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The Continuous Classical Boundary Optimal Control of Couple Nonlinear Hyperbolic Boundary Value Problem with Equality and Inequality Constraints

Jamil A. Ali Al-Hawasy

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Abstract:

The paper is concerned with the state and proof of the existence theorem of a unique solution (state vector) of couple nonlinear hyperbolic equations (CNLHEQS) via the Galerkin method (GM) with the Aubin theorem. When the continuous classical boundary control vector (CCBCV) is known, the theorem of existence a CCBOCV with equality and inequality state vector constraints (EIESVC) is stated and proved, the existence theorem of a unique solution of the adjoint couple equations (ADCEQS) associated with the state equations is studied. The Frcéhet derivative derivation of the Hamiltonian is obtained. Finally the necessary theorem (necessary conditions "NCs") and the sufficient theorem (sufficient conditions" SCs") for optimality of the state constrained problem are stated and proved.

Key words: Classical boundary optimal control, Nonlinear hyperbolic, Necessary, Sufficient conditions

Introduction:

The problems of optimal control (OCPs) have an important and vital role in many fields, such as in an electric power (1), economic (2), biology (3), robotics as in (4), and many other fields. This importance encouraged many researchers to be interested in the study of the OCPs for systems governed by nonlinear PDEs either of an elliptic type as in (5), or of a hyperbolic type as in (6) or by a parabolic type as in (7).

In the recent years, many studies about the classical optimal control problems (COCPs) governed by a couple of PDEs have been done, such as COCPs governed either by a couple of nonlinear elliptic PDEs as in (8), or by a couple of nonlinear parabolic PDEs as in (9), or by a couple of nonlinear hyperbolic PDEs as in (10). These studies and the studies of (11-13) in the boundary optimal control problems push us to study the continuous classical boundary optimal control problem (CCBOCP) governing by a couple of nonlinear PDEs of hyperbolic type.

This, work is concerned, at first, with the state and proof of the existence theorem of unique solution (state vector) of CNLHEQS using the GM when the CCBCV is given. Second the theorem of existence a CCBOCV governed by the considered CNLHEQS with EIESVC is stated and proved.

Department of Mathematics, College of Science, Al-Mustansiriyah University, Baghdad, Iraq.
E- mail: Jhawassy17@uomustansiriyah.edu.iq

The problem of the existence and uniqueness solution of the ADCEQS associated CNLHEQS is stated and studied. The "Fréchet derivative" of the Hamiltonian of this problem is derived. Finally the theorems of both the NCs and SCs of optimality of the state constrained problem are sated and proved.

Description of the problem: Let $Q = \Omega \times I$, where Ω be a bounded and open region in $\mathbb{R}^2$, with Lipschitz boundary $\Gamma = \partial\Omega$ and $I = [0, T]$,(with $T < \infty$) $\Sigma = \Gamma \times I$. Then the state equations are given by the following CNLHEQS:

$$y_{1tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\alpha_{ij}\, \frac{\partial y_1}{\partial x_j}) + \beta_1\, y_1 - \beta\, y_2 = h_1(y_1), \text{ in } Q \quad (1)$$

$$y_{2tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\beta_{ij}\, \frac{\partial y_2}{\partial x_j}) + \beta_2\, y_2 + \beta\, y_1 = h_2(y_2) \text{ ,in } Q \quad (2)$$

$$\frac{\partial y_1}{\partial v_\alpha} = w_1(x,t), \text{on } \Sigma \quad (3)$$

$$y_1(x,0) = y_1^0(x), \text{ and } y_{1t}(x,0) = y_1^1(x), \text{ on } \Omega \quad (4)$$

$$\frac{\partial y_2}{\partial v_\beta} = w_2(x,t), \text{on } \Sigma \quad (5)$$

$$y_2(x,0) = y_2^0(x), \text{ and } y_{1t}(x,0) = y_2^1(x), \text{ on } \Omega \quad (6)$$

where for all $x = (x_1, x_2)$, $(y_1, y_2) \in \left(H^1(\mathrm{Q})\right)^2$ is the state vector,$(w_1, w_2) \in \left(L^2(\Sigma)\right)^2$ is the continuous classical boundary control vector, $(h_1, h_2) \in \left(L^2(\mathrm{Q})\right)^2$ is a vector of a given function with $h_i(y_i) = h_i(x, t, y_i)$, $\alpha_{ij} = \alpha_{ij}(x,t)$,$\beta_{ij} = \beta_{ij}(x,t)$,$\beta = \beta(x,t)$ and $\beta_i = \beta_i(x,t) \in C^\infty(Q)$,

$\forall i=1,2$, and each of v_α, v_β is a unit vector normal outer to the boundary Σ.
The set of admissible controls is

$$\vec{W}_A=\{\vec{w}\in\vec{W}_c=L^2(\Sigma)\times L^2(\Sigma)\,|\,\vec{w}\in\vec{W}\text{ a.e. in }\Sigma,\ J_1(\vec{w})=0, J_2(\vec{w})\le 0\},\ \vec{W}\subset\mathbb{R}^2$$

The cost function is

$$J_0(\vec{w})=\int_Q[p_{01}(y_i)+p_{02}(y_i)\,]dxdt+\int_\Sigma[q_{01}(w_i)\ +q_{02}\,(w_i)]d\sigma \qquad (7)$$

The state (vector) constraints are

$$J_1(\vec{w})=\int_Q[p_{11}(y_i)\ +p_{12}(y_i)\,]dxdt+\int_\Sigma[q_{11}(w_i)\ +q_{12}(w_i)\,]d\sigma=0 \qquad (8)$$

$$J_2(\vec{w})=\int_Q[p_{21}(y_i)\ +p_{22}(y_i)]dxdt+\int_\Sigma[q_{21}(w_i)\ +q_{22}\,(w_i)]d\sigma\le 0 \qquad (9)$$

where $(y_1,y_2)=(y_{w1},y_{w2})$ is the solution of (1-6) corresponding to the boundary control (w_1,w_2), and $p_{li}(y_i)=p_{li}(x,t,y_i)$, and $q_{li}(w_i)=q_{li}(x,t,w_i)$, (for $l=0,1,2$ and $i=1,2$) are defined later.

The continuous optimal control problem is to find $\vec{w}\in\vec{W}_A$ such that $J_0(\vec{\bar{w}})=\min_{\vec{w}\in\vec{W}_A}J_0(\vec{w})$.

Let $\vec{U}=U\times U=\left\{\vec{u}:\vec{u}\in\left(H^1(\Omega)\right)^2,\text{with } u_1=u_2=0\text{ on }\partial\Omega\right\}$, $\vec{u}=(u_1,u_2)$. We denote by $(u,u)_\Omega$and $\|u\|_0$ (by $(u,u)_\Gamma$ and $\|u\|_\Gamma$) the inner product and the norm in $L^2(\Omega)$ (in $L^2(\Gamma)$), by $(u,u)_1$ and $\|u\|_1$ the inner product and the norm in $H^1(\Omega)$, by $(\vec{u},\vec{u})_\Omega$ and $\|\vec{u}\|_0$ (by $(\vec{u},\vec{u})_\Gamma$ and $\|\vec{u}\|_\Gamma$) the inner product and the norm in$(L^2(\Omega))^2$)(in $(L^2(\Gamma))^2$ by $(\vec{u},\vec{u})_1=(u_1,u_1)_1+(u_2,u_2)_1$ and $\|\vec{u}\|_1^2=\|u_1\|_1^2+\|u_2\|_1^2$ the inner product and the norm in $\vec{U}$ and finally $\vec{U}^*$ is the dual of $\vec{U}$.

The weak form (FW) of the problem (1-6) when $\vec{y}\in(H_0^1(Q))^2$ is given almost everywhere (a.e.) on I ($\forall u_1,u_2\in U, y_1(.,t),y_2(.,t)\in U$) by

$$\langle y_{1tt},u_1\rangle+\alpha_1(t,y_1,u_1)+(\beta_1y_1,u_1)_\Omega-(\beta y_2,u_1)_\Omega=(h_1,u_1)_\Omega+(w_1,u_1)_\Gamma, \qquad (10a)$$

$$(y_1^0,u_1)_\Omega=(y_1(0),u_1)_\Omega,\quad\text{and}\quad (y_1^1,u_1)_\Omega=(y_{1t}(0),u_1)_\Omega \qquad (10b)$$

$$\langle y_{2tt},u_2\rangle+\alpha_2(t,y_2,u_2)\ +(\beta_2y_2,u_2)_\Omega+(\beta y_1,u_2)_\Omega=(h_2,u_2)_\Omega+(w_2,u_2)_\Gamma, \qquad (11a)$$

$$(y_2^0,u_2)_\Omega=(y_2(0),u_2)_\Omega,\quad\text{and}\quad (y_2^1,u_2)_\Omega=(y_{2t}(0),u_2)_\Omega \qquad (11b)$$

where $\alpha_1(t,y_1,u_1)\ =\int_\Omega\sum_{i,j=1}^n\alpha_{ij}\frac{\partial y_1}{\partial x_i}\frac{\partial u_1}{\partial x_j}dx$, and $\alpha_2(t,y_2,u_2)=\int_\Omega\sum_{i,j=1}^n b_{ij}\frac{\partial y_2}{\partial x_i}\frac{\partial u_2}{\partial x_j}dx$.

The following assumptions are necessary to study the continuous classical boundary optimal control problem(CCBOCV):

Assumptions *(A)*:

(i)h_i on $Q\times\mathbb{R}$ is of "Carathéodory type" , and for each $i=1,2$ satisfies

$$|h_i(x,t,y_i)|\le\psi_i(x,t)+c_i|y_i|,$$

where$y_i\in\mathbb{R}$, $c_i>0$ and $\eta_i(x,t)\in L^2(Q,\mathbb{R})$.

(ii) h_i has "Lipschitz property" with respect to y_i, for each $i=1,2$, i.e.

$$|h_i(x,t,y_i)-h_i(x,t,\bar{y}_i)|\le L_i|y_i-\bar{y}_i|,$$

where$(x,t)\in Q$, $y_i,\bar{y}_i\in\mathbb{R}$ and $L_i>0$.

(iii)$s(t,\vec{y},\vec{u})=\ \alpha_1(t,y_1,u_1)+(\beta_1y_1,u_1)_\Omega+\alpha_2(t,y_2,u_2)+(\beta_2y_2,u_2)_\Omega$

$$t(t,\vec{y},\vec{u})=s(t,\vec{y},\vec{u})-(\beta y_2,u_1)_\Omega+(\beta y_1,u_2)_\Omega$$

and

$$|s(t,\vec{y},\vec{u})|\le a\|\vec{y}\|_1\|\vec{u}\|_1,\ \ s(t,\vec{y},\vec{y})\ge\bar{a}\|\vec{y}\|_1^2,$$
$$|s_t(t,\vec{y},\vec{u})|\le\alpha\|\vec{y}\|_1\|\vec{u}\|_1,\ s_t(t,\vec{y},\vec{y})\ge\bar{\alpha}\|\vec{y}\|_1^2,$$

where $a,\bar{a}$,α, $\bar{\alpha}$ are real positive constants.

Definition(1) (14)**:** A function $k(x,y):\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ is said to be of a **"Carathéodory type"** if it is continuous with respect to y for fixed $x\in\Omega$ and it is measurable with respect to $x\in\Omega$ for fixed $y\in\mathbb{R}^n$.

Definition(2) (14)**:** A mapping $f:\Omega\subset X\to Y$ from an open set Ω of a normed vector space X into a normed vector space Y is said to be has a **"Fréchet differentiable**" at a point $x\in\Omega$, if there exists an element $\varphi(x)\in Lin(X,Y)$ (linear and continuous), such that for $x+h\in\Omega$:
$f(x+h)=f(x)+\varphi(x)h+\varepsilon(h)\|h\|$, with $\lim_{\|h\|\to\infty}\|\varepsilon(h)\|=0$, or equivalent (with $h\neq 0$) $\lim_{\|h\|\to\infty}\frac{\|f(x+h)-f(x)-\varphi(x)h\|}{\|h\|}=0$. If there exists such an element $\varphi(x)$, then it is unique

Proposition (1) (15)**:** Suppose Ω be a measurable subset of $\mathbb{R}^d$ $(d=2,3)$, let $k:\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ is of a "Carathéodory type", satisfies

$$\|k(x,y)\|\le\varphi(x)+\psi(x)\|y\|^\alpha,$$

for each $(x,y)\in\Omega\times\mathbb{R}^n$, where $y\in L^p(\Omega\times\mathbb{R}^n)$, $\varphi(x)\in L^1(\Omega\times\ \mathbb{R})$,$\psi\in L^{\frac{p}{p-\alpha}}(\Omega\times\ \mathbb{R})$ and $\alpha\in[0,p]$, $\alpha\in\mathbb{N}$, if $p\in[1,\infty)$,and $\eta\equiv 0$, if $p=\infty$. Then the functional $K(y)=\int_\Omega k\big(x,y(x)\big)dx$ is continuous.

Proposition (2) (15)**:** Suppose Ω be a measurable subset of $\mathbb{R}^d$ $(d=2,3)$, let k, $k_y:\Omega\times\mathbb{R}^n\to\mathbb{R}^m$ be of a "Carathéodory type ", such that

$$\left\|k_y(x,y)\right\|\le\varphi(x)+\psi(x)\|y\|^{\frac{\beta}{q}},$$

for each $(x,y)\in\Omega\times\mathbb{R}^n$, where $\varphi\in L^q(\Omega\times\mathbb{R})$, $\frac{1}{p}+\frac{1}{q}=1$, $\psi\in L^{\frac{pq}{p-\beta}}(\Omega\times\ \mathbb{R})$, $\beta\in[0,p]$ if $p\neq\infty$, and $\eta\equiv 0$, if $p=\infty$. Then the **Fréchet derivative** of $K(y)=\int_\Omega k_y(x,y(x))dx$ exists for each $y\in L^p(\Omega\times\mathbb{R}^n)$ and is given by

$$\acute{K}(y)k=\int_\Omega k_y\big(x,y(x)\big)k(x)dx\ .$$

The Solution of the State Equations: In this section the theorem of existence a unique solution of the CNLHEQS under a suitable assumption is proved when the boundary control vector is given.

Theorem (1) :(Existence of a Uniqueness Vector Solution for the State Equations)

With assumptions (A), if the boundary control $\vec{w} \in (L_2(\Sigma))^2$is given, the WF (10-11) has a unique solution $\vec{y} = (y_1, y_2)$, such that $\vec{y} \in \left(L^2(I, U)\right)^2$, $\vec{y}_t = (y_{1t}, y_{2t}) \in \left(L^2(Q)\right)^2$, and $\vec{y}_{tt} = (y_{1tt}, y_{2tt}) \in \left(L^2(I, U^*)\right)^2$.

Proof: Let $\forall n$, $\vec{U}_n = U_n \times U_n \subset \vec{U}$ be the set of continuous and piecewise affine function in Ω. $\{\vec{U}_n\}_{n=1}^{\infty}$ be a sequence of subspaces of $\vec{U}$, s.t. $\forall\, \vec{u} = (u_1, u_2) \in \vec{U}$, there exists a sequence $\{\vec{u}_n\}$ with $\vec{u}_n = (u_{1n}, u_{2n}) \in \vec{U}_n$, $\forall n$, and $\vec{u}_n$ strongly in $\vec{U}$(which implies $\vec{u}_n \longrightarrow \vec{u}$ strongly in $\left(L^2(\Omega)\right)^2$). $\{u_j = (u_{1j}, u_{2j}) : j = 1,2, \dots, M(n)\}$ be a finite basis of $\vec{U}_n$ (where $\vec{u}_j$ is continuous and piecewise affine function in Ω, with $\vec{u}_j(x) = 0$ on the boundary Γ) and let $\vec{y}_n = (y_{1n}, y_{2n})$ be the Galerkin approximate solution to the exact solution $\vec{y} = (y_1, y_2)$ s. t.

$y_{1n} = \sum_{j=1}^{n} x_{1j}\,(t) u_{1j}(x)$,where $x_{1j}(t)$ is unknown function of t, $\forall j = 1,2, \dots, n$. (12a)

&

$y_{2n} = \sum_{j=1}^{n} x_{2j}(t) u_{2j}(x)$,where$x_{2j}(t)$ is unknown function of t,$\forall j = 1,2, \dots, n$. (12b)

The weak forms(10-11) are approximated with respect to x using the GM, then substituting $y_{int} = z_{in}$ $(i = 1,2)$

in the obtained equations, we get the following system of 1st order differential with their boundary conditions ($\forall\, u_1, u_2 \in U_n$)

$\langle z_{1nt}, u_1\rangle + \alpha_1(t, y_{1n}, u_1) + (\beta_1 y_{1n}, u_1)_\Omega - (\beta y_{2n}, u_1)_\Omega = (h_1(y_{1n}), u_1)_\Omega + (w_1, u_1)_\Gamma$ (12c)

$(y_{1n}^0, u_1)_\Omega = (y_1^0, u_1)_\Omega$, and $(y_{1n}^1, u_1)_\Omega = (y_1^1, u_1)_\Omega$ (12d)

$\langle y_{1nt}, u_{1n}\rangle = \langle z_{1n}, u_{1n}\rangle$ (12e)

$\langle z_{2nt}, u_2\rangle + \alpha_2(t, y_{2n}, u_2) + (\beta_2 y_{2n}, u_2)_\Omega + (\beta y_{1n}, u_2)_\Omega = (h_2(y_{2n}), u_2)_\Omega + (w_2, u_2)_\Gamma$ (12f)

$(y_{2n}^0, u_2)_\Omega = (y_2^0, u_2)_\Omega$, and $(y_{2n}^1, u_2)_\Omega = (y_2^1, u_2)_\Omega$ (12g)

$\langle y_{2nt}, u_{2n}\rangle = \langle z_{2n}, u_{2n}\rangle$ (12h)

where $y_{in}^0 = y_{in}(x,0) \in U_n$ (resp. $z_{in}^0 = y_{in}^1 = y_{int}(x,0) \in L^2(\Omega)$) be the projection of y_i^0 onto U(be the projection of $y_i^1 = y_{it}$ onto $L^2(\Omega)$), $\forall i = 1,2$, i.e.

$y_{in}^0 \longrightarrow y_i^0$ strongly in U , with $\|\vec{y}_n^0\|_{\mathbf{1}} \le b_0$ and $\|\vec{y}_n^0\|_{\mathbf{0}} \le b_0$ (13)

$y_{in}^1 \longrightarrow y_i^1$strongly in $L^2(\Omega)$ and$\|\vec{y}_n^1\|_{\mathbf{0}} \le b_1$ (14)

Substituting (12a) in (12c-d) and (12b) in (12f-g), setting$u_1 = u_{1i}$, $u_2 = u_{2i}$, the obtained equations are equivalent to the following nonlinear system of 1st ODES with their initial conditions which has unique solution $\vec{y}_n = (y_{1n}, y_{2n}) \in C(I, \vec{U})$, i.e.

$E_1 \acute{Y}_1(t) + F_1 X_1(t) - G X_2(t) = b_1\,(\overline{U}_1^T(x) X_1(t))$,

$E_1 \acute{X}_1(t) = E_1 Y_1(t)$, $E_1 X_1(0) = b_1^0$, $E_1 Y_1(0) = b_1^1$

$E_2 \acute{Y}_2(t) + F_2 X_2(t) + H X_1(t) = b_2\,(\overline{U}_2^T(x) X_2(t))$,

$E_2 \acute{X}_2(t) = E_2 Y_2(t)$,$E_2 X_2(0) = b_2^0$ & $E_2 Y_2(0) = b_2^1$

where $E_l = (e_{lij})_{n\times n}$,$e_{lij} = (u_{lj}, u_{li})_\Omega$, $F_l = (f_{lij})_{n\times n}$, $f_{lij} = [\alpha_l(t, u_{lj}, u_{li}) + (\beta_l(t) u_{lj}, u_{li})_\Omega]$, $G = (g_{ij})_{n\times n}$, $g_{ij} = (\beta(t) u_{2j}, u_{1i})_\Omega$, $H = (h_{ij})_{n\times n}$, $h_{ij} = (\beta(t) u_{1i}, u_{2i})_\Omega$, $X_l(t) = (x_{lj}(t))_{n\times 1}$, $Y_l(t) = (y_{lj}(t))_{n\times 1}$, $b_l = (b_{li})_{n\times 1}$,

$b_{li} = (h_l(U_l^T x_{li}(t), w_l), u_{li})_\Omega + (w_l, u_{li})_\Gamma$,

$b_l^k = (b_{lj}^k)$, $b_{lj}^0 = (y_l^k, u_{lj})_\Omega$, $k = 0{,}1$ and $l = 1{,}2$.

Then corresponding to the sequence$\{\vec{U}_n\}$, there exists a sequence of the following "approximation problems", i.e. for each $\vec{u}_n = (u_{1n}, u_{2n}) \subset \vec{U}_n$, and $n = 1,2, \dots$

$\langle y_{1ntt}, u_{1n}\rangle + \alpha_1(t, y_{1n}, u_{1n}) + (\beta_1 y_{1n}, u_{1n})_\Omega - (\beta y_{2n}, u_{1n})_\Omega = (h_1(y_{1n}), u_{1n})_\Omega + (w_1, u_{1n})_\Gamma$,

$\forall y_{1n}, y_{2n} \in L^2(I, U_n)$, a.e inI (15a)

$(y_{1n}^0, u_{1n})_\Omega = (y_1^0, u_{1n})_\Omega$,and $(y_{1n}^1, u_{1n})_\Omega = (y_1^1, u_{1n})_\Omega$, $\forall\, u_{1n} \in U_n$, $\forall n$ (15b)

$\langle y_{2ntt}, u_{2n}\rangle + \alpha_2(t, y_{2n}, u_{2n}) + (\beta_2 y_{2n}, u_{2n})_\Omega + (\beta y_{1n}, u_{2n})_\Omega = (h_2(y_{2n}), u_{2n})_\Omega + (w_2, u_{2n})_\Gamma$,

$\forall y_{1n}, y_{2n} \in L^2(I, U_n)$ a.e. in I (16a)

$(y_{2n}^0, u_{2n})_\Omega = (y_2^0, u_{2n})_\Omega$,$(y_{2n}^1, u_{2n})_\Omega = (y_2^1, u_{2n})_\Omega$,$\forall u_{2n} \in U_n$,$\forall n$ (16b)

which has a sequence of unique solutions $\{\vec{y}_n\}$. Substituting $u_{1n} = y_{1nt}$ in(15a) and $u_{2n} = y_{2nt}$ in (16a), adding the two obtained equations, using Lemma 1.2 in ref. (16) for the 1st term of the left hand side, to get

$\frac{d}{dt}\left[\|\vec{y}_{nt}(t)\|_0^2 + +s(t, \vec{y}_n, \vec{y}_n)\right] - s_t(t, \vec{y}_n, \vec{y}_n) = 2((\beta y_{2n}, y_{1nt})_\Omega - (\beta y_{1n}, y_{2nt})_\Omega + (h_1(y_{1n}), y_{1nt}) + (h_2(y_{2n}), y_{2nt})) + (w_1, y_{1nt})_\Gamma + (w_2, y_{2nt})_\Gamma)$ (17a)

Using assumption (A-iii) for the second term in the left hand side of (17a) and taking absolute value for both sides, then using assumption (A-i) for the right hand side of the obtained equation to get

$\frac{d}{dt}[\|\vec{y}_{nt}(t)\|_0^2 + \bar{\alpha}\|\vec{y}_n\|_1^2] \le \alpha\|\vec{y}_n\|_1^2 + 2(|(\beta y_{2n}, y_{1nt})_\Omega| + |(\beta y_{1n}, y_{2nt})_\Omega| + |(w_1, y_{1nt})_\Gamma| (h_1(y_{1n}), y_{1nt}) + |(h_2(y_{2n}), y_{2nt})| + |(w_2, y_{2nt})_\Gamma|)$ (17b)

Integrating both sides of (17b), on $[0,t]$, using the trace theorem and that $\|y_{in}\|_0 \le \|\vec{y}_n\|_0$, $\|y_{int}\|_0 \le \|\vec{y}_{nt}\|_0$, $\|y_{int}\|_0 \le \|y_{int}\|_1$, $\|\vec{y}_n\|_0 \le \|\vec{y}_n\|_1$, $\|w_1\|_\Gamma \le \|\vec{w}\|_\Gamma$, to get

$\int_0^t \frac{d}{dt}[\|\vec{y}_{nt}(t)\|_0^2 + \bar{\alpha}\|\vec{y}_n\|_1^2] dt$

$\leq \int_0^t 2b(\|\vec{y}_{nt}\|_0^2 + \|\vec{y}_n\|_1^2)\,dt + \int_0^t(\|\psi_1\|_0^2 + \|\psi_2\|_0^2)\,dt + \int_0^t(4\|\vec{y}_{nt}\|_0^2 + (c_1^2 + c_2^2 + \alpha)\|\vec{y}_n\|_1^2)\,dt + \int_0^t(2c_3\|\vec{y}_{nt}\|_0^2 + 2\|\vec{w}\|_\Gamma^2)dt$
$\leq \|\psi_1\|_Q^2 + \|\psi_2\|_Q^2 + 2\|\vec{u}\|_\Sigma^2 + c_5\int_0^t(\|\vec{y}_n\|_0^2 + \bar{a}\|\vec{y}_{nt}\|_1^2)\,dt,$
$\leq c_8 + c_5\int_0^t(\|\vec{y}_n\|_0^2 + \bar{a}\|\vec{y}_{nt}\|_1^2)\,dt$ (18)

with $\bar{a} = \frac{c_4}{c_5}$, where $c_4 = 2b + 4 + 2c_3$, $c_5 = 2b + (c_1^2 + c_2^2) + \alpha$, , $c_8 = c_6 + c_7$, $c_6 = \acute{b}_1 + \acute{b}_2$, with $\|\psi_i\|_Q^2 \leq \acute{b}_i$, $i = 1,2$. And $\|\vec{w}\|_\Gamma^2 \leq c_7$

Since $\|\vec{y}_n^0\|_\mathbf{1} \leq b_1$, and $\|\vec{y}_n^1\|_\mathbf{0} \leq b_0$, with $c_9 = b_0 + b_1 + c_9$, inequality (18) becomes

$\|\vec{y}_{nt}(t)\|_0^2 + \bar{a}\|\vec{y}_n(t)\|_1^2 \leq c_9 + c_5\int_0^t(\|\vec{y}_{nt}\|_0^2 + \bar{a}\|\vec{y}_n\|_1^2)\,dt$

Using the Belman-Gronwall (B-G) inequality, to get for each $t \in [0,T]$that

$\|\vec{y}_{nt}(t)\|_0^2 + \bar{a}\|\vec{y}_n(t)\|_1^2 \leq c_9 e^{c_5} = b^2(c) \Rightarrow$
$\|\vec{y}_{nt}(t)\|_0^2 \leq b^2(c)$, and $\|\vec{y}_n(t)\|_1^2 \leq b^2(c)$

Easily once can obtained that $\|\vec{y}_{nt}(t)\|_\mathbf{Q} \leq b_1(c)$ and $\|\vec{y}_n(t)\|_{\mathbf{L^2(I,V)}} \leq b(c)$.

Then applying the "Alaoglu's theorem", there exists a subsequence of $\{\vec{y}_n\}_{n\in N}$, for simplicity say again $\{\vec{y}_n\}_{n\in N}$ such that $\vec{y}_{nt} \longrightarrow \vec{y}$ weakly in $(L^2(Q))^2$ and $\vec{y}_n \longrightarrow \vec{y}$ weakly in $(L^2(I,U))^2$, and since

$(L^2(R,U))^2 \subset (L^2(R,\Omega))^2 \cong ((L^2(R,\Omega))^*)^2 \subset (L^2(R,U^*))^2$ (19)

Then the "Aubin theorem" in ref. (16) can be applied here to get that $\vec{y}_n \longrightarrow \vec{y}$ strongly in $(L^2(Q))^2$. Now, multiplying both sides of (15a) & (16a) by $\zeta_i(t) \in C^2[0,T]$, $\forall i = 1,2$ respectively , such that $\zeta_i(T) = \acute{\zeta}_i(T) = 0$, $\zeta_i(0) \neq 0$, $\acute{\zeta}_i(0) \neq 0$, $\forall i = 1,2$, integrating on $[0,T]$, finally integrate by parts twice the first term of each one of the obtained two equations, yield to

$-\int_0^T \frac{d}{dt}(y_{1n},u_{1n})\acute{\zeta}_1(t)dt + \int_0^T[\alpha_1(t,y_{1n},u_{1n}) + (\beta_1 y_{1n},u_{1n})_\Omega - (\beta y_{2n},u_{1n})_\Omega]\zeta_1(t)dt = \int_0^T(h_1(y_{1n}),u_{1n})_\Omega\zeta_1(t)dt + \int_0^T(w_1,u_{1n})_\Gamma\,\zeta_1(t)dt + (y_{1n}^1,u_{1n})_\Omega\zeta_1(0),$ (20a)

$\int_0^T(y_{1n},u_{1n})\acute{\acute{\zeta}}_1(t)dt + \int_0^T[\alpha_1(t,y_{1n},u_{1n}) + (\beta_1 y_{1n},u_{1n})_\Omega - (\beta y_{2n},u_{1n})_\Omega]\zeta_1(t)dt = \int_0^T(h_1(y_{1n}),u_{1n})_\Omega\zeta_1(t)dt + \int_0^T(w_1,u_{1n})_\Gamma\,\zeta_1(t)dt + (y_{1n}^1,u_{1n})_\Omega\zeta_1(0) + (y_{1n}^0,u_{1n})_\Omega\acute{\zeta}_1(0),$ (20b)

$-\int_0^T \frac{d}{dt}(y_{2n},u_{2n})\acute{\zeta}_2(t)dt + \int_0^T[\alpha_2(t,y_{2n},u_{2n}) + (\beta_2 y_{2n},u_{2n})_\Omega + (\beta y_{1n},u_{2n})_\Omega]\zeta_2(t)dt = \int_0^T(h_2(y_{2n}),u_{2n})_\Omega\zeta_2(t)dt + \int_0^T(w_2,u_{2n})_\Gamma\zeta_2(t)dt + (y_{2n}^1,u_{2n})_\Omega\zeta_2(0),$ (21a)

$\int_0^T(y_{2n},u_{2n})\acute{\acute{\zeta}}_2(t)dt + \int_0^T[\alpha_2(t,y_{2n},u_{2n}) + (\beta_2 y_{2n},u_{2n})_\Omega + (\beta y_{1n},u_{2n})_\Omega]\zeta_2(t)dt = \int_0^T(h_2(y_{2n}),u_{2n})_\Omega\zeta_2(t)dt + \int_0^T(w_2,u_{2n})_\Gamma\zeta_2(t)dt + (y_{2n}^1,u_{2n})_\Omega\zeta_2(0) + (y_{2n}^0,u_{2n})_\Omega\acute{\zeta}_2(0),$ (21b)

Since $\forall i = 1,2$the following convergences are satisfied: First

$u_{in} \longrightarrow u_i$ strongly in W $\Rightarrow$

$$\begin{cases} u_{in}\zeta_i(t) \longrightarrow u_i\zeta_i(t) & \text{strongly in } L^2(I,W) \\ u_{in}\acute{\zeta}_i(t) \longrightarrow u_i\acute{\zeta}_i(t) & \text{strongly in } L^2(I,W) \\ u_{in}\zeta_i(0) \longrightarrow u_i\zeta_i(0) & \text{strongly in } L^2(\Omega) \end{cases}$$

$u_{in} \longrightarrow u_i$ strongly in $L^2(\Omega)$ $\Rightarrow$

$$\begin{cases} v_{in}\acute{\zeta}_i(t) \longrightarrow \acute{\zeta}_i(t) & \text{strongly in} L^2(Q) \\ v_{in}\acute{\acute{\zeta}}_i(t) \longrightarrow v_i\acute{\acute{\zeta}}_i(t) & \text{strongly in } L^2(Q) \\ v_{in}\acute{\zeta}_i(0) \longrightarrow \acute{\zeta}_i(0) & \text{strongly in} L^2(\Omega) \end{cases}$$

Second, $y_{int} \longrightarrow y_{it}$ weakly in $L^2(Q)$ and $y_{in} \longrightarrow y_i$ weakly in $L^2(I,U)$ and strongly in $L^2(Q)$.

Third and on the other hand, let $\eta_{in} = u_{in}\zeta_i$ and $\eta_i = u_i\zeta_i$ then $\eta_{in} \longrightarrow \eta_i$ strongly in $L^2(Q)$ and then w_{in} is measurable with respect to (x,t), so using assumption (A-i), applying Proposition1.3, the integral $\int_Q h_i(x,t,y_{in})\eta_{in}dxdt$ is continuous with respect to (y_{in},η_{in}), then

$\int_0^T(h_i(y_{in}),u_{in})\zeta_i(t)dt \to \int_0^T(h_i(y_i),u_i)\zeta_i(t)dt$, $\forall i = 1,2$.

From these convergences, and (13) , (14d), we can passaged the limits in (20a,b), (21a,b) to get

$-\int_0^T(y_{1t},u_1)\acute{\zeta}_1(t)dt + \int_0^T[\alpha_1(t,y_1,u_1) + (\beta_1 y_1,u_1)_\Omega - (\beta y_2,u_1)_\Omega]\zeta_1(t)dt = \int_0^T(h_1(y_1),u_1)_\Omega\zeta_1(t)dt + \int_0^T(w_1,u_1)_\Gamma\,\zeta_1(t)dt + (y_1^1,u_1)_\Omega\zeta_1(0),$ (22a)

$\int_0^T(y_1,u_1)\acute{\acute{\zeta}}_1(t)dt + \int_0^T[\alpha_1(t,y_1,u_1) + (\beta_1 y_1,u_1)_\Omega - (\beta y_2,u_1)_\Omega]\zeta_1(t)dt = \int_0^T(h_1(y_1),u_1)_\Omega\zeta_1(t)dt + \int_0^T(w_1,u_1)_\Gamma\,\zeta_1(t)dt + (y_1^1,u_1)_\Omega\zeta_1(0) + (y_1^0,u_1)_\Omega\acute{\zeta}_1(0),$ (22b)

$-\int_0^T(y_{2t},u_2)\acute{\zeta}_2(t)dt + \int_0^T[\alpha_2(t,y_2,u_2) + (\beta_2 y_2,u_2)_\Omega + (\beta y_1,u_2)_\Omega]\zeta_2(t)dt = \int_0^T(h_2(y_2),u_2)_\Omega\zeta_2(t)dt + \int_0^T(w_2,u_2)_\Gamma\,\zeta_2(t)dt + (y_2^1,u_2)_\Omega\zeta_2(0),$ (22c)

$\int_0^T(y_2,u_2)\acute{\acute{\zeta}}_2(t)dt + \int_0^T[\alpha_2(t,y_2,u_2) + (\beta_2 y_2,u_2)_\Omega + (\beta y_1,u_2)_\Omega]\zeta_2(t)dt = \int_0^T(h_2(y_2),u_2)_\Omega\zeta_2(t)dt + \int_0^T(w_2,u_2)_\Gamma\,\zeta_2(t)dt + (y_2^1,u_2)_\Omega\zeta_2(0) + (y_2^0,u_2)_\Omega\acute{\zeta}_2(0),$ (22d)

Case1: $\forall i = 1,2$, choose $\varphi_i \in C^2[0,T]$, such that $\zeta_i(0) = \acute{\zeta}_i(0) = \zeta_i(T) = \acute{\zeta}_i(T) = 0$. Substituting in (22b), (22d), integration by parts twice the first terms in the LHS of each one of the obtained equation, yield to

$\int_0^T < y_{1tt}, u_1 > \zeta_1(t)dt + \int_0^T [\alpha_1(t, y_1, u_1) + (\beta_1 y_1, u_1)_\Omega - (\beta y_2, u_1)_\Omega]\zeta_1(t)dt = \int_0^T (h_1(y_1), u_1)_\Omega \zeta_1(t)dt + \int_0^T (w_1, u_1)_\Gamma \zeta_1(t)dt$
(23a)

$\int_0^T < y_{2tt}, u_2 > \zeta_2(t)dt + \int_0^T [\alpha_2(t, y_2, u_2) + (\beta_2 y_2, u_2)_\Omega + (\beta y_1, u_2)_\Omega]\zeta_2(t)dt = \int_0^T (h_2(y_2), u_2)_\Omega \zeta_2(t)dt + \int_0^T (w_2, u_2)_\Gamma \zeta_2(t)dt$
(23b)

Which give that y_1 & $y_2$are solutions of (10a) and (11a) respectively (a.e. on I).

Case2: For each $i = 1,2$, choose $\zeta_i \in C^2[0,T]$, such that $\zeta_i(T) = 0$ & $\zeta_i(0) \neq 0$. Multiplying both sides of (10a), (11a) by $\zeta_1(t)$, $\zeta_2(t)$ respectively, integrating on $[0,T]$, then integrating by parts the first term in the LHS of each resulting equation, then subtracting each one of these obtained equations from those in (22a) & (22c) respectively, once get

$(y_i^1, u_i)\zeta_i(0) = (y_{it}(0), u_i)\zeta_i(0)$.

Case3: Choose $\zeta_i \in C^2[0,T]$, such that $\zeta_i(0) = \zeta_i(T) = \acute{\zeta}_i(T) = 0$,$\acute{\zeta}_i(0) \neq 0$, $\forall i = 1,2$.

Multiplying both sides of (10a) and (11a) by $\zeta_1(t)$ and $\zeta_2(t)$ respectivly, integrating on $[0,T]$, then integrating by parts twice the first term in the LHS of the resulting equation, then subtracting each one of these obtains equations from those in (22b) & (22d) respectively, one gets

$(y_i^0, u_i)\acute{\zeta}_i(0) = (y_i(0), u_i)\acute{\zeta}_i(0)$.

From the last two cases easily one gets the initial conditions (10b) & (11b).

To prove that $\vec{y}_n \longrightarrow \vec{y}$ strongly in$\left(L^2(I,U)\right)^2$, we start with substituting $u_{1n} = y_{1n}$ in(15a)and and $u_{2n} = y_{2n}$ (16a), then adding the two obtained equations, applying Lemma 1.2 in (16) for the first term of the left hand side, and finally by integrating the resulting equation on $[0,T]$, to get

$\|\vec{y}_{nt}(T)\|_0^2 - \|\vec{y}_{nt}(0)\|_0^2 + s(t, \vec{y}_n, \vec{y}_n)(T) - s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_n)dt = 2\int_0^T (h_1(y_{1n}), y_{1nt}) + (h_2(y_{2n}), y_{2nt}))\,dt + (w_1, y_{1n})_\Gamma + (w_2, y_{2n})_\Gamma]dt$ (17c)

The same way which is used to get (17a,c), can be also used here when we have $\vec{y}$ and $\vec{y}_t$, i.e.

$\|\vec{y}_t(T)\|_0^2 - \|\vec{y}_t(0)\|_0^2 + s(t, \vec{y}, \vec{y})(T) - s(t, \vec{y}, \vec{y})(0) - \int_0^T s_t(t, \vec{y}, \vec{y}) = 2\int_0^T [\,(h_1(y_1), y_1)) + (h_2(y_2), y_2)) + (w_1, y_1)_\Gamma + (w_2, y_2)_\Gamma]dt$ (17d)

Since

$\|\vec{y}_{nt}(T) - \vec{y}_t(T)\|_0^2 - \|\vec{y}_{nt}(0) - \vec{y}_t(0)\|_0^2 + s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(T) - s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt =$ eq(17e1)-eq(17e2)-eq(17e3) (17e)

(17e1)= $\|\vec{y}_{nt}(T)\|_0^2 - \|\vec{y}_{nt}(0)\|_0^2 + s(t, \vec{y}_n, \vec{y}_n)(T) - s(t, \vec{y}_n, \vec{y}_n)(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y}_n)dt$

(17e2)=$\left(\vec{y}_{nt}(T), \vec{y}_t(T)\right) - \left(\vec{y}_{nt}(0), \vec{y}_t(0)\right) + s(t, \vec{y}_n, \vec{y})(T) - s(t, \vec{y}_n, \vec{y})(0) - \int_0^T s_t(t, \vec{y}_n, \vec{y})dt$

(17e3)= $\left(\vec{y}_t(T), \vec{y}_{nt}(T) - \vec{y}_t(T)\right) - \left(\vec{y}_t(0), \vec{y}_{nt}(0) - \vec{y}_t(0)\right) + s(t, \vec{y}, \vec{y}_n - \vec{y})(T) - s(t, \vec{y}, \vec{y}_n - \vec{y})(0) - \int_0^T s_t(t, \vec{y}, \vec{y}_n - \vec{y})dt$

Since $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$, $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(I,U)\right)^2$and $\vec{y}_{nt} \longrightarrow \vec{y}_t$ weakly in $\left(L^2(Q)\right)^2$, then from (17c) and the assumptions on h_1 and h_2, we obtain

(17e1)= $2\int_0^T (h_1(y_{1n}), y_{1n}) + (h_2(y_{2n}), y_{2n})) + (w_1, y_{1n})_\Gamma + (w_2, y_{2n})_\Gamma)dt \longrightarrow 2\int_0^T (h_1(y_1), y_1) + (h_2(y_2), y_2)) + (w_1, y_1)_\Gamma + (w_2, y_2)_\Gamma)\,dt$

by the same way that we used to get (14), once can get also that

$\vec{y}_{nt}(T) \rightarrow \vec{y}_t(T)$ strongly in $(L(\Omega))^2$ (17f)

On the other hand, since $\vec{y}_n \longrightarrow \vec{y}$ weakly in $\left(L^2(I,U)\right)^2$, then using (14,17f), to get

$(17e2) \rightarrow R.H.S.of\ (17d) = 2\int_0^T (h_1(y_1), y_1) + (h_2(y_2), y_2)) + (w_1, y_1)_\Gamma + (w_2, y_2)_\Gamma)\,dt$

and all the terms in (17e3) imply to zero, so as the first two terms in the LHS of (17e), hence (17e) gives

$\int_0^T s(t, \vec{y}_n - \vec{y}, \vec{y}_n - \vec{y})dt \rightarrow 0$

From assumption (A-iii), once get

$\bar{a}\int_0^t \|\vec{y}_n - \vec{y}\|_1^2 dt \rightarrow 0$ as $n \rightarrow \infty$, so we get that $\vec{y}_n \longrightarrow \vec{y}$ strongly in $\left(L^2(I,U)\right)^2$.

Uniqueness of the solution: Let $\vec{y} = (y_1, y_2)$ and $\vec{\bar{y}} = (\bar{y}_1, \bar{y}_2)$ be two solutions of the WF (10-11), in particular, i.e. y_1 and $\bar{y}_1$ are satisfied the WF (10a,b), subtracting each obtained equation from the other and then setting $v_1 = y_1 - \bar{y}_1$, yields to

$\langle (y_1 - \bar{y}_1)_{tt}, y_1 - \bar{y}_1 \rangle + \alpha_1(t, y_1 - \bar{y}_1, y_1 - \bar{y}_1) + (\beta_1(y_1 - \bar{y}_1), y_1 - \bar{y}_1)_\Omega - (\beta(y_2 - \bar{y}_2), y_1 - \bar{y}_1)_\Omega = (h_1(y_1) - h_1(\bar{y}_1), y_1 - \bar{y}_1)_\Omega$

$\left((y_1 - \bar{y}_1)(0), (y_1 - \bar{y}_1(0)\right)_\Omega = 0$,

And for$v_1 = (y_1 - \bar{y}_1)_t$, the following initial condition it holds

$\left((y_1 - \bar{y}_1)_t(0), (y_1 - \bar{y}_1)_t(0)\right)_\Omega = 0$

The same thing will be happened, for the solutions y_2 & $\bar{y}_2$ and (11a,b) , with $v_1 = y_2 - \bar{y}_2$, to get that

$\langle (y_2 - \bar{y}_2)_{tt}, y_2 - \bar{y}_2 \rangle + \alpha_2(t, y_2 - \bar{y}_2, y_2 - \bar{y}_2) + (\beta_2 y_2 - \bar{y}_2), y_2 - \bar{y}_2)_\Omega + (\beta(y_1 - \bar{y}_1), y_1 - \bar{y}_1)_\Omega = (h_2(y_{12}) - h_2(\bar{y}_2), y_2 - \bar{y}_2)_\Omega$,

$\left((y_2 - \bar{y}_2)(0), (y_2 - \bar{y}_2)(0)\right)_\Omega = 0$, and

$\left((y_2 - \bar{y}_2)_t(0), (y_2 - \bar{y}_2)_t(0)\right)_\Omega = 0$

Adding the above two equations, using Lemma 1.2 in ref. (16) for the 1^{st}in LHS of the obtained equation which will be positive, integrating both sides with respect to t from 0 to t, using the initial conditions, assumption (A- iii) on the LHS and assumption (A-ii) on the right hand side of the obtained equation, and finally applying the B -G inequality, to get

$\int_0^t [\frac{d}{dt}\|(\vec{y}-\vec{\bar{y}})_t(t)\|_0^2 + 2\bar{a}\|(\vec{y}-\vec{\bar{y}})\|_1^2]dt \le 2L$
$\int_0^t [\|(\vec{y}-\vec{\bar{y}})_t\|_0^2 + 2\bar{a}\|(\vec{y}-\vec{\bar{y}})\|_1^2]\,dt,$

where $L = L_1 + L_2$, $L_3 = \alpha + 2L$, $\bar{a} = \frac{L_3}{2L}$ $\Rightarrow$

$\|(\vec{y}-\vec{\bar{y}})(t)\|_1^2 = 0\,, \forall t \in I \Rightarrow$

$\|(\vec{y}-\vec{\bar{y}})(t)\|_{(L^2(I,U))^2} = 0$ ⇛the solution is unique.

Lemma (1): In addition to assumptions (A), if the boundary control vector is bounded, then the operator $\vec{w} \longmapsto \vec{y}_{\vec{w}}$ from $(L^2(\Sigma))^2$ into $(L^\infty(I, L^2(\Omega)))^2$or in to $(L^2(I,U))^2$ or in to $(L^2(Q))^2$ is continuous.

Proof: Let $\vec{w} = (w_1, w_2), \vec{\bar{w}} = (\bar{w}_1, \bar{w}_2) \in (L^2(\Sigma))^2$, set $\overrightarrow{\delta w} = \vec{\bar{w}} - \vec{w}$, then for $\varepsilon > 0$, $\vec{w}_\varepsilon = \vec{w} + \varepsilon\overrightarrow{\delta w} \in (L^2(\Sigma))^2$, then by Theorem 1, $\vec{y} = \vec{y}_{\vec{w}} = (y_1, y_2)$ and $\vec{y}_\varepsilon = \vec{y}_{\vec{u}_\varepsilon} = (y_{1\varepsilon}, y_{2\varepsilon})$ are their corresponding states solutions which are satisfied the WF (10-11), setting $\overrightarrow{\delta y}_\varepsilon = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, then (10-11), give

$\langle \delta y_{1\varepsilon tt}, v_1\rangle + \alpha_1(t, \delta y_{1\varepsilon}, u_1) + (\beta_1\delta y_{1\varepsilon}, u_1)_\Omega - (\beta\delta y_{2\varepsilon}, u_1)_\Omega = (h_1(y_1 + \delta y_{1\varepsilon}) - h_1(y_1)\,, u_1)_\Omega + (\varepsilon\delta w_1, v_1)_\Gamma$ (24a)

$\delta y_{1\varepsilon}(x,0) = 0$ and $\delta y_{1\varepsilon t}(x,0) = 0$ (24b)

$\langle \delta y_{2\varepsilon tt}, v_2\rangle + \alpha_2(t, \delta y_{2\varepsilon}, u_2) + (\beta_2\delta y_{2\varepsilon}, u_2)_\Omega + (\beta\delta y_{1\varepsilon}, u_2)_\Omega = (h_2(y_2 + \delta y_{2\varepsilon}) - h_2(y_2, u_2), u_2)_\Omega + (\varepsilon\delta w_1, u_2)_\Gamma$ (25a)

$\delta y_{2\varepsilon}(x,0) = 0$ and $y_{2\varepsilon t}(x,0) = 0,$ (25b)

Substituting $u_1 = \delta y_{1\varepsilon t}$ in (24a) and $u_2 = \delta y_{2\varepsilon t}$ in (25a), adding the two obtained equations, using Lemma 1.2 in (16) for the 1st term of the left hand side (LHS), to give

$\frac{d}{dt}\left[\|\overrightarrow{\delta y}_{\varepsilon t}(t)\|_0^2 + s(t, \overrightarrow{\delta y}_\varepsilon, \overrightarrow{\delta y}_{\varepsilon t})\right] - s_t(t, \overrightarrow{\delta y}_\varepsilon, \overrightarrow{\delta y}_{\varepsilon t}) = 2((\beta\delta y_{2\varepsilon}, \delta y_{1\varepsilon t})_\Omega - (\beta\delta y_{1\varepsilon}, \delta y_{2\varepsilon t})_\Omega + L_1(\delta y_{1\varepsilon}, \delta y_{1\varepsilon t}) + L_2(\delta y_{2\varepsilon}, \delta y_{2\varepsilon t}) + (w_1, \delta y_{1\varepsilon t})_\Gamma + (w_2, \delta y_{2\varepsilon t})_\Gamma)$

Integration both sides of the above equality on $[0, t]$, using assumptions (A-ii and iii), give

$\int_0^t \frac{d}{dt}[\|\overrightarrow{\delta y}_{\varepsilon t}(t)\|_0^2 + \bar{a}\|\overrightarrow{\delta y}_\varepsilon\|_1^2]\,dt \le \alpha\|\overrightarrow{\delta y}_\varepsilon\|_1^2 + 2\int_0^t\int_\Omega [b|\delta y_{1\varepsilon}||\delta y_{2\varepsilon t}| + L_1|\delta y_{1\varepsilon}||\delta y_{1\varepsilon t}| + b|\delta y_{2\varepsilon}||\delta y_{1\varepsilon t}| + L_2|\delta y_{2\varepsilon}||\delta y_{2\varepsilon t}|]dxdt + 2\int_0^t\int_\Gamma [\varepsilon|\delta w_1||\delta y_{1\varepsilon t}| + \varepsilon|\delta w_2||\delta y_{2\varepsilon t}|]d\gamma dt\,.$

Using assumption (A-i), the definitions of the norms and the relations between them, and then using the trace theorem, to get

$\|\overrightarrow{\delta y}_{\varepsilon t}(t)\|_0^2 + \bar{a}\|\overrightarrow{\delta y}_\varepsilon(t)\|_1^2 \le b_3\int_0^t (\|\overrightarrow{\delta y}_\varepsilon\|_0^2 + \|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2)\,dt + 2\varepsilon\int_0^T \|\overrightarrow{\delta w}\|_\Gamma^2\,dt + 2\varepsilon\int_0^t \|\overrightarrow{\delta y}_{\varepsilon t}\|_\Gamma^2\,dt \le \bar{L}_1\left(\|\overrightarrow{\delta w}(t)\|_\Sigma^2 + \|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2\right) + b_3\int_0^t \|\overrightarrow{\delta y}_\varepsilon\|_0^2\,dt + b_3\int_0^t \|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2\,dt$

$\le \bar{L}_1\|\overrightarrow{\delta w}(t)\|_\Sigma^2 + b_3\int_0^t (\|\overrightarrow{\delta y}_\varepsilon\|_0^2 + \bar{a}\|\overrightarrow{\delta y}_{\varepsilon t}\|_1^2)\,dt$

where $b_3 = 2b + L_1 + L_2, \bar{L}_1 = 2\varepsilon, \bar{L}_3 = b_3 + \bar{L}_1, \bar{a} = \frac{\bar{L}_3}{b_3}$.

Applying the B -G inequality, with $L^2 = \bar{L}_1 e^{b_3}$, to get

$\|\overrightarrow{\delta y}_{\varepsilon t}(t)\|_0^2 + \bar{a}\|\overrightarrow{\delta y}_\varepsilon(t)\|_1^2 \le L^2\|\overrightarrow{\delta u}(t)\|_\Sigma^2\,, \ \forall t \in \bar{I} \Rightarrow$

$\|\overrightarrow{\delta y}_\varepsilon(t)\|_1^2 \le L^2\|\overrightarrow{\delta u}(t)\|_\Sigma^2\,, L^2 = \frac{L^2}{\bar{a}}, \forall t \in \bar{I} \Rightarrow$

$\|\overrightarrow{\delta y}_\varepsilon\|_{L^\infty(I, L^2(\Omega))} \le L\|\overrightarrow{\delta w}\|_\Sigma$, $\|\overrightarrow{\delta y}_\varepsilon\|_{L^2(I,V)} \le L\|\overrightarrow{\delta w}\|_\Sigma$ and $\|\overrightarrow{\delta y}_\varepsilon\|_Q \le L\|\overrightarrow{\delta w}\|_\Sigma$

Form the above three inequalities the Lipschitz continuity of the operator $\vec{w} \longmapsto \vec{y}$ is obtained.

The Existence of a Classical Optimal Control: This section is concerned with the theorem of existence CCBOCV where satisfying EIESVC is proved. The following assumption and lemma will be needed.

Assumptions (B): Consider p_{li} and q_{li} (for each $l = 0,1,2$ and $i = 1,2$) is of "Carathéodory type " on $(Q \times \mathbb{R})$and on$(\Sigma \times \mathbb{R})$ respectively and satisfies the following sub quadratic condition with respect to y_i and u_i, i.e.

$|p(x,t,y_i,w_i)| \le P_{li}(x,t) + c_{li}y_i^2,$
$|q_{li}(x,t,w_i)| \le Q_{li}(x,t) + d_{li}(w_i)^2,$
where$y_i, w_i \in \mathbb{R}$ with $P_{li} \in L^1(Q)$,$Q_{li} \in L^1(\Sigma)$.

Lemma (2): With assumptions (B), and $\forall l = 0,1,2$ the functional $\vec{w} \longmapsto J_l(\vec{w})$, is continuous on $(L^2(\Sigma))^2$.

Proof: From assumptions(B), with using proposition 1, the integrals $\int_Q p_{li}(x,t,y_i)\,dxdt$ and $\int_\Sigma q_{li}(x,t,w_i)\,d\sigma$ are continuous on $L^2(Q)$ and $L^2(\Sigma)$ respectively $\forall i = 1,2$, and $\forall l = 0,1,2$, which gives $J_l(\vec{w})$ is continuous on $(L^2(\Sigma))^2$, $\forall l = 0,1,2$.

Theorem(2): In addition to the assumptions (A&B), if the set $\vec{W}$ is convex and compact, $\vec{W}_A \ne \emptyset$, g_{1i} is independent of w_i for each $i = 1,2$, p_{0i} and p_{2i} are convex w.r.t w_i for fixed (x,t,y_i) . Then there exists a CCBOCV.

Proof: From the assumptions on $\vec{W}$ and the "Egorov's theorem", once get that $\vec{W}_c$ is weakly compact. Since $\vec{W}_A \ne \emptyset$, then there is $\vec{\bar{w}} \in \vec{W}_A$ and there is a minimum sequence $\{\vec{w}_k\}$ with $\vec{w}_k \in \vec{W}_A\,, \forall k$, such that $\lim_{n\to\infty} J_0(\vec{w}_k) = \inf_{\vec{w}\in\vec{U}_A} J_0(\vec{\bar{w}})$. But $\vec{W}_c$ is weakly compact, then the sequence $\{\vec{w}_k\}$ has a subsequence for simplicity say again $\{\vec{w}_k\}$ such that $\vec{w}_k \longrightarrow \vec{w}$ weakly in $\vec{W}_c$ and $\|\vec{w}_k\|_\Sigma \leq c$, $\forall k$. From theorem 1, for each control $\vec{w}_k$ the weak form of the state equations has a unique solution $\vec{y}_k = \vec{y}_{\vec{w}_k}$, and the norms $\|\vec{y}_k\|_{L^2(I,V)}$, $\|\vec{y}_{kt}\|_{L^2(Q)}$ are bounded, then by "Alaoglu's theorem" there exist a subsequence of $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$ for simplicity say again $\{\vec{y}_k\}$ and $\{\vec{y}_{kt}\}$ such that

$\vec{y}_k \longrightarrow \vec{y}$ weakly in $\left(L^2(I,U)\right)^2$, and

$\vec{y}_{kt} \longrightarrow \vec{y}_t$ weakly in $\left(L^2(Q)\right)^2$.

Then by applying the "Aubin theorem" in (16), once get that there exists a subsequence of $\{\vec{y}_k\}$ for simplicity say again $\{\vec{y}_k\}$ such that $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$.

Now, Since for each k, $\vec{y}_k$ is a solutions of the WF (12c) - (12f), substituting this solution in the above indicate WF, then multiplying both sides of each one by $\zeta_1(t)$ and $\zeta_2(t)$ respectively (with $\zeta_i \in C^2[0,T]$, such that $\zeta_i(T) = \acute{\zeta}_i(T) = 0$, $\zeta_i(0) \neq 0, \acute{\zeta}_i(0) \neq 0$, $\forall i = 1,2$). Rewriting the first terms in the left hand side of each one of their, integrating both sides from 0 to , finally integrating by parts for these first terms, one has

$$\int_0^T \frac{d}{dt}(y_{1kt}, u_1)\zeta_1(t)dt + \int_0^T [\alpha_1(t, y_{1k}u_1) + (\beta_1 y_{1k}, u_1)_\Omega - (\beta y_{2k}, u_1)_\Omega]\zeta_1(t)]dt = \int_0^T (h_1(y_{1k}), u_1)_\Omega \zeta_1(t)dt + \int_0^T (w_{1k}, u_1)_\Gamma \zeta_1(t)\,dt + (y_{1k}(0), u_1)_\Omega \zeta_1(0) \quad (26)$$

$$\int_0^T \frac{d}{dt}(y_{2kt}, u_2)\zeta_2(t)dt + \int_0^T [\alpha_2(t, y_{2k}, u_2) + (\beta_2 y_{2k}, u_2)_\Omega + (\beta y_{1k}, u_1)_\Omega]\zeta_2(t)]dt = \int_0^T (h_2(y_{2k}), u_2)_\Omega \zeta_2(t)dt + \int_0^T (w_{2k}, u_2)_\Gamma \zeta_2(t)\,dt + (y_{2k}(0), u_2)_\Omega \zeta_2(0) \quad (27)$$

The limits in the LHS of (26) and (27) can be passaged using the same steps that are used in the proof of Theorem 1, so it remain the passage to the limits in the right hand side of (26) and (27) and this will be down as follows:

Let $\forall i = 1,2$, $u_i \in C[\overline{\Omega}]$, $w_i = u_i\zeta_i(t)$, then $\eta_i \in C[\overline{Q}] \in L^\infty(I,U) \subset L^2(Q)$, set $\bar{h}_{i1}(y_{1k}) = h_{i1}(y_{ik})\eta_i$, then $\bar{h}_{i1}: Q \times \mathbb{R} \to \mathbb{R}$ is of "Carathéodory type ", using Proposition 1, to get the integral $\int_Q h_{i1}(y_{ik})\eta_i\,dxdt$ is continuous with respect to y_{ik}, but $y_{ik} \longrightarrow y_i$ strongly in $L^2(Q)$ then

$$\int_Q h_{i1}(y_{1k})\eta_i\,dxdt \to \int_Q h_{i1}(y_i)\eta_i\,dxdt \quad (28a)$$

$,\forall \eta_i \in C[\bar{Q}]$, for $i = 1,2$

then it also are hold for every $u_i \in U$, $\forall i = 1,2$, since $C(\overline{\Omega})$ is dense in U.

On the other hand since, $\eta_{ik} \longrightarrow \eta_i$,weakly in $L^2(\Sigma)$ then $\forall u_i \in C(\overline{\Omega})]$, for $i = 1,2$

$$\int_\Sigma \eta_{ik}u_i\zeta_i(t)dxdt \to \int_\Sigma \eta_i u_i\,\zeta_i dtxdt, \quad (28b)$$

Hence from the above convergences the following two weak forms are obtained $\forall u_1, u_2 \in U$, a.e. on I

$$\langle y_{1tt}, u_1\rangle + \alpha_1(\mathrm{t}, y_1, u_1) + (\beta_1 z_1, u_1)_\Omega + (\beta y_2, u_1)_\Omega = (h_1(y_1), u_1)_\Omega + (w_1, u_1)_\Gamma, \quad (29a)$$

$$\langle y_{2tt}, u_2\rangle + \alpha_2(\mathrm{t}, y_2, u_2) + (\beta_2 y_2, u_2)_\Omega + (\beta y_1, u_2)_\Omega = (h_2(y_1), u_1)_\Omega + (w_2, u_2)_\Gamma, \quad (30a)$$

To pass the limits in the initial conditions which are associated with these weak forms, the same steps used in the proof of Theorem 1 can be also used here to get the requirement results for the initial conditions. Hence y_1 and y_2 are the solutions of the WF of the state equations.

On the other hand, since $J_1(\vec{w}_k) = \int_Q p_{11}(y_{1k})\,dxdt + \int_Q q_{12}(y_{2k})\,dxdt$, with p_{1i} (for $i = 1,2$) is independent of u_i and it is continuous wrt y_{ik}, then by Lemma2 $\int_Q p_{1i}(y_{ik})\,dxdt$ is continuous with respect to y_{ik}, but $\vec{y}_k \longrightarrow \vec{y}$ strongly in $\left(L^2(Q)\right)^2$, then from proposition 1

$J_1(\vec{w}) = \lim_{k\to\infty} J_1(\vec{w}_k) = 0.$

Again since $\forall i = 1,2$ and $\forall l = 0,2$, $p_{li}(y_{ik})$ is continuous with respect to y_{ik}, then from the proof of Lemma 2, one has

$$\int_Q p_{li}(y_{ik})\,dxdt \longrightarrow \int_Q p_{li}(y_i)\,dxdt \quad (31)$$

Now, from assumptions (B), $q_{li}(w_i)$ is weakly lower semi continuous with respect to w_i, $\forall i = 1,2$ and $l = 0,2$, then from (31), one has

$$\int_Q p_{li}(y_i)\,dxdt + \int_\Sigma q_{li}(w_i)\,d\sigma \leq \lim_{k\to\infty}\inf \int_\Sigma q_{li}(w_{ik})d\sigma + \int_Q p_{li}(y_i)\,dxdt = \lim_{k\to\infty}\inf \int_\Sigma (q_{li}(w_{ik})d\sigma + \lim_{k\to\infty} \int_Q (p_{li}(y_i) - p_{li}(y_{ik}))dxdt + \lim_{k\to\infty} \int_Q p_{li}(y_{ik})\,dxdt$$

$$= \lim_{k\to\infty}\inf \int_\Sigma q_{li}(w_{ik})\,d\sigma + \lim_{k\to\infty}\inf \int_Q p_{li}(y_{ik})\,dxdt$$

i.e. $J_l(\vec{w}) \leq \lim_{k\to\infty}\inf J_l(\vec{w}_k)$, (for each $l = 0,2$)

Then $J_2(\vec{w}) \leq 0$ (since $J_2(\vec{w}_k) \leq 0$, $\forall k$), which means $\vec{w} \in \vec{W}_A$ and

$J_0(\vec{w}) \leq \lim_{k\to\infty}\inf J_0(\vec{w}_k) = \lim_{k\to\infty} J_0(\vec{w}_k) = \inf_{\vec{\vec{u}} \in \vec{U}_A} J_0(\vec{\vec{w}}_k)$

Hence $\vec{w}$ is a CCBOCV.

Assumptions (C): If h_{iy_i}, $p_{l_iy_i}$and $q_{l_iw_i}$, ($\forall l = 0,1,2$ and $\forall i = 1,2$) are of "Carathéodory type" on $Q \times (\mathbb{R})$, $Q \times (\mathbb{R})$and on $\Sigma \times (\mathbb{R})$ respectively, such that

$\left|h_{iy_i}(x,t,y_i)\right| \leq \acute{L}_i$

$\left|p_{l_iy_i}(x,t,y_i,w_i)\right| \leq K_{li}(x,t) + m_{li}|y_i|$,

$\left|q_{l_iu_i}(x,t,y_i,w_i)\right| \leq L_{li}(x,t) + n_{li}|y_i|$

where $(x,t) \in Q$, $y_i, w_i \in \mathbb{R}$, $K_{li}(x,t) \in L^2(Q)$ $L_{li}(x,t) \in L^2(\Sigma)$, $\acute{L}_i, m_{li}, n_{li} \geq 0$.

Theorem(3):
Dropping the index l in p_{li} , q_{li} & J_l. With the assumptions (A), (B) and (C), the following ADCEQS $\vec{z} = (z_1, z_2)$ of the state equations (1-6) are given by:

$$z_{1tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\alpha_{ij} \frac{\partial z_1}{\partial x_i}) + \beta_1 z_1 + \beta z_2 = z_1 h_{1y_1}(y_1) + p_{1y_1}(y_1), \text{ in } \Omega \quad (32a)$$

$$\frac{\partial z_1}{\partial v_\alpha} = 0 \text{ on } \Sigma,\ z_1(x,T) = 0,\ z_{1t}(x,T) = 0 \text{ on } \Omega \quad (32b)$$

$$z_{2tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\beta_{ij} \frac{\partial z_2}{\partial x_i}) + \beta_2 z_2 - \beta z_1 = z_2 h_{2y_2}(y_2) + p_{2y_2}(y_2), \text{ in}\Omega \quad (33a)$$

$$\frac{\partial z_2}{\partial v_\beta} = 0, \text{ on}\Sigma,\ z_2(x,T) = 0,\ z_{2t}(x,T) = 0, \text{ on } \Omega \quad (33b)$$

where each of v_α, v_β is a unit vector normal outer on the boundary Σ
And the "Hamiltonian" is defined:

$$H(x,t,y_i,z_i,w_i) = \sum_{i=1}^{2}(z_i h_i(y_i) + p_i(y_i) + q_i(,w_i))$$

Where

$$J(\vec{w}) = \int_Q [p_1(y_1) + p_2(y_2)]dxdt + \int_\Sigma [q_1(w_1) + q_2(w_2)]d\gamma dt$$

Then for $\vec{w} \in \vec{U}$, the directional derivative of G is given by where

$$DJ(\vec{w},, \vec{\vec{w}} - \vec{w}) = \lim_{\varepsilon \to 0} \frac{J(\vec{w} + \varepsilon \overrightarrow{\delta w}) - J(\vec{w})}{\varepsilon} = \int_\Sigma \begin{pmatrix} z_1 + q_{1w_1} \\ z_2 + q_{2w_2} \end{pmatrix} \cdot \begin{pmatrix} \delta w_1 \\ \delta w_2 \end{pmatrix} d\sigma = H_{\vec{w}}(x,t,\vec{y},\vec{z},\vec{w})$$

Proof: At first let, the WF of the adjoint equations are given $\forall u_1, u_2 \in U$, by

$$\langle z_{1tt}, u_1 \rangle \alpha_1(t, z_1, u_1) + (\beta_1 z_1, u_1)_\Omega + (\beta z_2, u_1)_\Omega = (z_1 h_{1y_1}, u_1)_\Omega + (p_{1y_1}, u_1)_\Omega \text{ , a.e. on} I \quad (34a)$$

$$(z_1(T), u_1)_\Omega = (z_{1t}(T), u_1)_\Omega = 0, \quad (34b)$$

$$\langle z_{2t}, u_2 \rangle + \alpha_2(t, z_2, u_2) + (\beta_2 z_2, u_2)_\Omega - (\beta z_1, u_2)_\Omega = (z_2 h_{2y_2}, u_2)_\Omega + (p_{2y_2}, u_2)_\Omega, \text{ a.e. on} I \quad (35a)$$

$$(z_2(T), u_2)_\Omega = (z_{2t}(T), u_2)_\Omega = 0, \quad (35b)$$

From the given assumptions and using the same way which is used in the proof of Theorem1, once can prove that the weak from (34-35) has a unique solution $\vec{z} = (z_1, z_2) \in (L^2(Q))^2$.
Substituting $u_1 = \delta y_{1\varepsilon}$ (34a) and $u_2 = \delta y_{2\varepsilon}$ in (35a), integrating both sides on $[0,T]$, to get

$$\int_0^T \langle \delta y_{1\varepsilon}, z_{1tt} \rangle dt + \int_0^T [\alpha_1(t, z_1, \delta y_{1\varepsilon}) + (\beta_1 z_1, \delta y_{1\varepsilon})_\Omega + (\beta z_2, \delta y_{1\varepsilon})_\Omega]dt = \int_0^T [(z_1 h_{1y_1}, \delta y_{1\varepsilon})_\Omega + (p_{1y_1}, \delta y_{1\varepsilon})_\Omega]dt \quad (36)$$

$$\int_0^T \langle \delta y_{2\varepsilon}, z_{2tt} \rangle dt + \int_0^T [\alpha_2(t, z_2 \delta y_{2\varepsilon}) + (\beta_2 z_2, \delta y_{2\varepsilon})_\Omega - (\beta z_1, \delta y_{2\varepsilon})_\Omega]dt = \int_0^T [(z_2 h_{2y_2}, \delta y_{2\varepsilon})_\Omega + (p_{2y_2}, \delta y_{2\varepsilon})_\Omega]dt \quad (37)$$

Now, let $\vec{w}, \vec{\vec{w}} \in (L^2(Q))^2$, $\overrightarrow{\delta w} = \vec{\vec{w}} - \vec{w}$, for $\varepsilon > 0$, $\vec{w}_\varepsilon = \vec{w} + \varepsilon \overrightarrow{\delta w} \in (L^2(Q))^2$, then by theorem 1, $\vec{y} = \vec{y}_{\vec{w}}$ & $\vec{y}_\varepsilon = \vec{y}_{\vec{w}_\varepsilon}$ are their corresponding solutions. Setting $\overrightarrow{\delta y}_\varepsilon = (\delta y_{1\varepsilon}, \delta y_{2\varepsilon}) = \vec{y}_\varepsilon - \vec{y}$, substituting $u_1 = z_1$ and $u_2 = z_2$ in (24a) and (25a) respectively, integrating both sides on $[0,T]$, then Integrating by parts twice the first term in the left hand side of each one of the obtained equation, finding the "Fréchet derivatives" of f_1 and f_2 in the right hand side of each one them (which are exist from the assumptions(C), then from the result of Lemma 1 and the "Minkowiski inequality", once get

$$\int_0^T \langle \delta y_{1\varepsilon}, z_{1tt} \rangle dt + \int_0^T [\alpha_1(t, \delta y_{1\varepsilon}, z_1) + (\beta_1 \delta y_{1\varepsilon}, z_1)_\Omega - (\beta \delta y_{2\varepsilon}, z_1)_\Omega]dt = \int_0^T (h_{1y_1} \delta y_{1\varepsilon}, z_1)_\Omega dt + \int_0^T (\varepsilon \delta w_1, z_1)_\Gamma dt + O_{11}(\varepsilon) \quad (38)$$

$$\int_0^T \langle \delta y_{2\varepsilon}, z_{2tt} \rangle dt + \int_0^T [\alpha_2(t, \delta y_{2\varepsilon}, z_2) + (\beta_2 \delta y_{2\varepsilon}, z_2)_\Omega + (\beta \delta y_{1\varepsilon}, z_2)_\Omega]dt = \int_0^T (h_{2y_2} \delta y_{2\varepsilon}, z_2)_\Omega dt + \int_0^T (\varepsilon \delta w_2, z_2)_\Gamma dt + O_{12}(\varepsilon) \quad (39)$$

where $O_{1i}(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_{1i}(\varepsilon) = \|\delta y_{i\varepsilon}\|_Q$, for each $i = 1,2$

Subtracting (38), (39) from (36), (37) respectively, adding the two obtain equations, once get

$$\varepsilon \int_0^T [(\delta w_1, z_1)_\Gamma + (\delta w_2, z_2)_\Gamma]dt + O_1(\varepsilon) = \int_0^T [(p_{1y_1}, \delta y_{1\varepsilon}) + (p_{2y_2}, \delta y_{2\varepsilon})]dt \quad (40)$$

where $O_1(\varepsilon) = O_{11}(\varepsilon) + O_{12}(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_1(\varepsilon) = \|\overrightarrow{\delta y_\varepsilon}\|_Q$

On the other hand, from the assumptions on p_1 ,p_2, q_1, q_2 the definition of the "Fréchet derivative", the result of Lemma 1, and then using "Minkowiski inequality", we have

$$J_0(\vec{w}_\varepsilon) - J_0(\vec{w}) = \int_Q (p_{1y_1} \delta y_{1\varepsilon} + p_{2y_2} \delta y_{2\varepsilon}) dxdt + \varepsilon \int_\Sigma (q_{1w_1} \delta w_1 + q_{2w_2} \delta w_2) d\gamma dt + O_2(\varepsilon) , \quad (41)$$

where $O_2(\varepsilon) = \|\overrightarrow{\delta y_\varepsilon}\|_Q + \varepsilon \|\overrightarrow{\delta w}\|_\Sigma$, $O_2(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$
Now, by substituting (40) in (41), one have that

$$J_0(\vec{w}_\varepsilon) - J_0(\vec{w}) = \varepsilon \int_\Sigma [(z_1 + q_{1w_1}) \delta w_1 + (z_2 + q_{2w_2}) \delta w_2] \, dxdt + O_3(\varepsilon)$$

where $O_3(\varepsilon) = O_1(\varepsilon) + O_2(\varepsilon) \longrightarrow 0$, as $\varepsilon \longrightarrow 0$, with $O_3(\varepsilon) = 2\|\overrightarrow{\delta y_\varepsilon}\|_Q + \varepsilon \|\overrightarrow{\delta w}\|_\Sigma$
Finally, dividing both sides of the above equality by ε, then taking the limit $\varepsilon \longrightarrow 0$, once get

$$DJ(\vec{w},\vec{\bar{w}}-\vec{w}) = \int_\Sigma \begin{pmatrix} z_1 + q_{1w_1} \\ z_2 + q_{2w_2} \end{pmatrix} \cdot \begin{pmatrix} \delta w_1 \\ \delta w_2 \end{pmatrix} d\sigma.$$

Necessary and sufficient conditions for optimality: In this section the necessary and sufficient theorems for optimality under prescribed assumptions are proved as follows:

Theorem(4): (NCs for Optimality, or Multipliers Theorem):

a) with assumptions (A), (B) , (C) if $\vec{W}_c$ is convex, the control $\vec{w} \in \vec{W}_A$ is optimal, then there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$ such that the following Kuhn-Tucker-Lagrange (K.T.L.) conditions are satisfied:

$$\sum_{l=0}^{2} \lambda_l DJ_l(\vec{w},,\vec{\bar{w}}-\vec{w}) \geq 0,\ \forall \vec{\bar{w}} \in \vec{W}, \quad (42a)$$

$$\lambda_2 J_2(\vec{w}) = 0, \text{(Transversality condition)} \quad (42b)$$

(b) The inequality (42a) is equivalent to the (weak) pointwise minimum principle

$$H_{\vec{w}}(x,t,\vec{y},\vec{z},\vec{w}).\vec{w}(t) = \min_{\vec{\bar{w}} \in \vec{W}} H_{\vec{w}}(x,t,\vec{y},\vec{z},\vec{w}).\vec{\bar{w}}(t) \text{, a.e. on } Q \quad (43)$$

Where

$$H_{\vec{w}}(x,t,\vec{y},\vec{z},\vec{w}) = (z_1 + q_{1w_1}(t,w_1), z_2 + q_{2w_2}(t,w_2))$$

with $q_i = \sum_{l=0}^{2} \lambda_l q_{li}$ and $z_i = \sum_{l=0}^{2} \lambda_l z_{li}$, (for $i = 1,2$).

Proof: a) From Lemma 2, the functional $J_l(\vec{w})$ (for $l = 0,1,2$) is continuous and from Theorem 3, the functional DJ_l (for $l = 0,1,2$) is continuous wrt $\vec{\bar{w}}-\vec{w}$ and linear in $\vec{\bar{w}}-\vec{w}$, then DJ_l is M −differential for every M, then using the K.T.L. theorem in (16), there exist multipliers $\lambda_l \in \mathbb{R}$, $l = 0,1,2$ with $\lambda_0 \geq 0, \lambda_2 \geq 0$, $\sum_{l=0}^{2} |\lambda_l| = 1$, such that (42a-b) are satisfied, by using Theorem 3, then (42a) becomes

$\sum_{l=0}^{2} \int_\Sigma \sum_{i=1}^{2} \lambda_l (z_{li} + q_{liw_i}) \delta w_i d\gamma dt \geq 0$, which can be rewritten as

$$\int_\Sigma (z_1 + q_{1w_1}, z_2 + q_{2w_2}).(\vec{\bar{w}}-\vec{w}) d\gamma dt \geq 0, \forall \vec{\bar{w}} \in \vec{W} \quad (44)$$

where $q_i = \sum_{l=0}^{2} \lambda_l\, q_{li}$, $z_i = \sum_{l=0}^{2} \lambda_l\, z_{li}$, $\forall i = 1,2$

To prove the second part, let $\{\vec{\bar{w}}_k\}$ be a dense sequence in $\vec{W}$, and let $q \subset Q$ be a measurable set " with *Lebesgue* measure μ " such that

$$\vec{\bar{w}}(x,t) = \begin{cases} \vec{\bar{w}}_k(x,t) & , \ if\ (x,t) \in q \\ \vec{w}(x,t) & , \ if\ (x,t) \notin q \end{cases}$$

Therefore (44) becomes

$$\int_q (z_1 + q_{1w_1}, z_2 + q_{2w_2}).(\vec{\bar{w}}-\vec{w}) d\gamma dt \geq 0, \quad (44a)$$

which implies to

$$(z_1 + q_{1w_1}, z_2 + q_{2w_2}).(\vec{\bar{w}}_k-\vec{w}) \geq 0, \text{ a.e. on} \Sigma \quad (44b)$$

This means (44b) is satisfied on Σ/S_k" the boundary of the region Q except in a subset S_k " such that $\mu(S_k) = 0$, $\forall k$, i.e. (44b) satisfies on $\Sigma/\cup_k S_k$ with $\mu(\cup_k S_k) = 0$, but $\{\vec{\bar{w}}_k\}$ is a dense sequence in the control set $\vec{W}$, then there exists $\vec{\bar{w}} \in \vec{W}$ such that

$(z_1 + q_{1w_1}, z_2 + q_{2w_2}).(\vec{\bar{w}}-\vec{w}) \geq 0$, a.e. on Σ, $\forall \vec{\bar{w}} \in \vec{W}$

i.e. (42a) gives (44). The converse is clear.

Theorem (5): (SCs for Optimality): In Addition to the assumptions (A), (B) & (C). Suppose $\vec{W}_c$ is convex, with $\vec{W}_c$ convex, h_i & p_{1i} (h_{1i}) are affine wrt y_i (wrt $w_i, \forall (x,t) \in \Sigma$) $\forall (x,t) \in Q$, p_{0i}, p_{2i} (q_{0i}, q_{2i})are convex with respect to y_i(wrt$w_i \forall (x,t) \in \Sigma$), $\forall (x,t) \in Q$, $\forall i = 1,2$. Then the necessary conditions of Theorem 4 with $\lambda_0 > 0$ are also sufficient.

Proof: Assume $\vec{w} \in \vec{W}_A$ is satisfied the K.T.L. condition (42). Let $J(\vec{w}) = \sum_{l=0}^{2} \lambda_l J_l(\vec{w})$, then using Theorem 3, to get

$$DJ(\vec{w},\vec{\bar{w}}-\vec{w}) = \sum_{l=0}^{2} \lambda_l \int_\Sigma \sum_{i=1}^{2} (z_{li} + q_{liwi})\, \delta w_i\, dxdt \geq 0$$

Since

$$h_1(x,t,y_1) = h_{11}(x,t)y_1 + h_{12}(x,t) = h_{11}y_1 + h_{12} \quad ,$$

and

$$h_2(x,t,y_2,w_2) = h_{21}(x,t)y_2 + h_{22}(x,t) = h_{21}y_2 + h_{22}$$

Let $\vec{w} = (w_1,w_2)$ & $\vec{\bar{w}} = (\bar{w}_1,\bar{w}_2)$ are two given controls vectors, then $\vec{y} = (y_{w1},y_{w2}) = (y_1,y_2)$ & $\vec{\bar{y}} = (\bar{y}_{\bar{w}1},\bar{y}_{\bar{w}2}) = (\bar{y}_1,\bar{y}_2)$ are their corresponding stats solutions. Substituting the pair $(\vec{u},\vec{y})$ in equations (1-6) and multiplying all the obtained equations by $\gamma \in [0,1]$ once and then substituting the pair $(\vec{\bar{w}},\vec{\bar{y}})$ in (1-6) and multiplying all the obtained equations by $\gamma_1 = (1-\gamma)$ once again, finally adding each pair from the corresponding equations together one gets:

$$(\gamma y_1 + \gamma_1 \bar{y}_1)_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\alpha_{ij} \frac{\partial(\gamma y_1 + \gamma_1 \bar{y}_1)}{\partial x_j}) + \beta_1 (\gamma y_1 + \gamma_1 \bar{y}_1) - \beta (\gamma y_2 + \gamma_1 \bar{y}_2) = h_{11}(\gamma y_1 + \gamma_1 \bar{y}_1) + h_{12} \quad (45a)$$

$$\frac{\partial(\gamma y_1 + \gamma_1)\bar{y}_1)}{\partial n_\alpha} = (\gamma w_1 + \gamma_1 \bar{w}_1), \text{ on } \Sigma \quad (45b)$$

$$\gamma y_1(x,0) + \gamma_1 \bar{y}_1(x,0) = y_1^0(x), \quad \gamma y_{1t}(x,0) + \gamma_1, \bar{y}_{1t}(x,0) = y_1^1(x) \quad (45c)$$

$$(\gamma y_2 + \gamma_1 \bar{y}_2)_{tt} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(\beta_{ij} \frac{\partial(\gamma y_2 + \gamma_1 \bar{y}_2)}{\partial x_j}) + \beta_1(\gamma y_2 + \gamma_1 \bar{y}_2) + \beta(\gamma y_2 + \gamma_1 \bar{y}_2) = h_{21}(\gamma y_2 + \gamma_1 \bar{y}_2) + h_{22} \quad (46a)$$

$\frac{\partial(\gamma y_2+\gamma_1\bar{y}_2)}{\partial n_\beta} = (\gamma w_2 + \gamma_1\bar{w}_2)$, on Σ (46b)

$\gamma y_2(x,0) + \gamma_1\bar{y}_2(x,0) = y_2^0(x)$, $\gamma y_{2t}(x,0) + \gamma_1\bar{y}_{2t}(x,0) = y_2^1(x)$ (46c)

Equations (45) and (46), show that if the control vector is $\vec{\tilde{w}} = (\tilde{w}_1, \tilde{w}_2)$ with $\vec{\tilde{w}} = \gamma\vec{w} + \gamma_1\vec{\bar{w}}$ then its corresponding state vector is $\vec{\tilde{y}} = (\tilde{y}_1, \tilde{y}_2)$ with $\tilde{y}_i = y_{i\tilde{w}_i} = y_{i(\gamma w_i+\gamma_1\bar{w}_i)} = \gamma y_i + \gamma_1\bar{y}_i$, $\forall i = 1,2$. This means the operator $\vec{w} \longmapsto \vec{y}_{\vec{w}}$ is "convex – linear" wrt $(\vec{y},\vec{w})$ $\forall(x,t)) \in Q$.

On the other hand, the function $J_1(\vec{w})$ is "convex – linear" with respect to $(\vec{y},\vec{w})$ for each $(x,t) \in Q$, this back to the fact that the sum of two affine functions $p_{1i}(y_i)$ ($q_{1i}(w_i)$, $\forall i = 1,2$) with respect to y_i (w_i) is affine and the operator $\vec{w} \longmapsto \vec{y}_{\vec{w}}$ is convex-linear.

The functions $J_0(\vec{w})$, $J_2(\vec{w})$ are convex with respect to $(\vec{y},\vec{w})$, for each $(x,t) \in Q$ (from the assumptions on the functions p_{l1} p_{l2}, q_{l1} and q_{l2}, $\forall l$ =0,2 and from the sum of two integral of convex function is also convex). Hence $J(\vec{w})$ is convex with respect to $(\vec{y},\vec{w})$, for each $(x,t) \in Q$ in the convex set $\vec{W}$, and has a continuous "Fréchet derivative" satisfies

$DJ(\vec{w},,\vec{\bar{w}} - \vec{w}) \geq 0 \Rrightarrow J(\vec{w})$ has a minimum at $\vec{w}$

$\Rrightarrow J(\vec{w}) \leq J(\vec{\bar{w}}), \forall\vec{\bar{u}} \in \vec{W} \Rrightarrow$

$\lambda_0 J_0(\vec{w}) + \lambda_1 J_1(\vec{w}) + \lambda_2 J_2(\vec{w}) \leq$

$\lambda_0 J_0(\vec{\bar{w}}) + \lambda_1 J_1(\vec{\bar{w}}) + \lambda_2 J_2(\vec{\bar{w}})$, $\forall\vec{\bar{u}} \in \vec{W}$

Let $\vec{\bar{w}} \in \vec{W}_A$, with $\lambda_2 \geq 0$ and from Transversality condition , the above inequality becomes

$\lambda_0 J_0(\vec{w}) \leq \lambda_0 J_0(\vec{\bar{w}})$, $\forall\vec{\bar{w}} \in \vec{W} \Rrightarrow$ $J_0(\vec{w}) \leq J_0(\vec{\bar{w}})$, $\forall\vec{\bar{w}} \in \vec{W} \Rrightarrow\therefore \vec{w}$ is a boundary optimal control.

Conclusions:

The Galerkin method with the Aubin theorem are used successfully to prove the existence of unique "continuous state vector" solution for CNLHEQS when the CCBCV is given. The theorem of existence CCBOCV governing by the CNLHEQS with equality and inequality constraints is proved. The existence of unique solution of the ADCEQS associated with the CNLHEQS is studied. The Frcéhet derivation of the Hamiltonian is derived. The theorems of the NCs and the SCs for the (boundary) optimality of the constrained problem are proved.

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مسألة السيطرة الامثلية الحدودية التقليدية من النمط المستمر لزوج من المعادلات التفاضلية الجزئية غير الخطية من النمط الزائدي بوجود قيدي التساوي والتباين

جميل أمير علي الهواسي

قسم الرياضيات، كلية العلوم، الجامعة المستنصرية، بغداد، العراق.

الخلاصة:

يهتم هذا البحث بمسألة وجود ووحدانية الحل المتجه للحالة " State Vector" لزوج من المعادلات التفاضلية من النمط الزائدي باستخدام طريقة كاليركن "Galerkin" عندما يكون متجه السيطرة الحدودية التقليدية "Classical boundary control vector" ثابتا" . تم برهان مبرهنة الوجود لسيطرة امثلية حدودية تقليدية من النمط المستمر بوجود قيدي التساوي والتباين لمتجه الحالة . كذلك برهان مبرهنة وجود حل وحيد لزوج من المعادلات المرافقة "Adjoint equation" المصاحبة لمعادلات الحالة. تم ايجاد مشتقة فريشيه "Frcéhet " لدالة هاملتون الخاصة بهذه المسالة. ايضا تم برهان مبرهنتا الشروط الضرورية والكافية لوجود متجه سيطرة امثلية مستمرة تقليدية بوجود قيدي التساوي و والتباين.

الكلمات المفتاحية: سيطرة امثلية حدودية تقليدية مستمرة، معالدة تفاضلية جزئية غير خطية من لبنوع الزائدي،الشروط الضرورية والكافية للامثلية.