\alpha+\beta-1}{n-1} + (\gamma-1)\binom{\alpha+\gamma-1}{n-1}$$
$$+ \binom{\alpha+\beta-3}{n-1} + \binom{\alpha+\gamma-3}{n-1} + \alpha\binom{\beta+\gamma-2}{n-1} + (\beta+\gamma-2)\binom{\alpha-2}{n-1}. \quad ■$$

From Proposition 3.1 and (1.10) , we get the next corollary.

Corollary 3.3: For $\alpha,\beta,\gamma \ge 2$, we have :

$$H(G';x) = (\alpha+\beta+\gamma) + \frac{1}{2}[\alpha(\alpha-1)+2\beta\gamma+8]x + \frac{1}{2}[4(\alpha-4)+\beta(\beta+3)+\gamma(\gamma+3)]x^2$$
$$+ (\alpha-2)(\beta+\gamma-2)x^3 .$$

And, Wiener index of G' is $W(G') = \frac{1}{2}\alpha(\alpha-5)+\beta(\beta-3)+\gamma(\gamma-3)+3\alpha(\beta+\gamma)+\beta\gamma$ ■

4. The Composite Graph $K_\alpha \boxtimes W_\beta$:

Let K_α , $\alpha \ge 2$, and W_β, $\beta \ge 4$ be complete and wheel graphs respectively , then the composite graph $K_\alpha \boxtimes W_\beta$ has order $p = \alpha+\beta$, $q = \frac{1}{2}\{\alpha(\alpha-1)+4(\beta+1)\}$, and diameter 4 , for $\beta \ge 6$ and $\alpha \ge 3$.

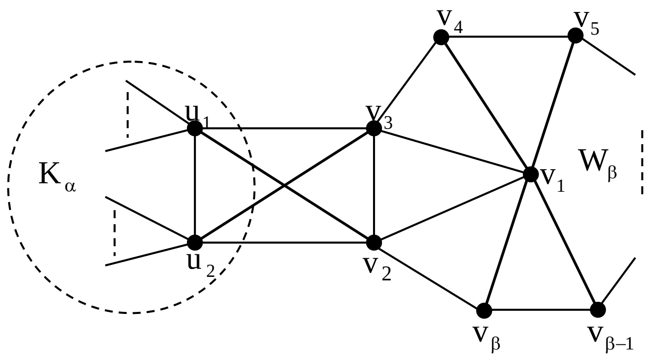

Fig.4.1 $K_\alpha \boxtimes W_\beta$

We denote $K_\alpha \boxtimes W_\beta$ by G''. From Fig.4.1 , with $\alpha \geq 3,\ \beta \geq 6$ we notice that :

$$\operatorname{diam}_n G'' = \begin{cases} 4, & \text{if } 2 \leq n \leq \max\{\alpha-1, \beta-4\}, \\ 3, & \text{if } \max\{\alpha, \beta-3\} \leq n \leq \max\{\alpha+1, \beta-1\},. \\ 2 \text{ or } 1, & \text{if } n > \max\{\alpha+1, \beta-1\}. \end{cases}$$

In the next proposition , we obtain the n-Hosoya polynomial of G'':

Proposition 4.1: For $2 \leq n \leq p = \alpha + \beta + \gamma$, $\alpha \geq 3, \beta \geq 6$, we have

$$H_n(G''; x) = p\binom{p-1}{n-2} + \sum_{k=1}^{4} C_n(G'', k)x^k ,$$

where,

$$C_n(G'',1) = p\binom{p-1}{n-1} - (\beta-3)\binom{p-4}{n-1} - 2\binom{p-6}{n-1} - \binom{\alpha}{n-1} - 2\binom{\beta-2}{n-1} - (\alpha-2)\binom{\beta}{n-1},$$

$$C_n(G'',2) = (\beta-3)\binom{p-4}{n-1} + 2\binom{p-6}{n-1} + (\alpha-2)\binom{\beta}{n-1} - (\beta-6)\binom{\alpha}{n-1} - (\alpha-4)\binom{\beta-2}{n-1} - 2\binom{\beta-5}{n-1} - 3\binom{\alpha-2}{n-1},$$

$$C_n(G'',3) = (\alpha-2)\binom{\beta-2}{n-1} + (\beta-5)\binom{\alpha}{n-1} - (\alpha-4)\binom{\beta-5}{n-1} - (\beta-8)\binom{\alpha-2}{n-1},$$

$$C_n(G'',4) = (\alpha-2)\binom{\beta-5}{n-1} + (\beta-5)\binom{\alpha-2}{n-1}.$$

Proof: From (1.5), we get $C_n(G'',1)$. To find the coefficient $C_n(G'',2)$, we , first find $C_n(G'',k)$, $k = 3, 4$.

For k=3 , there are three vertices namely , v_1, v_4, v_β of distance 3 from u_i , $3 \leq i \leq \alpha$, and there are $\beta - 5$ vertices of distance more than 3 from u_i. Hence, by (1.11)

$$C_n(u_i, G'', 3) = \binom{\beta-2}{n-1} - \binom{\beta-5}{n-1} ,\ 3 \leq i \leq \alpha \ . \qquad \ldots(4.1.1)$$

But, there are two vertices namely u_1 and u_2 of distance 3 from v_i, $5 \leq i \leq \beta - 1$, and there are $\alpha - 2$ vertices of distance more than 3 from v_i. Hence, by (1.11)

$$C_n(v_i, G'', 3) = \binom{\alpha}{n-1} - \binom{\alpha-2}{n-1} ,\ 5 \leq i \leq \beta - 1 \ . \qquad \ldots(4.1.2)$$

Finally , there are $\beta - 5$ vertices namely , $v_5, v_6, \ \ldots \ , v_{\beta-1}$ of distance 3 from u_i , $i = 1, 2$, and there is no vertex of distance more than 3 from u_i, and there are $\alpha - 2$ vertices, namely $u_3, u_4, \ \ldots \ , u_\alpha$ of distance 3 from v_i, $i = 1, 4, \beta$, and there is no vertex of distance more than 3 from v_i. Hence,

$$C_n(u_i, G'', 3) = \binom{\beta-5}{n-1} ,\ i = 1, 2 \ , \qquad \ldots(4.1.3)$$

and,

$$C_n(v_i,G'',3)=\binom{\alpha-2}{n-1}\ ,\ i=1,4,\beta. \qquad \ldots(4.1.4)$$

Hence, from (4.1.1)-(4.1.4) we get $C_n(G'',3)$.

Now , for k=4 , there are $\beta-5$ vertices , namely $v_5, v_6, \ \ldots\ , v_{\beta-1}$ of distance 4 from u_i , $3\le i\le\alpha$, and there is no vertex of distance more than 4 from u_i , then

$$C_n(u_i,G'',4)=\binom{\beta-5}{n-1}\ ,\ 3\le i\le\alpha. \qquad \ldots(4.1.5)$$

And , there are $\alpha-2$ vertices, namely $u_3, u_4, \ \ldots\ , u_\alpha$ of distance 4 from v_i, $5\le i\le\beta-1$, and there is no vertex of distance more than 4 from v_i, then

$$C_n(v_i,G'',4)=\binom{\alpha-2}{n-1}\ ,\ 5\le i\le\beta-1. \qquad \ldots(4.1.6)$$

Hence, from (4.1.5) and (4.1.6) we get $C_n(G'',4)$.

From the relation $\sum_{k=2}^{4}C(G'',K)=\sum_{v\in V(G'')}\binom{p-1-\deg v}{n-1}$, we obtain $C_n(G'',2)$ as it is given in Proposition 4.1. ■

Remark 1:

- If $\alpha=2$ and $\beta\ge 6$, Proposition 4.1 holds with $C_n(G'',4)=0$ **.**
- If $\alpha\ge 2$ and $\beta=5$, Proposition 4.1 holds with $C_n(G'',4)=0$ **.**
- If $\alpha\ge 2$ and $\beta=4$, we have $K_\alpha \boxtimes K_4$ which is given in Proposition 2.1.

Corollary 4.2: For $2\le n\le p=\alpha+\beta+\gamma$, $\alpha\ge 3,\beta\ge 6$, we have

$$W_n(G'')=p\binom{p-1}{n-1}+(\beta-3)\binom{p-4}{n-1}+2\binom{p-6}{n-1}+(\alpha-2)\binom{\beta}{n-1}+(\beta-4)\binom{\alpha}{n-1}$$
$$+\alpha\binom{\beta-2}{n-1}+(\beta-2)\binom{\alpha-2}{n-1}+\alpha\binom{\beta-5}{n-1}. \ ■$$

From Proposition 4.1 and (1.10) , we get the next corollary.

Corollary 4.3: For, $\alpha\ge 2,\beta\ge 5$, the Hosoya polynomial of G'' of order $\alpha+\beta$ is given by:

$$H(G'';x)=(\alpha+\beta)+\frac{1}{2}[\alpha(\alpha-1)+4(\beta+1)]x+\frac{1}{2}[4(\alpha+2)+\beta(\beta-5)]x^2+[3\alpha+2\beta-16]x^3$$
$$+(\alpha-2)(\beta-5)x^4.$$

And, Wiener index of G'' is

$$W(G'')=\frac{1}{2}\alpha(\alpha-15)+\beta(\beta-5)+4\alpha\beta+2. \quad ■$$

Remark 2 : If $\alpha\ge 2$ and $\beta=4$, we have

$$H(G'';x)=(\alpha+4)+\frac{1}{2}[\alpha(\alpha-1)+20]x+2\alpha x^2+2(\alpha-2)x^3,$$

and,

$W(G'') = \frac{1}{2}\alpha(\alpha+19) - 2$.

5. The Composite Graph $K_\alpha \boxtimes C_\beta$:

Let C_β be a cycle graph of order $\beta, \beta \geq 4$ and let $v_1 v_\beta \in E(C_\beta)$ and , K_α be a complete graph of order $\alpha, \alpha \geq 2$ and let $u_1 u_2 \in E(K_\alpha)$, then the composite graph $K_\alpha \boxtimes C_\beta$ has order $p = \alpha + \beta$, size $q = \frac{1}{2}\{\alpha(\alpha-1) + 2\beta + 8\}$, and diameter $\left\lceil \frac{\beta}{2} \right\rceil + 1$, as depicted in Fig.5.1.

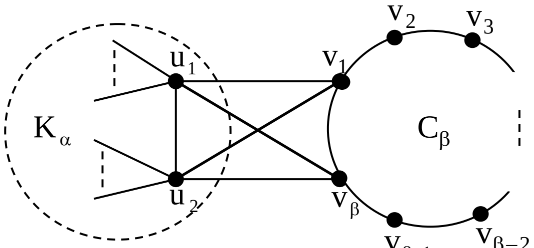

Fig.5.1. $K_\alpha \boxtimes C_\beta$

We denote $K_\alpha \boxtimes C_\beta$ by G'''. The n-diameter of G''' is determined in the following proposition:

Proposition 5.1: For $2 \leq n \leq p = \alpha + \beta + \gamma$, $\alpha \geq 3, \beta \geq 4$, then

$$\operatorname{diam}_n G''' = \begin{cases} \left\lceil \frac{\beta}{2} \right\rceil + 1, & \text{if } 2 \leq n \leq \alpha - 1, \\ \left\lceil \frac{\beta}{2} \right\rceil, & \text{if } n = \alpha \text{ or } \alpha + 1, . \\ \left\lfloor \frac{p-n}{2} \right\rfloor + 1, & \text{if } \alpha + 2 \leq n \leq p. \end{cases} \qquad \ldots(5.1.1)$$

Proof : Let S be an (n-1) - subset of V(G'''), and let w be a vertex of V(G''') , such that $d_n(w,S) = \operatorname{diam}_n G'''$. Since the n-diameter is the maximum of the n-distances $d_n(v,S)$, $v \in V(G''')$, $S \subseteq V(G''')$, $|S| = n-1$, then w must be a vertex of C_β that is furthest from $\{v_1, v_\beta\}$, that is

$$w = \begin{cases} v_{\frac{\beta}{2}} \text{ or } v_{\frac{\beta}{2}+1} , & \text{if } \beta \text{ is even}, \\ v_{\frac{\beta+1}{2}} , & \text{if } \beta \text{ is odd}, \end{cases}$$

and S consists of the first n-1 vertices from the sequence $u_3, u_4, \ldots, u_\alpha, u_1, u_2, v_1, v_\beta, v_2, v_{\beta-1}, \ldots$. Therefore :

$$d_n(w,S) = \begin{cases} \left\lceil \frac{\beta}{2} \right\rceil + 1, \text{ if } 2 \leq n \leq \alpha - 1, S \subseteq V(K_\alpha) - \{u_1, u_2\}, \\ \left\lceil \frac{\beta}{2} \right\rceil , \quad \text{if } n = \alpha \text{ or } \alpha + 1, S \subseteq V(K_\alpha). \end{cases} \qquad \ldots(5.1.2)$$

Since $diam_n C_\beta = \left\lfloor \frac{\beta - n}{2} \right\rfloor + 1$, for $2 \le n \le \beta$, [1], then

$d_n(w,S) = \left\lfloor \frac{\beta - (n-\alpha)}{2} \right\rfloor + 1$, if $\alpha + 2 \le n \le \alpha + \beta$, for which

$$S = V(K_\alpha) \cup \{V(C_\beta) - \{w\}\}. \qquad ...(5.1.3)$$

From (5.1.2) and (5.1.3), we get (5.1.1). ■

To find the coefficients of the n-Hosoya polynomial of the composite graph G''', we denote $V(K_\alpha)$ by U, and $V(C_\beta)$ by V , and notice, for $2 \le k \le diam_n G'''$, that $C_n(G''',k) = C_n(U,G''',k) + C_n(V,G''',k)$.

In the next lemmas , we obtain the coefficients of the n-Hosoya polynomial of G'''.

Lemma 5.2: For $3 \le n \le \beta + 1$, $\alpha \ge 3, \beta \ge 4$, and $2 \le k \le \left\lfloor \frac{\beta - n + 1}{2} \right\rfloor + 2$,we have

$$C_n(U,G''',k) = \begin{cases} (\alpha-2)\binom{\beta-2k+4}{n-1} - (\alpha-4)\binom{\beta-2k+2}{n-1} - 2\binom{\beta-2k}{n-1}, & \text{if } 2 \le k \le \left\lfloor \frac{\beta-n+1}{2} \right\rfloor + 1 \text{ , } n < \beta \\ (\alpha-2)r & \text{, if } k = \left\lfloor \frac{\beta-n+1}{2} \right\rfloor + 2 \text{ ,} \end{cases} \qquad \ldots(5.2.1)$$

where $r = \begin{cases} 1 \text{ , if } \beta - n + 1 \text{ is even ,} \\ n \text{ , if } \beta - n + 1 \text{ is odd .} \end{cases}$

Proof: For $3 \le n \le \beta + 1$ and $2 \le k \le \left\lfloor \frac{\beta - n + 1}{2} \right\rfloor + 2$, there are two vertices , namely v_{k-1} and $v_{\beta-k+2}$ of distance k from u_i, $3 \le i \le \alpha$, and there are $\beta - 2k + 2$ vertices of distance more than k from u_i. Thus, by (1.11)

$$C_n(u_i,G''',k) = \binom{\beta-2k+4}{n-1} - \binom{\beta-2k+2}{n-1}, \text{ for } 3 \le i \le \alpha . \qquad \ldots(5.2.2)$$

And, for $3 \le n \le \beta - 1$ and $2 \le k \le \left\lfloor \frac{\beta - n + 1}{2} \right\rfloor + 1$, there are two vertices , namely v_k and $v_{\beta-k+1}$ of distance k from u_i, $i = 1,2$, and there are $\beta - 2k$ vertices of distance more than k from u_i. Thus, by (1.11)

$$C_n(u_i,G''',k) = \binom{\beta-2k+2}{n-1} - \binom{\beta-2k}{n-1}, \text{ for } i = 1,2 . \qquad \ldots(5.2.3)$$

Moreover , if $n = \beta$ or $\beta + 1$, then $k = 1$.

From (5.2.2) and (5.2.3), we get (5.2.1) . ■

We note that (5.2.1) is not satisfied for n = 2 , therefore we can obtain $C_2(U,G''',k)$ from Fig. 5.1 , in the next remark:

Remark I: For $n = 2$, $\alpha \ge 3$, $\beta \ge 4$, then

$C_2(U,G''',k) = 2(\alpha - 2) + 4$, if $2 \le k \le \left\lfloor \frac{\beta - 1}{2} \right\rfloor$,

$C_2(U,G''',\left\lfloor\frac{\beta-1}{2}\right\rfloor+1)=2(\alpha-2)+2h$, $C_2(U,G''',\left\lfloor\frac{\beta-1}{2}\right\rfloor+2)=h(\alpha-2)$,

where $h=\begin{cases}2 \text{ , if } \beta \text{ is even ,}\\ 1 \text{ , if } \beta \text{ is odd .}\end{cases}$

Remark II: For $n\geq\beta+2$, $k\geq 2$, $C_n(U,G''',k)=0$.

Lemma 5.3: For $3\leq n\leq p=\alpha+\beta$, $\alpha\geq 3,\beta\geq 5$, we have

$$C_n(V,G''',2)=(\beta-2)\binom{p-3}{n-1}-(\beta-6)\binom{p-5}{n-1}-2\left[\binom{p-7}{n-1}+\binom{\beta-5}{n-1}\right]. \qquad ...(5.3.1)$$

Proof: If $v\in V(C_\beta)$, $S\subseteq V(G''')$, $|S|=n-1$, and $d_2(v,S)=2$ then there are three cases:

Case 1: There are α vertices , namely $u_3,u_4,\ ...\ ,u_\alpha,v_3,v_{\beta-1}$, of distance 2 from vertex v_1, and there are $\beta-5$ vertices of distance more than 2 from v_1. Thus, by (1.11)

$$C_n(v_1,G''',2)=\binom{p-5}{n-1}-\binom{\beta-5}{n-1}.$$

Since, $C_n(v_1,G''',2)=C_n(v_\beta,G''',2)$,(by symmetry), then we have

$$C_n(V_1,G''',2)=2\left[\binom{p-5}{n-1}-\binom{\beta-5}{n-1}\right], \text{ where } V_1=\{v_1,v_\beta\}. \qquad ...(5.3.2)$$

Case 2: There are four vertices , namely u_1,u_2,v_4,v_β,of distance 2 from vertex v_2, and there are $p-7$ vertices of distance more than 2 from v_2. Thus, by (1.11)

$$C_n(v_2,G''',2)=\binom{p-3}{n-1}-\binom{p-7}{n-1}.$$

Since, $C_n(v_2,G''',2)=C_n(v_{\beta-1},G''',2)$, (by symmetry), then we have

$$C_n(V_2,G''',2)=2\left[\binom{p-3}{n-1}-\binom{p-7}{n-1}\right], \text{ where } V_2=\{v_2,v_{\beta-1}\}. \qquad ...(5.3.3)$$

Case 3: There are two vertices , namely v_{i-2},v_{i+2},of distance 2 from vertex v_i, $i=3,4,\ ...\ ,\beta-2$, and there are $p-5$ vertices of distance more than 2 from v_i. Thus, by (1.11)

$$C_n(v_i,G''',k)=\binom{p-3}{n-1}-\binom{p-5}{n-1},\ 3\leq i\leq\beta-2 \text{ , then}$$

$$C_n(V_3,G''',k)=(\beta-4)\left[\binom{p-3}{n-1}-\binom{p-5}{n-1}\right], V_3=\{v_i\ :i=3,4,\ ...\ ,\beta-2\}. \qquad ...(5.3.4)$$

Then, from (5.3.2) ,(5.3.3) , and (5.3.4) we get (5.3.1). ■

From Lemma 5.3 , we note that (5.3.1) is satisfied when n = 2 , that is

$$C_2(V,G''',2)=(\beta-2)(p-3)-(\beta-6)(p-5)-2[p-7+\beta-5]=2(\alpha+\beta). \qquad ...(5.3.5)$$

Lemma 5.4: For $3\leq n\leq p=\alpha+\beta$, $\alpha\geq 3,\beta\geq 7$, and for $3\leq k\leq\left\lceil\frac{\beta}{2}\right\rceil-1$, we have

$$C_n(V,G''',k)=2(k-2)\binom{\beta-2k+1}{n-1}-2(k-1)\binom{\beta-2k-1}{n-1}-(\beta-2k-2)\binom{p-2k-1}{n-1}$$
$$+(\beta-2k+2)\binom{p-2k+1}{n-1}-2\binom{p-2k-3}{n-1}. \qquad ...(5.4.1)$$

Proof: For $3\le k\le\left\lceil\frac{\beta}{2}\right\rceil-1$, there are four cases for partitioning $V(C_\beta)$ corresponding to such values of k .

Case I: There are two vertices , namely $v_{k+i}, v_{\beta-k+i}$,of distance k from vertex v_i, $i=1,2,\ \ldots\ ,k-2$, and there are $\beta-2k-1$ vertices of distance more than k from v_i. Thus by (1.11)

$$C_n(v_i,G''',k)=.\binom{\beta-2k+1}{n-1}-\binom{\beta-2k-1}{n-1},\ i=1,2,\ \ldots\ ,k-2.$$

Since, $C_n(v_i,G''',k)=C_n(v_{\beta-i+1},G''',k)$, $i=1,2,\ \ldots\ ,k-2$ (by symmetry), for $2\le k\le\left\lceil\frac{\beta}{2}\right\rceil-1$, then we have

$$C_n(V_I,G''',k)=2(k-2)\left[\binom{\beta-2k+1}{n-1}-\binom{\beta-2k-1}{n-1}\right],$$
$$V_I=\{v_i,v_{\beta-i+1}\ :i=1,2,\ \ldots\ ,k-2\} \qquad ...(5.4.2)$$

Case II: There are α vertices , namely $u_3,u_4,\ \ldots\ ,u_\alpha,v_{2k-1},v_{\beta-1}$, of distance k from vertex v_{k-1}, and there are $\beta-2k-1$ vertices of distance more than k from v_{k-1}. Thus, by (1.11)

$$C_n(v_{k-1},G''',k)=\binom{p-2k-1}{n-1}-\binom{\beta-2k-1}{n-1}.$$

Since, $C_n(v_{k-1},G''',k)=C_n(v_{\beta-k+2},G''',k)$, for $2\le k\le\left\lceil\frac{\beta}{2}\right\rceil-1$, we have

$$C_n(V_{II},G''',k)=2\left[\binom{p-2k-1}{n-1}-\binom{\beta-2k-1}{n-1}\right],V_{II}=\{v_{k-1},v_{\beta-k+2}\}. \qquad \ldots(5.4.3)$$

Case III: There are four vertices , namely u_1,u_2,v_{2k},v_β ,of distance k from vertex v_k, and there are $p-2k-3$ vertices of distance more than k from v_k. Thus, by (1.11)

$$C_n(v_k,G''',k)=\binom{p-2k+1}{n-1}-\binom{p-2k-3}{n-1}.$$

Since, $C_n(v_k,G''',k)=C_n(v_{\beta-k+1},G''',k)$, for $2\le k\le\left\lceil\frac{\beta}{2}\right\rceil-1$, we have

$$C_n(V_{III},G''',k)=2\left[\binom{p-2k+1}{n-1}-\binom{p-2k-3}{n-1}\right],V_{III}=\{v_k,v_{\beta-k+1}\}. \qquad \ldots(5.4.4)$$

Case IV: There are two vertices , namely v_{i-k},v_{i+k} ,of distance k from vertex v_i, $i=k+1,k+2,\ \ldots\ ,\beta-k$, and there are $p-2k-1$ vertices of distance more than k from v_i. Thus, by (1.11)

$$C_n(v_i,G''',k)=\binom{p-2k+1}{n-1}-\binom{p-2k-1}{n-1},\ k+1\le i\le \beta-k\ .$$

Then,

$$C_n(V_{IV},G''',k)=(\beta-2k)\left[\binom{p-2k+1}{n-1}-\binom{p-2k-1}{n-1}\right],$$

$$V_{IV}=\{v_i\ :i=k+1,k+2,\ \ldots\ ,\beta-k\} \qquad \ldots(5.4.5)$$

From (5.4.2) - (5.4.5) we get (5.4.1) . ■

Also , from Lemma 5.4 , we note that (5.4.1) is satisfied when n = 2, that is

$$C_2(V,G''',k)=2(k-2)(\beta-2k+1)-2(k-1)(\beta-2k-1)-(\beta-2k-2)(p-2k-1)$$
$$+(\beta-2k+2)(p-2k+1)-2(p-2k-3)=2(\alpha+\beta)\ ,\text{for } 3\le k\le\left\lceil\frac{\beta}{2}\right\rceil-1. \quad \ldots(5.4.6)$$

Lemma 5.5: For $3\le n\le p=\alpha+\beta$, $\alpha\ge 3,\beta\ge 5$, then

$$C_n(V,G''',\left\lceil\frac{\beta}{2}\right\rceil)=\begin{cases}2\left[\binom{\alpha+1}{n-1}+\binom{\alpha-2}{n-2}\right], & \text{if } \beta \text{ is even}\\ \binom{\alpha}{n-1}+\binom{\alpha-2}{n-1}, & \text{if } \beta \text{ is odd}\end{cases}.$$

Proof:(i) If β is even, then there are $\alpha-1$ vertices , namely $u_3,u_4,\ \ldots\ ,u_\alpha\ ,v_{\beta-1}$ of distance $\frac{\beta}{2}$, from vertex $v_{\frac{\beta}{2}-1}$, and there is no vertex of distance more than $\frac{\beta}{2}$ from $v_{\frac{\beta}{2}-1}$. Then,

$$C_n(v_{\frac{\beta}{2}-1},G''',\frac{\beta}{2})=\binom{\alpha-1}{n-1}, \qquad \ldots(5.5.1)$$

And there are three vertices , namely u_1,u_2,v_β, of distance $\frac{\beta}{2}$ from vertex $v_{\frac{\beta}{2}}$, and there are $\alpha-2$ vertices of distance more than $\frac{\beta}{2}$, from $v_{\frac{\beta}{2}}$.Then, by (1.11)

$$C_n(v_{\frac{\beta}{2}},G''',\frac{\beta}{2})=\binom{\alpha+1}{n-1}-\binom{\alpha-2}{n-1}. \qquad \ldots(5.5.2)$$

It is clear that $C_n(v_{\frac{\beta}{2}-r},G''',\frac{\beta}{2})=C_n(v_{\frac{\beta}{2}+r+1},G''',\frac{\beta}{2})$. r=0,1 .

Moreover, one may easily check that

$$C_n(w,G''',\frac{\beta}{2})=0\ ,\text{ for } w\in V(C_\beta)-\{v_{\frac{\beta}{2}-1},v_{\frac{\beta}{2}},v_{\frac{\beta}{2}+1},v_{\frac{\beta}{2}+2}\}. \qquad \ldots(5.5.3)$$

Therefore, from (5.5.1) and (5.5.2) we get

$$C_n(V, G''', \frac{\beta}{2}) = 2\left[\binom{\alpha+1}{n-1} + \binom{\alpha-2}{n-2}\right]. \qquad \ldots(5.5.4)$$

(ii). If β is odd, then there are $\alpha-2$ vertices , namely $u_3, u_4, \ \ldots \ , u_\alpha$, of distance $k = \frac{\beta+1}{2}$ from vertex $v \in \{v_{\frac{\beta-1}{2}}, v_{\frac{\beta+3}{2}}\}$, and there is no vertex of distance more than $k = \frac{\beta+1}{2}$ from v. Then,

$$C_n(v, G''', \frac{\beta+1}{2}) = \binom{\alpha-2}{n-1}, \ v \in \{v_{\frac{\beta-1}{2}}, v_{\frac{\beta+3}{2}}\}. \qquad \ldots(5.5.5)$$

And, there are two vertices , namely u_1 and u_2 , of distance $\frac{\beta+1}{2}$ from vertex $v_{\frac{\beta+1}{2}}$, and there are $\alpha-2$ vertices of distance more than $\frac{\beta+1}{2}$ from $v_{\frac{\beta+1}{2}}$. Then,

by (1.11)

$$C_n(v_{\frac{\beta+1}{2}}, G''', \frac{\beta+1}{2}) = \binom{\alpha}{n-1} - \binom{\alpha-2}{n-1}. \qquad \ldots(5.5.6)$$

Moreover, $C_n(w, G''', \frac{\beta+1}{2}) = 0$, for $w \in V(C_\beta) - \{v_{\frac{\beta-1}{2}}, v_{\frac{\beta+1}{2}}, v_{\frac{\beta+3}{2}}\}$.

Hence, from (6.5.5) ,(6.5.6) , we get :

$$C_n(V, G''', \frac{\beta+1}{2}) = \binom{\alpha}{n-1} + \binom{\alpha-2}{n-1}.$$

This completes the proof . ■

From Lemma 5.5 , we get

$$C_2(V, G''', \left\lceil\frac{\beta}{2}\right\rceil) = \begin{cases} 2(\alpha+2)+\beta-4 \ , & \text{if } \beta \text{ is even} \\ 2\alpha-2 \ , & \text{if } \beta \text{ is odd} \end{cases}. \qquad \ldots(5.5.7)$$

Since, $C_n(w, G''', \frac{\beta}{2}) = 1$, for $w \in V(C_\beta) - \{v_{\frac{\beta}{2}-1}, v_{\frac{\beta}{2}}, v_{\frac{\beta}{2}+1}, v_{\frac{\beta}{2}+2}\}$ in (5.5.3) when $n = 2$.

Lemma 5.6: For $3 \le n \le p = \alpha+\beta$, $\alpha \ge 3, \beta \ge 5$, we have

$$C_n(V, G''', \left\lceil\frac{\beta}{2}\right\rceil + 1) = \begin{cases} 2\binom{\alpha-2}{n-1} \ , & \text{if } \beta \text{ is even}, \\ \binom{\alpha-2}{n-1} \ , & \text{if } \beta \text{ is odd}. \end{cases} \qquad \ldots(5.6.1)$$

Proof: To simplify the notations, let $m = \left\lceil\frac{\beta}{2}\right\rceil$ .Then, for k=m+1, there are $\alpha-2$ vertices , namely $u_3, u_4, \ \ldots \ , u_\alpha$, of distance m+1 from vertex v_m, and there is no vertex of distance more than m+1
from v_m. Then,

$C_n(v_m, G''', m+1) = \binom{\alpha-2}{n-1}$.

Since, $C_n(v_m, G''', m+1) = C_n(v_{\beta-m+1}, G''', m+1)$, this is for even β only $(\because v_{\beta-m+1} \equiv v_{m+1})$, but for odd β , $v_{\beta-m+1} \equiv v_m$

Moreover, $C_n(w, G''', m+1) = 0$, for $w \in V(C_\beta) - \{v_m, v_{\beta-m+1}\}$.

Hence,

$$C_n(V, G''', \left\lceil \frac{\beta}{2} \right\rceil + 1) = r' \binom{\alpha-2}{n-1}, \text{ where } r' = \begin{cases} 2 & \text{if } \beta \text{ is even} \\ 1 & \text{if } \beta \text{ is odd} \end{cases}.$$

This completes the proof . ■

From Lemma 5.6 , we note that (5.6.1) is satisfied when $n = 2$, that is

$$C_2(V, G''', \left\lceil \frac{\beta}{2} \right\rceil + 1) = \begin{cases} 2(\alpha-2) \ , & \text{if } \beta \text{ is even}, \\ \alpha - 2 \ , & \text{if } \beta \text{ is odd}. \end{cases} \qquad \ldots(5.6.2)$$

Theorem 5.7: For $3 \le n \le p = \alpha + \beta + \gamma$, $\alpha \ge 3$, $\beta, \ge 5$, and $2 \le k \le \delta_n = \text{diam}_n G'''$ we have

$$H_n(G''; x) = p\binom{p-1}{n-2} + \left[p\binom{p-1}{n-1} - (\alpha-2)\binom{\beta}{n-1} - 2\binom{\beta-2}{n-1} - 2\binom{p-5}{n-1} - (\beta-2)\binom{p-3}{n-1} \right] x$$
$$+ \sum_{k=2}^{\delta_n} C_n(G''', k) x^k ,$$

and,

$$W_n(G'') = \left[p\binom{p-1}{n-1} - (\alpha-2)\binom{\beta}{n-1} - 2\binom{\beta-2}{n-1} - 2\binom{p-5}{n-1} - (\beta-2)\binom{p-3}{n-1} \right]$$
$$+ \sum_{k=2}^{\delta_n} k C_n(G''', k) ,$$

where, $C_n(G''', k) = C_n(U, G''', k) + C_n(V, G''', k)$, for $2 \le k \le \delta_n$, and $C_n(U, G''', k)$, $C_n(V, G''', k)$ are given in Lemmas 5.2- 5.6 . ■

Remark III: If $\beta = 4$, $\alpha \ge 3$, then,

$$H_n(G'''; x) = (\alpha+4)\binom{\alpha+3}{n-2} + \left[(\alpha+4)\binom{\alpha+3}{n-1} - (\alpha-2)\binom{4}{n-1} - 2\binom{2}{n-1} - 2\binom{\alpha-1}{n-1} - 2\binom{\alpha+1}{n-1} \right] x$$
$$+ \left[2\binom{\alpha+1}{n-1} + (\alpha-2)\binom{4}{n-1} - (\alpha-4)\binom{2}{n-1} + 2\binom{\alpha-2}{n-2} \right] x^2 + \left[(\alpha-2)\binom{2}{n-1} + 2\binom{\alpha-2}{n-1} \right] x^3 .$$

Remark IV: From (5.3.5) , (5.4.6) , (5.5.7) and (5.6.2), we get

$$C_2(V, G''', k) = 2(\alpha+\beta) , \qquad \text{if } 2 \le k \le \left\lceil \frac{\beta}{2} \right\rceil - 1$$

$$C_2(V, G''', \left\lceil \frac{\beta}{2} \right\rceil) = \begin{cases} 2(\alpha+2) + \beta - 4 \ , & \text{if } \beta \text{ is even} \\ 2\alpha - 2 \ , & \text{if } \beta \text{ is odd} \end{cases}$$

$$C_2(V,G''',\left\lceil \frac{\beta}{2} \right\rceil+1)=\begin{cases} 2(\alpha-2) \ , & \text{if } \beta \text{ is even}, \\ \alpha-2 \ , & \text{if } \beta \text{ is odd}. \end{cases}$$

Since, $C(G''',k)=\frac{1}{2}[C_2(U,G''',k)+C_2(V,G''',k)]$, for $2\le k\le \left\lceil \frac{\beta}{2} \right\rceil+1$, where, $C_2(U,G''',k)$ and $C_2(V,G''',k)$ are given in Remarks II and IV, we get the next corollary.

Corollary 5.8: For, $\alpha\ge 3, \beta\ge 5$, the Hosoya polynomial of G'' of order $p=\alpha+\beta$ is given by:

$$H(G''';x)=(\alpha+\beta)+\frac{1}{2}[\alpha(\alpha-1)+2\beta+8]x+(2\alpha+\beta)\sum_{k=2}^{\lceil \beta/2\rceil-1} x^k$$

$$+\begin{cases} (2\alpha+\beta/2)x^{\beta/2}+2(\alpha-2)x^{\beta/2+1} \ , & \text{if } \beta \text{ is even} \\ 2(\alpha-1)x^{(\beta+1)/2}+(\alpha-2)x^{(\beta+1)/2+1} \ , & \text{if } \beta \text{ is odd} \end{cases}.$$

And, Wiener index of G''' is

$$W(G''')=\frac{1}{2}\alpha(\alpha-1)+\frac{\beta}{2}(3\alpha-4)+\frac{\beta^2}{4}(\alpha+\frac{\beta}{2}), \text{ if } \beta \text{ is even},$$

$$W(G''')=\frac{1}{2}\alpha(\alpha-\frac{1}{2})+\frac{\beta}{2}(3\alpha-\frac{17}{4})+\frac{\beta^2}{4}(\alpha+\frac{\beta}{2}), \text{ if } \beta \text{ is odd}. \ \blacksquare$$

Remark III: If $\beta=4$, $\alpha\ge 3$, then,

- $H(G''';x)=(\alpha+4)+\frac{1}{2}[\alpha(\alpha-1)+16]x+(2\alpha+2)x^2+2(\alpha-2)x^3$,
- $W(G''')=\frac{1}{2}\alpha(\alpha+19)$.

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