An Application of He's Variational Iteration Method for Solving Duffing - Van Der Pol Equation

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ABSTRACT

In this paper, we apply He's variational iteration method (VIM) and the Adomian decomposition method (ADM) to approximate the solution of Duffing-Van Der Pol equation (DVP). In VIM, a correction functional is constructed by a general Lagrange multiplier which can be identified via a variational theory. The VIM yields an approximate solution in the form of a quickly convergent series. Comparisons of the two series solutions with the classical Runge-Kutta order four RK45 method show that the VIM is a powerful method for the solution of nonlinear equations. The convergent of He's variational iteration method to this equation is also considered.

Keywords: He's variational iteration method, Adomian decomposition method, Duffing-Van Der Pol equation, Runge-Kutta order four, approximate solution.

تطبيق طريقة التغايير التكرارية لحل معادلة Duffing - Van Der Pol

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الملخص

تم في هذا البحث تطبيق طريقتين الـ VIM و الـ ADM لإيجاد الحل التقريبي لمعادلة Duffing-Vav Der Pol. في طريقة الـ VIM تعتمد دالـة التصـحيح بصـورة عامـة علـى مضـروب لاكرانج الـذي يـتم إيجـاده باستخدام نظرية التغاير. المقارنة العددية للحلول المتمثلة بشكل متسلسلة تمت مقارنتها مع طريقة RK45 القياسية وقد أثبتت طريقة الـ VIM كفاءتها في حل هذه النوعية من المعادلات. هذا البحث أيضا تضمن برهان التقارب لطريقة VIM المستخدمة لحل معادلة Duffing Van Der Pol.

الكلمات المفتاحية: طريقة التغايير التكرارية,طريقة تحليل ادومـاين, معادلـة Duffing-Van Der Pol, طريقـة رانج–كوتا من الرتبة الرابعة, الحل التقريبي.

1. Introduction

Chaotic systems have received a flurry of research effort in the past few decades. Such systems which are nonlinear by nature, can occur in various natural and manmade systems, and are characterized by a great sensitivity to initial conditions [16].

The Duffing .Van der Pol equation provides an important mathematical model for dynamic systems having a single unstable fixed point, along with a single stable limit cycle and is governed by the non-linear differential equation

$$u'' - \mu(1-u^2)u' + u + \beta u^3 = 0, \quad t > 0 \qquad \ldots(1)$$

with the initial conditions

$$u(0) = 1 \ , \ u'(0) = 0 \qquad \ldots(2)$$

where, the over dot represents the derivative with respect to time, μ and β which are positive coefficients.
It generates the limit cycle for small values of μ, developing into relaxation oscillations when μ becomes large which can be evaluated through the Lindsted's perturbation method [6]. Examples of such phenomena arise in all of the natural and engineering sciences [17,18], in many physical problems, as well [8,11]. Most scientific problems in solid mechanic are inherently non-linear. Except a limited number of these problems, most of them do not have analytical solution. Some of them are solved by using numerical techniques and some are done so the analytical perturbation method [19].
Recently introduced variational iteration method by He [7,12-15] which gives rapidly convergent successive approximations of the exact solution if such a solution exists, has proved successful in deriving analytical and approximate solutions of linear and nonlinear differential equations. This method is preferable over numerical methods as it is free from rounding off errors and neither requires large computer power/memory.
He [13,14,22] has applied this method for solving analytical solutions of autonomous ordinary differential equation, non-linear partial differential equations with variable coefficients and integro - differential equations. The variational iteration method was successfully applied to Burger's and coupled Burger's equations [1], to Schruodinger-KdV, generalized KdV and shallow water equations [2], to linear Helmholtz partial differential equation [9], to seventh order Sawada - Kotera equation [10], to Van der Pol–Duffing Oscillators [21], Linear and nonlinear wave equations, KdV, K(2,2), Burgers, and cubic Boussinesq equations have been solved by Wazwaz [23,24] by using the variational iteration method.

In the present paper we employ VIM method for solving Duffing .Van der Pol equation.

Further, we compare the result with the given solutions by using Adomian Decomposition Method [3,4, 20] and we prove the convergence of the method.

2. Adomian Decomposition Method for Solving Duffing – Van der Pol Equation

To solve eq. (1), ADM is employed. We rewrite it in the following form

$$Au(t) = 0 \qquad \ldots(3)$$

in a real Hilbert space H, where $A = H \to H$ is either a linear or a nonlinear operator. The principle of the ADM is based on the decomposition of the non-linear operator A in the following form: $A = L + R + N$ with

$$Lu(t) = u''$$

$$Ru(t) = u - \mu u'$$

$$Nu(t) = \mu u^2 u' + \beta u^3$$

Where, L+R is linear, N non-linear, L invertible with L^{-1} as inverse defined by

$$L^{-1}u(t) = \int_0^t \int_0^s u(z)dzds$$

$$L^{-1}Ru(t) = \int_0^t \int_0^s (u(z) - \mu u'(z))dzds$$

$$L^{-1}Nu(t) = \int_0^t \int_0^s [\mu u^2(z)u'(z) + \beta u^3(z)]dzds$$

As usual in ADM the solutions of Eq. (3) can be considered to be as the sum of the following infinite series

$$u(t)=\sum_{n=0}^{\infty}u_n(t), \qquad \ldots(4)$$

From Eq. (1), we have:

$$u(t)=L^{-1}Lu(t)-L^{-1}Ru(t)-L^{-1}Nu(t) \qquad \ldots(5)$$

where,

$$L^{-1}Lu(t)=u(0)+tu'(0)$$

$$\therefore\, u(t)=u(0)+tu'(0)-L^{-1}Ru(t)-L^{-1}Nu(t) \qquad \ldots(6)$$

From which we define the following scheme

$$u_0(t)=u(0)+tu'(0)=1,$$

$$u_{n+1}(t)=-L^{-1}Ru(t)-L^{-1}A_n$$

$$=\int_0^t\int_0^s[\mu u_n(z)-u_n(z)]dzds-\int_0^t\int_0^s A_n(z)dzds \qquad n=0,1,2...... \qquad \ldots(7)$$

Where A_n are called Adomian Polynomials [3,4,5].

2.1 Algorithm (Computing Adomian Polynomials)

Input: The Equation

$$F=F(u,u',u'')$$

Set $n=N$, $m=M$, $k=K$; the input of Adomian Polynomials is needed .

Output: Aj ; the Adomian Polynomials

Step 1: set $j=1$

Step 2: while $j\le n$ do steps (3) and (4)

Step 3: $F(\lambda)=F(u_j(\lambda))$

Step 4: $F=F(\lambda)$

Step 5: s = expansion of $F(\lambda)$ w.r.t. λ

$ft=s(\lambda)$

Step 6: while $j\le k$ and while $j\le m$

$$A_j=\frac{\partial}{\partial\lambda}(ft_j)(0)=D(ft_j)(0)$$

Step 7: output Aj (the Adomian Polynomials)

Step 8: end.

2.2 Computing Adomian Polynomials for Equation (1)

Computing Adomian Polynomial by Algorithm (2.1) yields to

$$A_0(u_0)=N(u_0)=\beta$$

$$A_1(u_0,u_1)=-\mu(1+\beta)t-\frac{3}{2}\beta(1+\beta)t^2$$

$$A_2(u_0,u_1,u_2)=\frac{1}{3}\mu(1+\beta)(2+3\beta)t^3+\frac{1}{8}\beta(1+\beta)(7+9\beta)t^4$$

$$A_3(u_0,u_1,u_2,u_3) = -\frac{1}{4}\mu^2(1+\beta)^2 t^4 - \frac{1}{120}\mu(1+\beta)(61+222\beta+165\beta^2)t^5 - $$
$$-\frac{1}{240}\beta(1+\beta)((61+204\beta+147\beta^2)t^6 \quad \ldots(8)$$

$\vdots$

Now, substituting (8) in (7) yields :

$u_0(t) = 1$

$u_1(t) = -\frac{1}{2}(1+\beta)t^2$

$u_2(t) = \frac{1}{24}(1+\beta)(1+3\beta)t^3 \quad \ldots(9)$

$u_3(t) = -\frac{1}{20}\mu(1+\beta)^2 t^3 - \frac{1}{720}(1+\beta)(1+24\beta+27\beta^2)t^6$

$u_4(t) = \frac{1}{840}\mu(1+\beta)(11+34\beta+23\beta^2)t^7 + \frac{1}{40320}(1+\beta)(1+207\beta+639\beta^2+441\beta^3)t^8$

and so on …

The five terms of the approximations to the solutions are considered as

$u(t) \approx u_0 + u_1 + u_2 + u_3 + u_4$

for the convergence of the method, we refer the reader to [5] in which the problem of convergence has been discussed briefly .

3. He's Variational Iteration Method for Solving Duffing – Van der Pol Equation

To explain the basic idea of He's variational iteration method (VIM), we consider a general nonlinear oscillator with specified initial conditions (2) as follows (more general form can be considered without the loss of generality)

$F(u,u',u'') := u'' + f(u)u' + g(u,u',u'')u = 0$

and for the Eq.(1) we have

$F(u,u',u'') := u'' - \mu(1-u^2)u' + (1+\beta u^2)u = 0 \quad \ldots(10)$

where, f andare continuous nonlinear operators with respect to their arguments, g and $u(t)$ is an unknown variable. We first consider Eq. (10) as

$L[u(t)] + N[u(t)] = 0 \quad \ldots(11)$

and for Eq.(1) we have:

$L[u(t)] = u'' + w^2 u$ and $N[u(t)] = -\mu(1-u^2)u' + (1+\beta u^2)u - w^2 u \quad \ldots(12)$

$= f(u)u' + g(u,u',u'')u - w^2 u$

Where, L with the property $Lf \equiv 0$ when $f \equiv 0$ denotes the linear operator with respect to u and N is a non-linear operator with respect to u .We then construct a correction functional for Eq.(11) as [12]

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_{(t,s)}(u'' + w^2 u_n(s) + N[\tilde{u}_n(s)])ds \quad \ldots(13)$$

where, $u_0(t)$ is the initial guess and the subscript n denotes the n-th iteration, $\lambda_{(t,s)} \neq 0$ denote the Lagrange multiplier, which can be identified efficiently via the variational theory, and $\tilde{u}_n$ is considered as a restricted variation i.e. $\delta\tilde{u}_n = 0$

Taking the variation with respect to the independent variable $u_n = 0$, we notice that $\delta u_n(0) = 0.$ Afterward, we make the correction functional stationary, and we obtain $\delta u_{n+1}(t) = 0;$ therefore , we have

$$\delta u_{n+1}(t) = \delta u_n(t) + \delta \int_0^t \lambda_{(t,s)} (u''(s) + w^2 u_n(s) + N[\tilde{u}_n(s)])ds$$

$$= \delta u_n(t) + \delta\lambda u_n'(s)\big|_{s=t} - \frac{\partial \lambda}{\partial s}\delta u_n(s)\big|_{s=t} + \int_0^t (\frac{\partial^2 \lambda}{\partial s^2} + w^2\lambda)\delta u_n(s)ds \qquad \ldots(14)$$

$$= (1 - \frac{\partial \lambda}{\partial s})\delta u_n(s)\big|_{s=t} + \lambda \delta u_n'(s)\big|_{s=t} + \int_0^t (\frac{\partial^2 \lambda}{\partial s^2} + w^2\lambda)\delta u_n(s)ds = 0$$

As a result, we have the following stationary conditions:

$$\lambda_{(t,s)}\big|_{s=t} = 0 \qquad \ldots(15)$$

$$\frac{\partial \lambda_{(t,s)}}{\partial s}\bigg|_{s=t} = 1 \qquad \ldots(16)$$

$$\frac{\partial^2 \lambda_{(t,s)}}{\partial s^2} + w^2 \lambda_{(t,s)} = 0 \qquad \ldots(17)$$

The Lagrange multiplier can be readily identified as

$$\because \ \frac{\partial \lambda_{(t,s)}}{\partial s}\bigg|_{s=t} = 1 \quad \Rightarrow \frac{\partial \lambda_{(t,s)}}{\partial s} = \cos w(s-t)\big|_{s=t}$$

$$\therefore \ \lambda_{(t,s)} = \frac{1}{w}\sin(s-t) \qquad \ldots(18)$$

Moreover, we have the following variational iteration formula:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda_{(t,s)} F(u_n(s), u_n'(s), u_n''(s))ds$$

$$u_{n+1}(t) = u_n(t) + \int_0^t \frac{1}{w}\sin w(s-t)[\frac{\partial^2 u_n}{\partial s^2} - \mu(1-u^2)\frac{\partial u_n}{\partial s} + (1+\beta u_n^2)u_n]ds \qquad \ldots(19)$$

Accordingly, the successive approximations $u_n(t), n \geq 0$ of VIM will be readily obtained by choosing all the above parameters as follows

$$u_0(t) = 1$$

$$u_1(t) = u_0(t) + \int_0^t \frac{1}{w}\sin w(s-t)(1+\beta)ds$$

$$= 1 - \frac{1}{w^2}(1+\beta)[1-\cos wt]$$

$$u_2(t) = u_1(t) + \int_0^t \frac{1}{w}\sin w(s-t)[\frac{\partial^2 u_n}{\partial t^2} - \mu(1-u_1^2)\frac{\partial u_1}{\partial t} + (1+\beta u_1^2)u_1]ds \qquad \ldots(20)$$

$$= 1 - \frac{1}{w^2}(1+\beta)[1-\cos wt] - [-(1+\beta)\cos wt +$$

$$+ \mu[1 - (1 - \frac{1}{w^2}(1+\beta)(1-\cos wt))^2]\frac{1}{w}(1+\beta)\sin wt +$$

$$+ (1 + \beta[1 - \frac{1}{w^2}(1+\beta)(1-\cos wt)]^2)(1 - \frac{1}{w^2}(1+\beta)(1-\cos wt)][\frac{1}{w^2} - \frac{1}{w^2}\cos wt]]$$

$\vdots$

and so on…

4. Numerical Results

The numerical approach for (9) and (20) is computed by using Matlab. We consider the following four cases

Case 1: $\mu = 2, \beta = 2, w = 0.75$, and the initial conditions $u(0) = u_0(t) = 1$

Case 2: $\mu = 5, \beta = 2, w = 0.75$, and the initial conditions $u(0) = u_0(t) = 1$

Case 3: $\mu = 10, \beta = 2, w = 0.75$, and the initial conditions $u(0) = u_0(t) = 1$

Case 4: $\mu = 10, \beta = 0.5, w = 0.75$, and the initial conditions $u(0) = u_0(t) = 1$

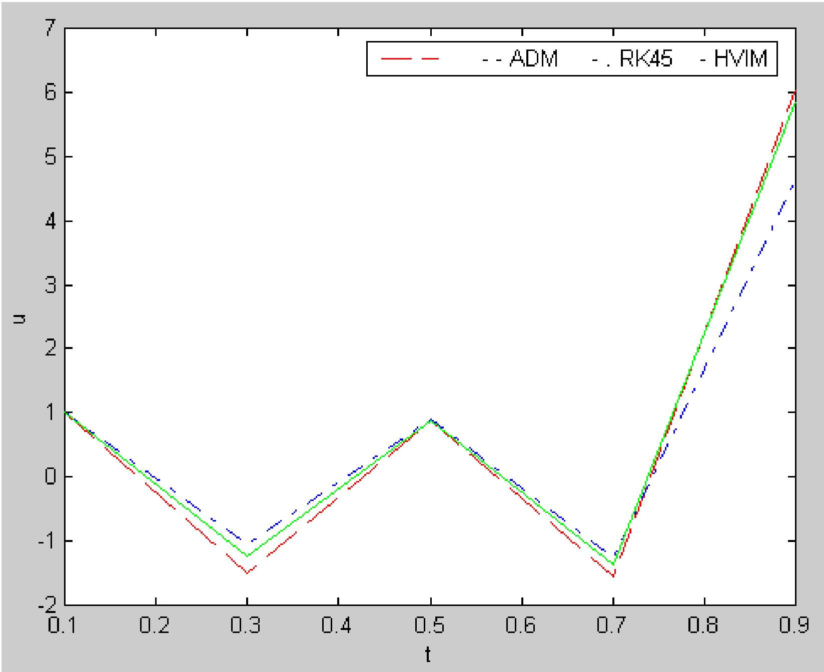

Fig. (1) Comparison of the Solution of Eq. (1) u at time t for case 1 using classical ADM, VIM and RK45.

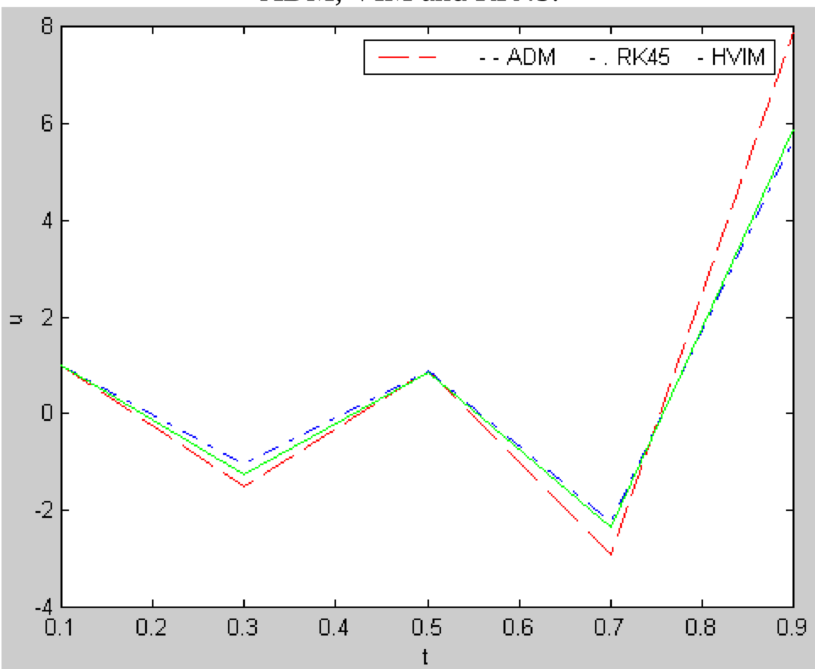

Fig. (2) Comparison of the Solution of Eq. (1) u at time t for case 2 using classical ADM, VIM and RK45.

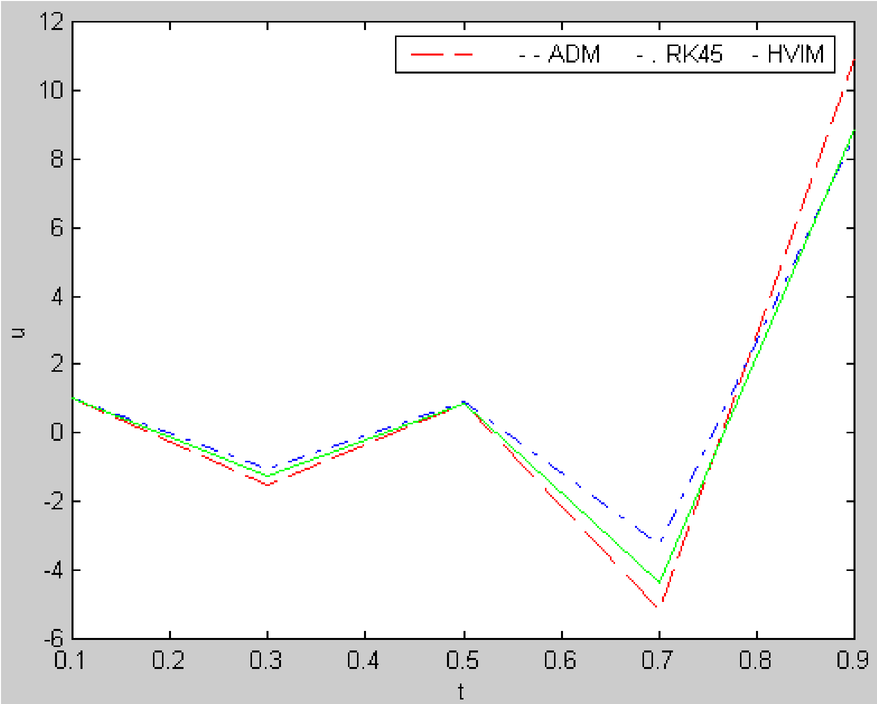

Fig. (3) Comparison of the Solution of Eq. (1) u at time t for case 3 using classical ADM, VIM and RK45.

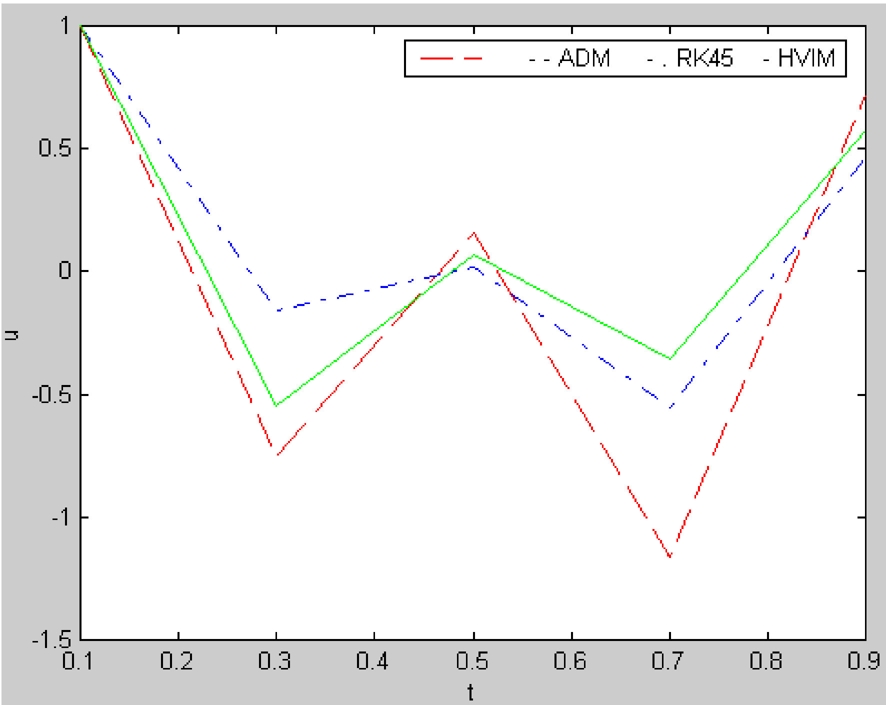

Fig. (4) Comparison of the Solution of Eq. (1) u at time t for case 4 using classical ADM, VIM and RK45.

5. Convergence Analysis:

The He's variational iteration formula makes a recurrence sequence $\{u_n(t)\}$. Obviously, the limit of the sequence will be the solution of eq.(10) if the sequence is convergent. In this section, we give a new proof of convergence of He's variational iteration method in details by introducing a new iterative formulation of this procedure. Here $C^n[0, T]$ denotes the class of all real valued functions defined on [0,T], which have continuous nth order derivatives.

Lemma(1)

If for any $n, u_n \in C^2[0,T]$, then the He's variational iteration formula Eq.(19) is equivalent to the following iterative relation

$$L[u_{n+1}(t) - u_n(t)] = -[u'' - \mu(1-u^2)u' + (1+\beta u^2)u] \quad \ldots(21)$$

Where L is as noted in (12).

Proof

Suppose u_n and u_{n+1} satisfy the variational iteration formula (19). Applying $\frac{d^2}{dt^2}$ to both sides of (19) results in

$$\frac{d^2}{dt^2}[u_{n+1}(t) - u_n(t)] = \int_0^t \frac{\partial^2}{\partial t^2}(\frac{1}{w}\sin w(s-t))[u'' - \mu(1-u^2)u' + (1+\beta u^2)u]ds +$$

$$+\frac{\partial}{\partial t}(\frac{1}{w}\sin w(s-t)\Big|_{s=t})[u'' - \mu(1-u^2)u' + (1+\beta u^2)u] +$$

$$+\frac{d}{dt}(\frac{1}{w}\sin w(s-t))[u'' - \mu(1-u^2)u' + (1+\beta u^2)u] \quad \ldots(22)$$

Now ,using the conditions (15)-(17) and $\frac{\partial}{\partial t}(\frac{1}{w}\sin(s-t))\Big|_{s=t} = -1$ we will get

$$\frac{d^2}{dt^2}[u_{n+1}(t) - u_n(t)] + w^2[u_{n+1}(t) - u_n(t)] = -[u'' - \mu(1-u^2)u' + (1+\beta u^2)u]$$

From the definition (12) of L, we obtain

$$L[u_{n+1}(t) - u_n(t)] = -[u'' - \mu(1-u^2)u' + (1+\beta u^2)u] \quad \ldots(23)$$

Conversely, suppose u_n and u_{n+1} satisfy (21). In view of the definition L and $\frac{1}{w}\sin w(s-t) \neq 0$. Multiplying eq. (21) by $\frac{1}{w}\sin w(s-t)$ and then integrating from both sides of the resulted term from 0 to t yields

$$\int_0^t \frac{1}{w}\sin w(s-t)[u''_{n+1}(s) - u''_n(s)]ds + \int_0^t w^2(\frac{1}{w}\sin w(s-t))[u_{n+1}(s) - u_n(s)]ds = -$$

$$-\int_0^t (\frac{1}{w}\sin w(s-t))[u'' - \mu(1-u^2)u' + (1+\beta u^2)u]ds \quad \ldots(24)$$

Using integration by part, the expression (24) becomes

$$\frac{1}{w}\sin w(s-t)\Big|_{s=t}[u'_{n+1}(t) - u'_n(t)] - \frac{\partial}{\partial s}(\frac{1}{w}\sin w(s-t)\Big|_{s=t}[u_{n+1}(t) - u_n(t)] +$$

$$+\int_0^t [\frac{\partial^2}{\partial s^2}(\frac{1}{w}\sin w(s-t)) + w^2\frac{1}{w}\sin w(s-t)][u_{n+1}(s) - u_n(s)]ds$$

$$= -\int_0^t (\frac{1}{w}\sin w(s-t))[u'' - \mu(1-u^2)u' + (1+\beta u^2)u]ds \quad \ldots(25)$$

Which exactly results in (19) upon the conditions (15)-(17),i.e.

$$u_{n+1}(t) = u_n(t) + \int_0^t (\frac{1}{w}\sin w(s-t))[u'' - \mu(1-u^2)u' + (1+\beta u^2)u]ds \qquad \ldots(26)$$

and this ends the proof.

Theorem (1):

If the sequence $u(t) = \lim_{u\to\infty} u_n(t)$ converges, where $u_n(t)$ is produced by the variational iteration formula of Eq. (19), then it must be the solution of the equation (10)

Proof:

If the sequence $u_n(t)$ converges, we can write

$$v(t) = \lim_{n\to\infty} u_n(t) \qquad \ldots(27)$$

and it holds

$$v(t) = \lim_{n\to\infty} u_{n+1}(t) \qquad \ldots(28)$$

Using the expressions (27) and (28) and the definition of L in (12), we can easily gain

$$\lim_{n\to\infty} L[u_{n+1}(t) - u_n(t)] = L \lim_{n\to\infty}[u_{n+1}(t) - u_n(t)] = 0 \qquad \ldots(29)$$

From (29) and according to the lemma (1), we obtain

$$L \lim_{n\to\infty}[u_{n+1}(t) - u_n(t)] = -\lim_{n\to\infty}[u_n'' - \mu(1-u_n{}^2)u_n' + (1+\beta u_n{}^2)u_n] = 0 \qquad \ldots(30)$$

Which gives us

$$\lim_{n\to\infty}[u_n'' - \mu(1-u_n{}^2)u_n' + (1+\beta u_n{}^2)u_n] = 0 \qquad \ldots(31)$$

From Eq.(31) and continuity of f and g operators, it holds

$$\lim_{n\to\infty}[u_n'' - \mu(1-u_n{}^2)u_n' + (1+\beta u_n{}^2)u_n] = \lim_{n\to\infty}[u_n'' + f(u_n)u_n' + g(u_n, u_n', u_n'')u_n] \qquad \ldots(32)$$

$$= (\lim_{n\to\infty} u_n)'' + f(\lim_{n\to\infty} u_n)(\lim_{n\to\infty} u_n)' + g(\lim_{n\to\infty} u_n, (\lim_{n\to\infty} u_n)', (\lim_{n\to\infty} u_n)'')\lim_{n\to\infty} u_n$$

$$= v'' + f(v)v' + g(v, v', v'')v$$

From the equations (31) and (32), we have

$$v'' + f(v)v' + g(v, v', v'')v = 0, t \geq 0 \qquad \ldots(33)$$

On the other hand, using the specified initial conditions and the definition of the initial guess, we have

$$v(0) = \lim_{n\to\infty} u_n(0) = 1, \; since \quad u_n(0) = 1, n \geq 0 \qquad \ldots(34)$$

$$v'(0) = \lim_{n\to\infty} u_n'(0) = 0, \; since \quad u_n{}'(0) = 0, n \geq 0 \qquad \ldots(35)$$

Therefore according to the above three expressions (33),(34) and (35), v(t) must be the solution of the Eq.(10).This ends the proof.

6. Conclusions

In this work, we have given a new proof of convergence of He's Variational Iteration Method by presenting a new formulation of He's method. We have compared this method with ADM and RK45, and we can conclude that the main property of this method is in its flexibility and ability to solve Duffing –Van Der Pol accurately and conveniently without decomposing the non-linear terms, which are very complex. This technique gives an accurate and easy computable solution by means of a truncated series whose convergence is fast.

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