

Solution of Nonlinear 2nd Order Multi-Point BVP By Semi-Analytic Technique

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ABSTRACT

In this paper, we present new algorithm for the solution of the nonlinear second order multi-point boundary value problem with suitable multi boundary conditions. The algorithm is based on the semi-analytic technique and the solutions are calculated in the form of a rapid convergent series. It is observed that the method gives more realistic series of solutions that converge very rapidly in physical problems. Illustrative examples are provided to demonstrate the efficiency and simplicity of the proposed method in solving this type of multipoint boundary value problems.

Keywords: Differential Equation, Multi-point Boundary Value Problem.

حل مسألة غير خطية من الرتبة الثانية ذات شروط حدودية متعددة بالنقاط باستخدام التقنية شبه التحليلية

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المخلص

في هذا البحث، نعرض خوارزمية جديدة لحل معادلات تفاضلية اعتيادية من الرتبة الثانية ذات شروط حدودية متعددة النقاط، الخوارزمية تعمل على أساس التقنية شبه التحليلية والحل تم إيجاده بصيغة متسلسلة سريعة التقارب وهذا يتضح أكثر في المسائل الفيزيائية، أيضا تناولنا بعض الأمثلة التوضيحية لإثبات الدقة والكفاءة وسهولة أداء الطريقة المقترحة في حل هذا النوع من المسائل الحدودية متعددة النقاط. الكلمات المفتاحية: المعادلات التفاضلية، مسائل قيم حدودية متعددة النقاط.

1. Introduction

Nonlinear problems, which have wide range of application in science and engineering, have usually been solved by perturbation methods. These methods have some limitations, e.g., the approximate solution involves a series of small parameters which posed difficulty since the majority of nonlinear problems have no small parameters at all. Although, appropriate choices of small parameters do lead to ideal solution while in most other cases, unsuitable choices lead to serious effects in the solutions. The semi-analytic technique employed here, is a new approach for finding solution that does not require small parameters, thus over-coming the limitations of the traditional perturbation techniques. The method was first proposed by Grundy (2003) and successfully applied by other researchers like Grundy (2003- 2007) he examined the feasibility of using two points Hermite interpolation as a systematic tool in the analysis of initial-boundary value problems for nonlinear diffusion equations. In 2005 Grundy analyses initial - boundary value problems involving nonlocal nonlinearities by using two points Hermite interpolation [1], also, in 2006 showed how two-points Hermite interpolation can be used to construct polynomial representations of solutions to some initial-boundary value problems for inviscid Proudman- Johnson equation. In 2008, Maqbool [2] used a semi-analytical method to model effective SINR spatial distribution in WiMAX networks. Also, in 2008, Debabrata [3] studied Elasto-plastic

strain analysis by a semi-analytical method. In 2009, Mohammed [4] investigated the feasibility of using osculatory interpolation to solve two points second order boundary value problems. In 2010, S. Das[5] used the efficient of homotopy perturbation approach (HPM) to solve the nonlinear second order multipoint BVP, where this problem was recently solved by Geng and Cui (2009 [6]) in a rather complicated manner by combining the HPM and variational iteration method (VIM). The elegance of [5] lies in the simplicity of the solution, using only the HPM. In 2011, Yassien [7] used semi-analytic technique for solving High order ordinary two point BVPs.

Kwong[8], studied a multiple solutions of Two and multi-point BVPs of nonlinear second order ODE as fixed points of a cone mapping. Kosmatov [9], applied a coincidence degree theorem of Mawhin to show the existence of at least one symmetric solution of the nonlinear second order multipoint BVP. Gupta [10] studied the existence of solutions for the generalized multi-point BVP in the non-resonance case.

In this paper, we use two-point osculatory interpolation, essentially, this is a generalization of interpolation by using Taylor polynomials. The idea is to approximate a function y by a polynomial P in which values of y and any number of its derivatives at given points are fitted by the corresponding function values and derivatives of P .

We are, particularly, concerned with fitting function values and derivatives at the two end points of a finite interval, say $[0, 1]$ where a useful and succinct way of writing osculatory interpolant P_{2n+1} of degree $2n + 1$ was given, for example, by Phillips [11] as:

$$P_{2n+1}(x) = \sum_{j=0}^n \{ y^{(j)}(0) q_j(x) + (-1)^j y^{(j)}(1) q_j(1-x) \} \quad \dots(1)$$

$$q_j(x) = (x^j / j!)(1-x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s = Q_j(x) / j! \quad \dots(2)$$

So that (1) with (2) satisfies:

$$y^{(j)}(0) = {}^{(j)}P_{2n+1}(0), \quad y^{(j)}(1) = {}^{(j)}P_{2n+1}(1), \quad j = 0, 1, 2, \dots, n.$$

Implying that P_{2n+1} agrees with the appropriately truncated Taylor series for y about $x = 0$ and $x = 1$. We observe that (1) can be directly written in terms of the Taylor coefficients a_i and b_i about $x = 0$ and $x = 1$ respectively, as:

$$P_{2n+1}(x) = \sum_{j=0}^n \{ a_j Q_j(x) + (-1)^j b_j Q_j(1-x) \} \quad \dots(3)$$

2. Suggested Solution of Multi-Point 2nd Order BVP's for ODE

A general form of 2nd - order BVP's is :-

$$y''(x) = f(x, y, y') \quad , \quad 0 \leq x \leq 1 \quad \dots(4a)$$

subject to the boundary condition :

$$y(0) = A, \quad y(1) = \sum_{i=0}^4 B y(\eta_i) \quad , \quad \text{where } \eta_j \in (0, 1) \quad , \quad B \in \mathbb{R} \quad \dots(4b)$$

The simple idea of semi - analytic method uses a two - point polynomial interpolation to replace y in problem (4) by a P_{2n+1} which enables any unknown derivatives of y to be computed, the first step is therefore, to divide the interval $[0, 1]$ into $m+1$ subinterval by η_j , i.e., $[0, \eta_1], [0, \eta_2], \dots, [0, \eta_m], [\eta_m, 1]$, then we apply a suggested method for each subinterval, as the following: firstly construct the P_{2n+1} , to do this, we need evaluate Taylor coefficients of y about $x = 0$:

$$y = \sum_{i=0}^{\infty} a_i x^i \quad \exists a_i = y^{(i)}(0) / i! \quad \dots(5a)$$

Then, insert the series form (5a) into (4a) and put $x=0$ and equate the coefficients of powers of x to obtain a_i , $i \geq 2$. Also, evaluate Taylor coefficients of y about $x = \eta_1$:

$$y = \sum_{i=0}^{\infty} c_i (x - \eta_1)^i \quad \exists c_i = y^{(i)}(\eta_1) / i! \quad \dots(5b)$$

Then, insert the series form (5b) into (4a) and put $x = \eta_1$ and equate coefficients of powers of $(x - \eta_1)$, to obtain c_i , $i \geq 2$. Also, evaluate Taylor coefficients of y about $x = \eta_i$, $i = 2, \dots, m$. Then, repeat the above step for each η_i , to obtain $y^{(i)}(\eta_i) / i!$, $i \geq 2$. Also, evaluate Taylor coefficients of y about $x = 1$:

$$y = \sum_{i=0}^{\infty} b_i (x - 1)^i \quad \exists b_i = y^{(i)}(1) / i! \quad \dots(5c)$$

Then, insert the series form (5c) into (4a) and put $x = 1$ and equate coefficients of powers of $(x-1)$, to obtain b_i , $i \geq 2$, then derive equation (4a) with respect to x to obtain a new form of equation say (6)

$$y^{(3)}(x) = \frac{df(x, y, y')}{dx} \quad \dots(6)$$

Now, insert the series form (5a) into (6) and put $x = 0$ and equate coefficients of powers of x , to obtain a_3 , also insert the series form (5b) into (6) and put $x = \eta_i$ and equate coefficients of powers of x , to obtain

$y^{(3)}(\eta_i) / 3!$, also insert the series form (5c) into (6) and put $x = 1$ and equate coefficients of powers of x , to obtain b_3 , now iterate the above process any times to obtain a_4 , $y^{(4)}(\eta_i) / 4!$, b_4 , then a_5 , $y^{(5)}(\eta_i) / 5!$, b_5 and so on, that is, we can get a_i , $y^{(i)}(\eta_i) / i!$ and b_i for all $i \geq 2$, the resulting equations can be solved by using MATLAB to obtain a_i , $y^{(i)}(\eta_i) / i!$ and b_i for all $i \geq 2$, the notation implies that the coefficients depend only on the indicated unknown a_0 , a_1 , $y(\eta_i)$, $y'(\eta_i)$, b_0 , b_1 , and we get a_0 , b_0 denoted by $y(\eta_i)$, by the boundary conditions. Then, construct a $P_{2n+1}(x)$ for each subinterval from these coefficients (a_i 's, $y^{(i)}(\eta_i) / i!$ and b_i 's) by the following :

$$P_{2n+1}(x) = \sum_{j=0}^n \{a_j Q_j(x) + (-1)^j c_j Q_j(\eta_1 - x)\} + \sum_{j=0}^n \{c_j R_j(x - \eta_1) + (-1)^j b_j R_j(\eta_2 - x)\} + \dots + \sum_{j=0}^n c_j K_j(x - \eta_m) + (-1)^j b_j K_j(1 - x) \quad \dots(7a)$$

$$\text{Where, } Q_j(x) / j! = (x^j / j!) (\eta_1 - x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} x^s \quad \dots(7b)$$

$$R_j(x) / j! = ((x - \eta_1)^j / j!) (\eta_2 - x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} (x - \eta_1)^s$$

$$K_j(x) / j! = ((x - \eta_2)^j / j!) (1 - x)^{n+1} \sum_{s=0}^{n-j} \binom{n+s}{s} (x - \eta_2)^s$$

We see that (7a) has $2m+2$ unknown coefficients a_1 , $y'(\eta_i)$, b_1 and

$$b_0 = \sum_{i=0}^4 B y(\eta_j) .$$

Now, to evaluate the remainder coefficients integrate equation (4a) on $[0, x]$ to obtain:

$$y'(x) - y'(0) - \int_0^x f(x, y, y') dx = 0$$

$$\text{i.e., } y'(x) - a_1 - \int_0^x f(x, y, y') dx = 0 \quad \dots(8a)$$

and again integrate equation (8a) on $[0, x]$ to obtain:

$$y(x) - y(0) - a_1 x - \int_0^x (1-x)f(x, y, y') dx = 0 \quad ,$$

$$\text{i.e., } y(x) - a_0 - a_1 x - \int_0^x (1-x)f(x, y, y') dx = 0 \quad \dots(8b)$$

and again integrate equation (4a) on $[\eta_1, x]$ to obtain :

$$y'(x) - y'(\eta_1) - \int_{\eta_1}^x f(x, y, y') dx = 0 \quad \dots(9a)$$

and again integrate equation (9a) on $[\eta_1, x]$ to obtain :

$$y(x) - y(\eta_1) - y'(\eta_1) x + \eta_1 y'(\eta_1) - \int_{\eta_1}^x (1-x)f(x, y, y') dx = 0 \quad \dots(9b)$$

and repeat the above steps on $[\eta_i, x]$, $i= 2, \dots, m$, use P_{2n+1} as a replacement of y, y' in (8) and put $x= \eta_i$ in equation (8a) and (8b) and put $x= \eta_2$ in equation (9a) and (9b), and so on put $x= \eta_{i+1}$, in each integration defined on $[\eta_i, x]$, $i= 2, \dots, m-1$, and put $x=1$ in the integration defined on $[\eta_m, x]$, then we have the system of $2m+2$ equations (8), (9), ... , with $2m+2$ unknown coefficients which can be solved by using the **MATLAB** package, version 7.11, to get the unknown coefficients, thus insert it into (7a), thus (7a) represents the solution of (4).

Now, we introduce some examples of 2nd order multi-point BVP's for ODE to illustrate the suggested method. Accuracy and efficiency of the suggested method are established through the comparison with other methods.

Example 1

Consider the following nonlinear, 2nd order, 6 point BVP's :

$$y''(x) + \frac{x^2(1-x)}{2} y'(x) + y^2(x) = x^3 + 2 \quad , \quad 0 \leq x \leq 1 \quad ,$$

$$\text{Subject to the BC: } y(0) = 0, y(1) = \sum_{i=0}^4 \frac{1}{1+i} y\left(\frac{i}{5}\right) + 0.708667,$$

The exact solution for this problem is: $y(x) = x^2$

Now, we solve this equation by using semi-analytic technique from equation (7) we have: $P_3 = - 0.0000002708684265 x^3 + 1.000000163 x^2 + 0.0000009564047771 x$

For more details, table (1) gives the results for different nodes in the domain, for $n = 1$, i.e. P_3 and errors obtained by comparing it with the exact solution, and Figure (1) illustrates the accuracy of the suggested method P_3

Example 2

Consider the following linear, 2nd order, 6 point BVP's:

$$y''(x) + x y(x) y'(x) - 2 y^2(x) = x^3 - x^2 + 2 \quad , \quad 0 \leq x \leq 1$$

With BC: $y(0) = 0$; $y(1) = \sum_{i=0}^4 \frac{1}{1+i} y(\frac{i}{5}) + 0.252$

The exact solution for this problem is $y(x) = x(x-1)$

Now, we solve this equation using semi - analytic method from equations (7) we have: $P_3 = - 0.0000000000000000156514 x^3 + 1.00000000000000002348 x^2 - x$.

For more details, table (2) gives the results for different nodes in the domain, for $n = 1$, i.e. P_3 and errors obtained by comparing it with the exact solution, and Figure (2) illustrates the accuracy of the suggested method P_3 . Das [5] solved this example by HPM and gave the following series solution:

$$u(x) = - 0.00118096 x + 0.000871127x^4 - 0.000214393x^5 + 0.0000772396x^7 + 0.0000077 x^8 - 0.000101922 x^9 .$$

Example 3

Consider the linear, 2nd order, 6 point BVP's.

$$y''(x) + y(x) y'(x) = (\cos x - 1) \sin x$$

With BC: $y(0) = 0$, $y(1) = \sum_{i=0}^4 \frac{1}{1+i} y(\frac{i}{5}) + 0.3277$

The exact solution is: $y = \sin(x)$

Now, we solve this equation by using semi-analytic method from the equation (7), if $n=7$, we have:

$$P_{15} = 2.18915110^{-13} x^{15} - 1.1627290310^{-11} x^{14} - 1.0514471510^{-12} x^{13} + 0.00000000209094434 x^{12} - 4.4374915110^{-12} x^{11} - 0.000000275569894 x^{10} - 1.3142893210^{-12} x^9 + 0.0000248015875 x^8 - 0.00138888889 x^6 + 0.0416666667 x^4 - 0.5 x^2 + 0.61602371 x$$

Higher accuracy can be obtained by evaluating higher n , now, we take $n= 8$, i.e.,

$$P_{17} = 5.4232483810^{-13} x^{17} - 4.5600265210^{-12} x^{16} + 1.7175433110^{-11} x^{15} - 4.81974610^{-11} x^{14} + 4.931124910^{-11} x^{13} + 0.0000000020450742 x^{12} + 2.3147561410^{-11} x^{11} - 0.000000275580432 x^{10} + 9.9945801210^{-13} x^9 + 0.0000248015873 x^8 - 0.00138888889 x^6 + 0.0416666667 x^4 - 0.5 x^2 + 0.61602371 x$$

Now, increase n , to get higher accuracy, let $n = 9$, i.e.,

$$P_{19} = - 2.0533508410^{-12} x^{19} + 1.9479615610^{-11} x^{18} - 8.2373427510^{11} x^{17} + 2.0390193210^{-10} x^{16} - 3.25501246 10^{-10} x^{15} + 3.3645645510^{-10} x^{14} - 2.4910193110^{-10} x^{13} + 0.0000000022029493 x^{12} - 3.13130846 10^{-11} x^{11} - 0.000000275569384 x^{10} + 0.0000248015873 x^8 - 0.00138888889 x^6 + 0.0416666667 x^4 - 0.5 x^2 + 0.61602371 x$$

For more details, table (3) gives the results for different nodes in the domain, for $n = 9$, i.e. P_{19} and errors obtained by comparing it with the exact solution. Higher accuracy can be obtained by evaluating higher n .

Das [5] solved this example by HPM and gave the following series solution:

$$u(x) = - 0.0086 x - 0.000023 x^3 - 0.00073 x \cos 2x + 0.051 \sin x + \cos(0.1165 + 0.251 \sin x) - 0.02083 \sin 3x + 0.00097 \sin 4x .$$

3. Conditioning of BVP's

In particular, BVP's for which a small change to the ODE or boundary conditions results in a small change to the solution must be considered, a BVP's that has this property is said to be well-conditioned. Otherwise, the BVP's is said to be ill-

conditioned [12]. To be useful in applications, a BVP's should be well posed. This means that given the input to the problem there exists a unique solution, which depends continuously on the input. Consider the following 2nd order BVP's

$$y''(x) = f(x, y(x), y'(x)) \quad , \quad x \in [0, 1] \quad \dots(10a)$$

$$\text{With BC: } y(0) = A, y(1) = B y(\eta) \quad , \quad \text{where } \eta \in (0, 1) \quad \dots(10b)$$

For a well-posed problem, we make the following assumptions:

1. Equation (10) has an approximate solution $P \in C^n[0, 1]$, with this solution and $\rho > 0$, we associate the spheres :

$$S_\rho(P(x)) = \{ y \in \mathbb{R}^n : |P(x) - y(x)| \leq \rho \}$$

2. $f(x, P(x), P'(x))$ is continuously differentiable with respect to P , and $\partial f / \partial P$ is continuous.

This property is important due to the error associated with the approximate solutions to BVP's, depending on the semi-analytic technique, approximate solution \check{y} to the linear 2nd order BVP's (10) may exactly satisfy the perturbed ODE:

$$\check{y}'' = d(x) \check{y}' + q(x) \check{y} + r(x) \quad ; \quad 0 < x < 1 \quad \dots(11a)$$

Where, $r: \mathbb{R} \rightarrow \mathbb{R}^m$, and the linear BC:

$$B_0 \check{y}(0) + B_1 \check{y}(1) = \beta + \alpha \quad ; \quad \dots(11b)$$

Where, $\beta + \alpha = \sigma$, $\sigma \in \mathbb{R}^m$ and $\{\alpha, \beta, \sigma\}$ are constants. If \check{y} is a reasonably good approximate solution to (10), then $\|r(x)\|$ and $\|\sigma\|$ are small. However, this may not imply that \check{y} is close to the exact solution y . A measure of conditioning for linear BVP's that relates both $\|r(x)\|$ and $\|\sigma\|$ to the error in the approximate solution can be determined. The following discussion can be extended to nonlinear BVP's by considering the variational problem on small sub domains of the nonlinear BVP's [13].

Letting: $e(x) = |\check{y}(x) - y(x)|$; then subtracting the original BVP's (10) from the perturbed BVP's (11) results in:

$$e''(x) = \check{y}''(x) - y''(x) \quad ; \quad \dots(12a)$$

$$e''(x) = d(x) e'(x) + q(x) e(x) + r(x) \quad ; \quad 0 < x < 1 \quad \dots(12b)$$

$$\text{With BC: } B_0 e(0) + B_1 e(1) = \sigma \quad \dots(12c)$$

However, the form of the solution can be furthered simplified by letting: $\Theta(x) = Y(x) Q^{-1}$; where Y is the fundamental solution and Q is defined in (7b). Then, the general solution can be written as:

$$e(x) = \Theta(x) \sigma + \int_0^1 G(x, t) r(t) dt \quad \dots(13)$$

Where, $G(x, t)$ is Green's function [14], taking norms of both sides of (13) and using the Cauchy - Schwartz inequality [14] results in:

$$\|e(x)\|_\infty \leq k_1 \|\sigma\|_\infty + k_2 \|r(x)\|_\infty \quad \dots(14)$$

$$\text{Where, } k_1 = \|Y(x)Q^{-1}\|_\infty \quad ; \quad \text{and } k_2 = \sup_{0 \leq x \leq 1} \int_0^1 \|G(x, t)\|_\infty dt \quad ,$$

In (14), the L_∞ norm, sometimes called a maximum norm, is used due to the common use of this norm in numerical BVP's software. For any vector $v \in \mathbb{R}^N$, the L_∞ norm is defined as: $\|v\|_\infty = \max_{1 \leq i \leq N} |v_i|$.

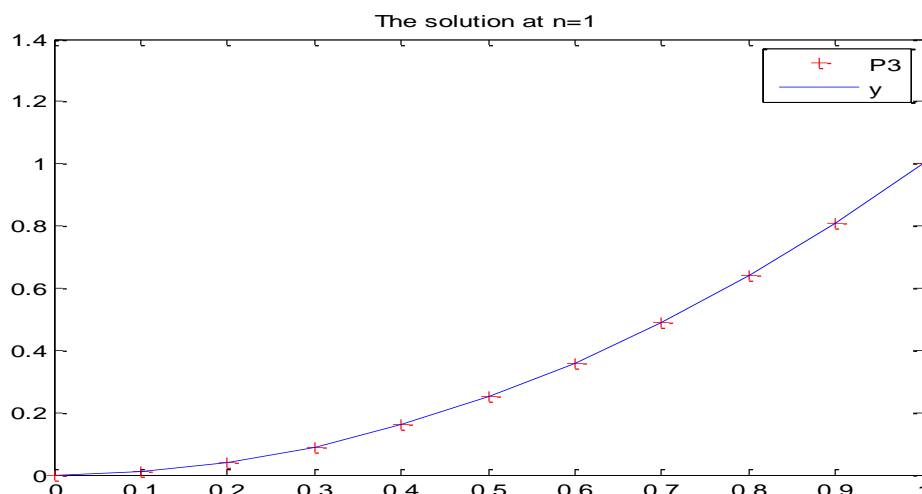
The measure of conditioning is called the conditioning constant k , and it is given by: $k = \max(k_1, k_2)$... (15)

When the conditioning constant is of moderate size, then the BVP's is said to be well-conditioned.

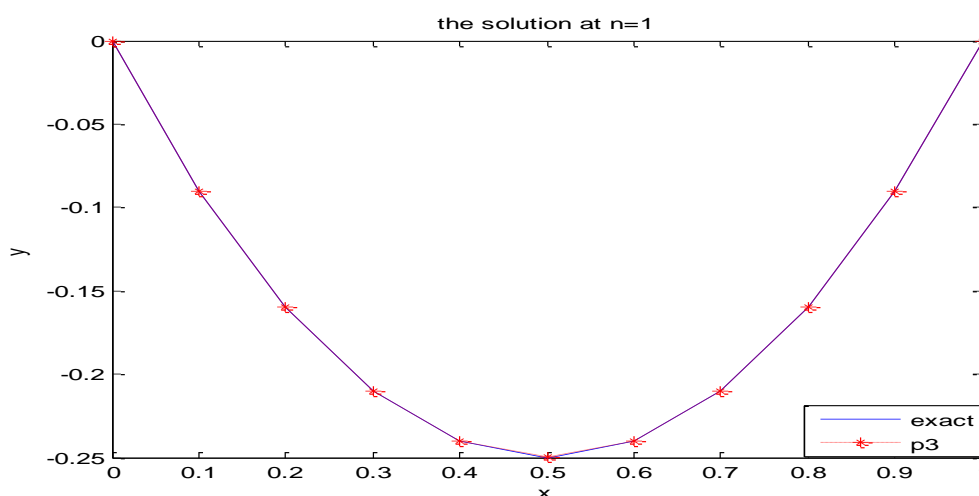
Referring again to (14), the constant k thus, provides an upper bound for the norm of the error associated with the perturbed solution,

$$\|e(x)\|_{\infty} \leq k [\|\sigma\|_{\infty} + \|r(x)\|_{\infty}] \quad \dots(16)$$

It is important to note that the conditioning constant only depends on the original BVP's and not the perturbed BVP's. As a result, the conditioning constant provides a good measure of conditioning that is independent of any numerical technique that may cause such perturbations. The well conditioned nature of a BVP's and the local uniqueness of its desired solution are assumed in order to, numerically, solve the problem.



Fig(1): A Comparison between Exact & P_3 of Example1



Fig(2): A Comparison between Exact & P_3 Solution of Example2

Table(1): A Comparison between Exact & Suggest Solution P_3

X	exact y(x)	Semi-analytic P_3	$y(x) - P_3$
0	0	0	0
0.1	0.01	0.0100000969892815	9.69892814846329e-08
0.2	0.04	0.0400001955947546	1.95594754552608e-07
0.3	0.09	0.0900002941926108	2.94192610772459e-07
0.4	0.16	0.160000391159042	3.91159041795985e-07

0.5	0.25	0.250000484870239	4.84870239125801e-07
0.6	0.36	0.360000573702394	5.73702394424114e-07
0.7	0.49	0.490000656031699	6.56031699131088e-07
0.8	0.64	0.640000730234345	7.30234344992198e-07
0.9	0.81	0.810000794686524	7.94686523586385e-07
1	1	1	0
S.S.E= 3.165303485474155e-012			

Das [5] solved this example by HPM and gave the following series solution:
 $u(x) = 0.003025x - 0.0009x^4 - 0.0010237x^5 + 0.000075x^7 + 0.00035x^8 - 0.000043x^9$

Table(2): A Comparison between Exact & Suggested Solution P_3

x	Exact solution y(x)	Suggest solution (p ₃)	Error y - P ₃
0	0	0	0
0.1	-0.0900000000000000	-0.0900000000000000	0
0.2	-0.1600000000000000	-0.1600000000000000	0
0.3	-0.2100000000000000	-0.2100000000000000	0.027755575615629e-015
0.4	-0.2400000000000000	-0.2400000000000000	0.027755575615629e-015
0.5	-0.2500000000000000	-0.2500000000000000	0.055511151231258e-015
0.6	-0.2400000000000000	-0.2400000000000000	0.055511151231258e-015
0.7	-0.2100000000000000	-0.2100000000000000	0.055511151231258e-015
0.8	-0.1600000000000000	-0.1600000000000000	0.111022302462516e-015
0.9	-0.0900000000000000	-0.0900000000000000	0.111022302462516e-015
1	0	0	0
Max. E			1.110223024625157e-016
S.S.E			3.543711097672514e-032

Table(3): A Comparison between Exact & Suggested Solution P_{19}

X	Exact solution	Suggest solution P ₁₉	Error y - P ₁₉
0	0	0	0
0.1	0.056606536300412	0.056606536299945	0.046630754813037e-011
0.2	0.103271319886014	0.103271319885081	0.093242080723144e-011
0.3	0.140143602192764	0.140143602191365	0.139896427775454e-011
0.4	0.167470478092429	0.167470478090564	0.186506365906780e-011
0.5	0.185594417002303	0.185594416999971	0.233144059613721e-011
0.6	0.194949841043994	0.194949841041197	0.279767875532855e-011
0.7	0.196058784441191	0.196058784437926	0.326411120354919e-011
0.8	0.189525677526253	0.189525677522523	0.373037711831614e-011
0.9	0.176031307472138	0.176031307467942	0.419647649962940e-011
1	0.156326016092000	0.156326016087337	0.466285343669881e-011
S.S.E		8.370778904341119e-023	
Max.E		4.662853436698811e-012	

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