On Completely YJ-injective Rings

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ABSTRACT

A ring R is called completely right YJ-injective (briefly, right CYJ injective) if every homomorphic image of R is right YJ-injective. In this paper, we study completely right YJ-injective rings and their connection with Von Neumann regular rings. In addition, we also study regularity of rings whose ring homomorphic images are right YJ-injective as right R-modules

Keywords: Completely YJ-injective Rings, homorphic image, strongly π- regular ring.

حول الحلقات الغامرة التامة من النمط – YJ

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الملخص

يقال للحلقة R أنها حلقة غامرة يمنى تامة من النمط – YJ إذا كان كل صورة تشاكلية لـ R تكون حلقة غامرة يمنى من النمط – YJ . في هذا البحث قمنا بدراسة الحلقات الغامرة اليمنى التامة من النمط – YJ وعلاقتها مع الحلقات المنتظمة حسب مفهوم Von Neumann فضلا عن ذلك درسنا الانتظام للحلقات التي هي صورة تشاكلية للحلقات الغامرة اليمنى التامة من النمط – YJ باعتبارها مقاسات يمنى.

الكلمات المفتاحية: الحلقات الغامرة التامة من النمط – YJ , صورة تشاكلية, الحلقات المنتظمة من النمط – π

.1. Introduction:

Throughout this paper, R denoted an associative ring with identity, and all modules are unitary. For $a\in R$, r(a) and l(a) denote the right annihilator and the left annihilator of a, respectively. We write J(R), Y(R) (Z(R)) and N(R) for the Jacobson radical, the right (left) singular ideal and the set of nilpotent elements respectively. A right R-module M is called YJ-injective if for every $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$ and any right R-homomorphism $f:a^nR\to M$ can be extended to $R\to M$. A ring R is called right YJ-injective if R is YJ-injective as a right R-module. This notion was firstly introduced by **[** 8 **]**. A ring R is (Von Neumann) regular provided that for every $a\in R$ there exists $b\in R$ such that a=aba **[** 7 **]**. R is called a strongly regular ring if for each $a\in R$, $a\in a^2R$. In [9], proved that a ring R is Von Neumann regular if and only if every cyclic right R-module is YJ-injective. So Von Neumann regular rings are right YJ-injective rings, but the converse is not true in general. Following [3], call R abelian if every idempotent element of R is central. A ring R is said to be reversible if ba=0 implies ab=0 for $a,b\in R$. A ring R is called reduced if contains no non-zero nilpotent elements. A ring R is said to be 2-primal **[2]** if N(R) =P(R), where P(R) is the prime radical of R.

2. Completely YJ-injective :

Following [5], a ring R is called π-regular (strongly π-regular) if for every element $a \in R$, there exists a positive integer n and $b \in R$ such that $a^n = a^n b a^n$ ($a^{n+1}b$). We begin with the following definition.

Definition 2.1:

A ring R is called completely right (left) YJ-injective (briefly, right (left) CYJ-injective) if every homomorphic image of R is right (left) YJ-injective.

Obviously right (left) CYJ-injective rings are right (left) YJ-injective, but the converse is not true by Example (1).

Proposition 2.2:

A ring R is right CYJ-injective if and only if every factor ring R / I is a right YJ-injective rings, where I is a two-sided ideal of R.

Proof :

Given an ideal I of R, the natural map $\pi : R \rightarrow R / I$ define by f(r)=r+I, is an epimorphism since if R is CYJ-injective, then the homomorphic image R / I is right YJ-injective by Definition 2.1.

Conversely, assume that every factor ring R / I is a right YJ-injective ring, where I is a two-sided ideal of R. Then the argument above, every homomorphic image of R is right YJ-injective, concluding that R is right CYJ-injective. ■

Example (1):

Let Z_2 be the ring of integers modulo 2. Let A ={ < a_i > / $a_i \in Z_2$ and a_i is eventually a constant }. Then A is a commutative Von Neumann regular ring. If $k \in Z_2$ and < a_1 , a_2 , … , a_n , a , a , a , … > $\in$ A, let k. < a_1 ,a_2 ,…, a_n ,a , a, a, …> = ka, then Z_2 is a right A-module. Consider a ring $R = \begin{bmatrix} Z_2 & Z_2 \\ 0 & A \end{bmatrix}$, then by [4, example 1] R is aright YJ-injective. We claim that R is not right CYJ-injective. Let B= { < a_i > $\in$ A / a_i is eventually zero }. Notice that B is a proper two –sided ideal of A. Let $I = \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix}$. Then I is a two sided ideal of R. Consider a factor ring R / I. Then $R / I \cong \begin{bmatrix} z_2 & z_2 \\ 0 & z_2 \end{bmatrix}$ as ring. However, $\begin{bmatrix} z_2 & z_2 \\ 0 & z_2 \end{bmatrix}$ is not right YJ-injective. Hence a factor ring R / I is not right YJ-injective and therefore R is not right CYJ-injective. ■

Remark (1) :

Regular rings are right CYJ-injective but the converse is not true by the following example. ■

Example (2) :

Let $R = Z_{q^2}$, where Z_{q^2} is the ring of integers modulo q^2 and q is a prime number. Then R is a commutative CYJ-injective ring but it is not Von Neumann regular. ■

Lemma 2.3 : [2]

A ring R is 2-primal if and only if R / P(R) is reduced ■

Lemma 2.4 : [1 **]**

R is strongly regular ring if and only if R is reduced YJ-injective ring. ■

Lemma 2.5 : [4 **]**

For a ring R, the following statements are equivalent:
(1) R is strongly π-regular.
(2) Each prime factor ring of R is strongly π-regular. ■

So we have the following

Proposition 2.6 :

Let R be a 2-primal ring. If R is right CYJ-injective, then R is strongly π - regular.

Proof:

Since R is 2-primal, R / P(R) is reduced by Lemma 2.3 and by Lemma 2.4, R / P(R) is strongly regular, so every prime factor ring of R is strongly π-regular. Therefore R is strongly π-regular by Lemma 2.5■

Following [1] , a ring R is called quasi-strongly right bounded (briefly QSRB) if every non zero maximal right ideal contains a non zero two sided ideal.

Lemma 2.7: [1]

Let R be a QSRB-ring. Then R / Y(R) is reduced ring. ■

Lemma 2.8: [9]

If R is a right YJ-injective, then J(R) = Y(R).

Theorem 2.9 :

Let R be abelian QSRB-ring with nil J(R). If R is a CYJ-injective, then R is strongly π-regular ring.

Proof :

Since R is QSRB. Then by Lemma 2.7 R / Y(R) is reduced ring. And R / J(R) is regular Lemmas (2.4and 2.8). Hence for any $x \in R$, there exists $y \in R$ such that $x-xyx \in J(R)$. Write $x + J(R) = \bar{x}$. since J(R) is nil, implies that there exists $e^2 = e \in R$ such that $\bar{e} = \bar{x}\bar{y}$ and $\bar{x} = \bar{e}\bar{x}$.

But x-ex is nilpotent so there exists a positive integer n such that $(x-ex)^n = 0$. Thus $x^n \in eR$ because e is central i.e. $x^n \subseteq eR$.

Next since $\bar{e} = \bar{x}\bar{y}$ is central in R/J(R) we have $\bar{e} = \bar{x}\bar{y} = \bar{x}\bar{y}\bar{x}\bar{y} = \bar{x}^2\bar{y}^2 = ... = \bar{x}^n\bar{y}^n$. Thus $e-x^ny^n \in J(R)$, and so $(e-x^ny^n)^m = 0$ for some positive integer m because J(R) is nil. Consequently $e \in x^nR$ i.e. $eR \subseteq x^nR$; hence we obtain $x^nR = eR$. But eR is a two-sided ideal in R and therefore R is strongly π-regular ring.■

3. Strongly CYJ-Injective Rings:

In this section we investigate the Von Neumann regularity of rings R whose homomorphic images are YJ-injective as right R-modules.

Definition 3.1 :

A ring R is called strongly right CYJ-injective if every ring homomorphic image of R is YJ-injective as a right R-module.

Proposition 3.2 :

If R is a strongly right CYJ-injective ring, then R is right CYJ-injective.

Proof :

Let $\overline{R}=R/I$, where I is two sided ideal of R. Since $\overline{R}$ is a YJ-injective right R-module and for any a $\in$ R, then there exists a positive integer n and $\overline{c}\in\overline{R}$ such that $f(a^n b)=\overline{c}a^n b$ for all b$\in$R. Now, let $\overline{f}$ be any non-zero $\overline{R}$-homomorphism $\overline{f}:\overline{a}^n\overline{R}\to\overline{R}$, we claim that, there exists $\overline{c}\in\overline{R}$ such that $f(\overline{a}^n\overline{b})=\overline{c}\overline{a}^n\overline{b}$ for all $\overline{b}\in\overline{R}$. Let $\pi:a^nR\to\overline{a}^n\overline{R}$ denote the natural map. Then $f=\overline{f}o\pi:a^nR\to\overline{R}$. Since $f(a^n b)=\overline{c}a^n b$ for all b $\in$ R. Thus $\overline{f}(\overline{a}^n\overline{b})=\overline{f}o\pi(a^n b)=f(a^n b)=\overline{c}a^n b=\overline{c}a^n b=\overline{c}\overline{a}^n\overline{b}$. So $\overline{R}$ is YJ-injective as a right $\overline{R}$ -module. Therefore R is right CYJ-injective. ■

Recall that a ring R is called ZI-ring if for every a,b $\in$R, ab=0 implies aRb=0. Clearly if R is ZI-ring then any right annihilator of R is two sided ideal [6].

Theorem 3.3 :

Let R be a strongly CYJ-injective ring, then the following conditions are equivalent:
(1) R is reduced.
(2) R is reversible.
(3) R is a ZI-ring.

Proof :

Obviously (1) $\Rightarrow$ (2) $\Rightarrow$ (3)
(3) $\Rightarrow$ (1) Let $0 \neq a \in$ R such that $a^2 = 0$. Since R is ZI-ring then r(a) is two-sided ideal of R. Since R is CYJ-injective, and so any R-homomorphism of aR into R / r(a) extends to one of R into R / r (a). Let f : aR $\to$ R / r(a) be defined by f(ax) = x + r(a) for all x $\in$ R. Then f is well-defined. Since R / r(a) is YJ-injective, there exists c $\in$ R such that f (ax) = (c + r(a)) ax = cax + r(a). Hence f(a) = 1+r(a) = ca+r(a). Therefore 1-ca $\in$ r(a). But ca $\in$ r(a) which implies 1 $\in$ r(a). So that R must be reduced. ■

The following gives conditions on strongly right CYJ-injective to be strongly regular ring.

Theorem 3.4 :

Let R be a ZI-ring. Then the following conditions are equivalent:
(1) R is strongly regular ring
(2) R is strongly CYJ-injective ring.

Proof :

(1) $\Rightarrow$ (2) It is clear.
(2)$\Rightarrow$(1) for all a$\in$R**,** r(a) is two-sided. Therefore R/r(a) is YJ-injective and so there exists a positive integer n such that $a^n\neq0$ and any R-homomorphism of a^nR into R/r(a) extended to one of R into R/r(a).

Let $f : a^nR \to R / r(a)$ be defined by $f(a^nr) = r + r(a)$, then f is well-defined R-homomorphism. Indeed if $a^nr_1 = a^nr_2$, then $r_1 - r_2 \in r(a^n)$. Since R ZI-ring, then by Theorem 3.3 R is reduced. Hence $r(a^n) = r(a)$. Therefore $r_1 + r(a) = r_2 + r(a)$. Since $R / r(a)$ is YJ-injective, there exists $c \in R$ such that $f(a^nr) = (c + r(a))(a^nr) = ca^nr + r(a)$ so that $f(a^n)= 1 + r(a) = ca^n + r(a)$ which implies that $1-ca^n \in r(a) = l(a)$. Hence $a = ba^2$, where $b = ca^{n-1}$. So that R is strongly regular rings. ■

Recall that a ring R is MERT [6], if every maximal essential right ideal of R is a two sided ideal of R. Now, the following result is given.

Theorem 3.5

Let R be MERT strongly right CYJ-injective ring, then $Y(R) = 0$.

Proof :

If $Y(R) \neq 0$, then there exists $0 \neq x \in R$ such that $x^2 = 0$. Since $r(x) \neq R$, there exists a maximal right ideal M of R such that $r(x) \subseteq M$. Thus M is an essential right ideal and so M is two-sided ideal. Let $f:xR \to R / M$ defined by $f(xr) = r + M$. Note that f is well-defined R-homomorphism. Since R / M is YJ-injective, there exists $y \in R$ such that $1 + M = f(x) = yx + M$. Thus $1-yx \in M$, whence $1 \in M$, which is a contradiction. Therefore $Y(R) = 0$. ■

Theorem 3.6 :

Let R be a MERT-ring. Then the following conditions are equivalents:

(1) R is strongly regular ring.

(2) R is a strongly right CYJ-injective.

Proof :

By Theorem3.5, Y(R)=0,since R is YJ-injective then J(R)=Y(R)=0. Therefore R is semi-prime. We have to prove that R is reduced. Let $0 \neq a \in R$ such that $a^2 = 0$. Then there exists a maximal right ideal M of R containing r(a). Since R is a semi-prime, then M must be essential right ideal and so M is a two-sided. By similar methods used in Theorem 3.5 we have $1 \in M$, which is a contradiction, so that R is reduced and by Theorem 3.4 we have R is a strongly regular ring. ■

Theorem 3.7 :

The following conditions are equivalent for any ring R :

(1) R is strongly regular ring.

(2) R is SLB and strongly right CYJ-injective ring.

Proof :

(1) ⇒ (2) It is clear.

(2) ⇒ (1) We will show that R is reduced. Suppose that $a^2 = 0$ with $a \neq 0$. Then lr(a) is is non-zero left ideal of R. Thus there exists a non-zero two-sided ideal I of R such that $I \subseteq lr(a)$ and so $r(a) \subseteq r(I)$.

Now, R / r(I) is YJ-injective right R-module. Let $f : aR \to R / r(I)$ be defined by $f(ar) = r + r(I)$. Then f is a well-defined right R-homomorphism. Since R / r(I) is YJ-injective right R-module, there exists $c \in R$ such that $1 + r(I) = f(a) = ca + r(I)$. Thus $1-ca \in r(I)$; whence $1 \in r(I)$, which is a contradiction. Therefore R is reduced and hence R is strongly regular. ■

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