

## A Series of Saddle - Node Bifurcation and Chaotic Behavior of a Family of a Semi - Triangular Maps

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### ABSTRACT

This paper studies the bifurcations in dynamics of a family of semi-triangular maps  $S = \{S_\alpha(x) = \alpha x \sin(x) : \alpha \in \mathbb{R}\}$ . We will prove that this family has a series of Saddle-node bifurcations and a period doubling bifurcation. Also, we show that for some value of the parameter the functions  $S_\alpha$  will be chaotic.

**Keywords:** Bifurcation, Chaotic, Semi-Triangular Maps.

سلسلة من تشعب العقدة والسلوك الفوضوي لعائلة من خرائط شبه ثلاثية

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### الملخص

يتناول هذا البحث التشعب في ديناميكية عائلة من الدوال شبه المثلثية  $S = \{S_\alpha(x) = \alpha x \sin(x) : \alpha \in \mathbb{R}\}$ . وسوف يتم إثبات أن هذه العائلة من الدوال تمتلك سلسلة من تشعب العقدة السرجية وتشعب تفرعات مضاعفة الرتبة. وكذلك سيتم تبيان انه عند قيم محددة من المعلمة  $\alpha$  فان الدالة  $S_\alpha$  تكون فوضوية. الكلمات المفتاحية: تشعب، فوضى، دوال شبه مثلية.

### 1. Introduction

The term "bifurcation" refers to significant changes in the set of fixed or periodic points or other sets of dynamics interest. In fact, in dynamical systems, the object of bifurcation theorem is to study the changes that maps undergo as parameters changes. There are several types of bifurcations like saddle – node bifurcation ,period doubling bifurcation pitch fork bifurcation, and others.

Our goal in this paper ,is to study how and when the periodic points of the family of maps  $S = \{S_\alpha(x) = \alpha x \sin(x) : \alpha \in \mathbb{R}\}$  change, i.e. the bifurcation that this family undergoes.

We will prove that our family has a series of saddle node bifurcations which is route to chaos. Also, we will show that this family has a period doubling bifurcation when the parameter meets the value  $\alpha \approx 1.327295$ .

Finally, we show that this family has a chaotic behavior on  $\mathbb{R}$  when the parameter is  $\alpha > 2$ .

### 2. Definitions

Let  $f$  be any function .Then,

1. A point  $x$  is called fixed point of the function  $f$  if  $f(x) = x$ , [1].
2. A point  $x$  is called periodic point if  $\exists n \in \mathbb{N}$  such that  $f^n(x) = x$ . And we say that  $x$  of period  $n$ . Note that the fixed point is a periodic point of period 1, [1].

3. A point  $x$  is called critical point if  $f'(x) = 0$ . The critical point  $x$  is called degenerate if  $f''(x) \neq 0$ . And  $x$  is called non-degenerate if  $f''(x) = 0$ , [1].

4. A periodic point  $x$  is called hyperbolic if  $|f'(x)| \neq 1$ , and the number  $|f'(x)|$  is the multiplier.

The periodic point  $x$  is called the attracting fixed point (sink) if  $|f'(x)| < 1$ . And  $x$  is called repelling (source) if  $|f'(x)| > 1$ , [1].

5. Saddle-node bifurcation

Let  $\{f_\alpha : \alpha \in \mathbb{R}\}$  be a family of mappings we say that  $f$  has saddle-node bifurcation if for some  $\alpha \in \mathbb{R}$ , say,  $\alpha = \alpha_0$ , the following satisfied :

1. For  $\alpha < \alpha_0$ , then  $f_\alpha$  has no fixed point.
2. For  $\alpha = \alpha_0$ , then  $f_\alpha$  has one fixed point.
3. For  $\alpha > \alpha_0$ , then  $f_\alpha$  has two fixed points; one of them attracting and the other is repelling, [1].

6. Period doubling bifurcation

We say that the family  $\{f_\alpha : \alpha \in \mathbb{R}\}$  has a period doubling bifurcation if this bifurcation involves a change from an attracting (or repelling) to repelling (or attracting) periodic points of period two when  $\alpha$  passes through  $\alpha_0$ , [1].

7. Let  $J$  be an interval, and suppose that  $f : J \rightarrow J$ . Then,  $f$  has sensitive dependence on initial conditions at  $x$ , or just sensitive dependence at  $x$  if there is an  $\varepsilon > 0$  such that for each  $\delta > 0$ , there is a  $y$  in  $J$  and a positive integer  $n$  such that

$$|x - y| < \delta \quad \text{and} \quad |f^n(x) - f^n(y)| > \varepsilon$$

If  $f$  has sensitive dependence on initial condition at each  $x$  in  $J$ , we say that  $f$  has sensitive dependence on initial conditions on  $J$ , or that  $f$  has sensitive dependence on  $J$ , [2].

8. Let  $J$  be a bounded interval, and  $f : J \rightarrow J$  continuously differentiable on  $J$ . Fix  $x$  in  $J$ , and let  $\lambda(x)$  be defined by

$$\lambda(x) = \frac{1}{n} \ln |(f^{[n]})'(x)|$$

provided that the limit exists. In this case,  $\lambda(x)$  is the Lyapunov exponent of  $f$  at  $x$ . If  $\lambda(x)$  is independent of  $x$  wherever  $\lambda(x)$  is defined, then the common value of  $\lambda(x)$  is denoted by  $\lambda$ , and is the Lyapunov exponent of  $f$ , [2].

9. A function  $f$  is chaotic if it satisfies, at least, one of the following :

i.  $f$  has a positive Lyapunov exponent at each point in its domain that is not eventually periodic.

Or

ii.  $f$  has a sensitive dependence on initial conditions on its domain, [2].

### 3. Properties and the fixed points of the family $S_\alpha$ :

Let  $S = \{S_\alpha(x) = \alpha x \sin(x) : \alpha \in \mathbb{R}\}$ . First, we study the properties of  $S_\alpha, \alpha \in \mathbb{R}$ .

### 3.1: Critical Points of $S_\alpha$ :

Taking the derivative of  $S_\alpha$  with respect to  $x$  ,we get  
 $S'_\alpha(x) = \alpha(\sin(x) + x \cos(x))$

$$S'_\alpha(x) = 0, \text{ then } \sin(x) + x \cos(x) = 0$$

This implies that  $\sin(x) + x \cos(x) = 0$ . Therefore, the critical points of  $S_\alpha$  are  $x_0 = 0, x_1 = 2.208$  which is a maximum for  $S_\alpha$  and  $x_2 = 4.913$  which is a minimum for  $S_\alpha$  and all of them lie in the interval  $[0, 2\pi]$  .

The following proposition gives the fixed points of the function in the family  $S = \{S_\alpha(x) = \alpha x \sin(x) : \alpha \in \mathbb{R}\}$  .

### 3.2: Proposition

Let  $S_\alpha(x) = \alpha x \sin(x), \alpha > 0, x \geq 0$ , then the fixed points for this family of functions are  $x = 0$  ,  $x = \sin^{-1}(1/\alpha)$ , and  $x = \pi - \sin^{-1}(1/\alpha)$  ,  $\forall \alpha > 0, x \in [0, 2\pi]$

#### Proof

Let  $x_p$  be a fixed point for this family, thus  
 $\alpha x_p \sin(x_p) = x_p$  and  $\alpha x_p \sin(x_p) - x_p = 0$ , then  
 $x_p (\alpha \sin(x_p) - 1) = 0$ . Therefore, either  $x_p = 0$  ,  $\forall \alpha > 0$  or  $\alpha \sin(x_p) - 1 = 0$  and this implies that  $x_p = \sin^{-1}(1/\alpha)$  is fixed point  $\forall \alpha > 1$

Since  $\sin(x_p) = \sin(\pi - x_p) = 1/\alpha$ , then  $\pi - x_p = \sin^{-1}(1/\alpha)$

Therefore,  $x_p = \pi - \sin^{-1}(1/\alpha)$  is a fixed point  $\forall \alpha > 1$

Hence, the fixed points of  $S_\alpha$  in  $[0, 2\pi]$  are  $x_p = 0 \quad \forall \alpha > 0$  ,  $x_p = \sin^{-1}(1/\alpha)$  ,  $x_p = \pi - \sin^{-1}(1/\alpha) \quad \forall \alpha > 1$  .

### 3.3: Remark

The fixed point  $x = 0$  is attracting fixed point  $\forall \alpha > 0$ .

#### Proof

Taking the derivative of the function  $S_\alpha(x) = \alpha x \sin(x)$  with respect to  $x$ . Then,  
 $S'_\alpha(x) = \alpha(\sin(x) + x \cos(x))$ .

Put  $x = 0$

$$\begin{aligned} |S'_\alpha(0)| &= |\alpha * (\sin(0) + 0 * \cos(0))| \\ &= |\alpha * (0 + 0)| \\ &= |0| < 1. \end{aligned}$$

Thus, the fixed point  $x = 0$  is attracting fixed point  $\forall \alpha > 0$  .

The following theorem gives the number of fixed points and their natures of  $S_\alpha$  .

### 3.4: Theorem

Let  $S_\alpha(x) = \alpha x \sin(x)$ , Then :

1. If  $\alpha < 1$  , then  $S_\alpha(x)$  has only the attracting fixed point  $x = 0$  ,  $\forall \alpha > 0$ .

2. If  $\alpha = 1$ , then  $S_\alpha(x)$  has infinite number of fixed points and all of them are not hyperbolic .(one fixed point in each interval  $(2n\pi, (2n+1)\pi)$ ,  $n = 0,1,2,3, \dots$ )
3. If  $\alpha > 1$ , then the number of fixed points in 2. are doubled
4. The general forms of the fixed points for  $S_\alpha$  are
 
$$x_{1n} = \sin^{-1}(1/\alpha) + 2n\pi \quad n = 0,1,2,3,\dots\dots\dots$$

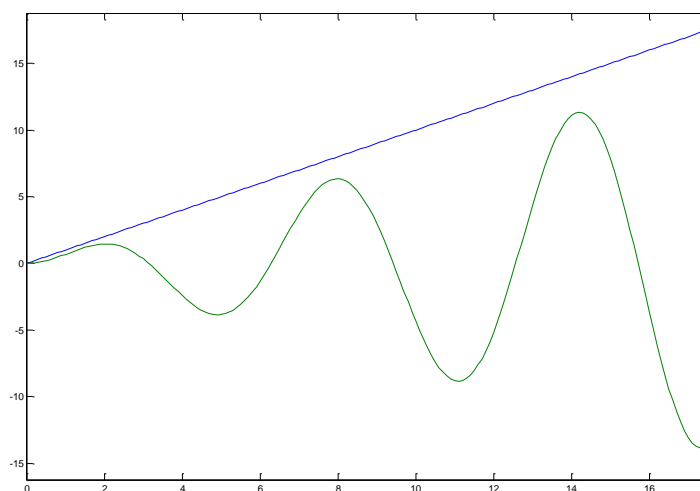
$$x_{2n} = (2n+1)\pi - \sin^{-1}(1/\alpha) \quad n = 0,1,2,3,\dots\dots\dots$$
 on the interval  $[2n\pi, 2(n+1)\pi]$  for  $n = 0,1,2,3,\dots\dots\dots$
5. The fixed points  $x_{1n}$  are attracting in the interval  $(-2/\alpha * \cos(\sin^{-1}(1/\alpha)), 0)$  and repelling out of this interval .The fixed points  $x_{2n}$  are attracting in the interval  $(0, 2/\alpha * \cos(\sin^{-1}(1/\alpha)))$  and repelling out of this interval .

**Proof**

1. From proposition (3.1),  $(S_\alpha)$  has the fixed points

$x = 0$ ,  $x = \sin^{-1}(1/\alpha)$  and  $x = \pi - \sin^{-1}(1/\alpha)$ . Now  $x = 0$  is attracting fixed point (remark 3.2)

Moreover,  $\alpha < 1$  implies  $1/\alpha > 1$ . This means that  $x = \sin^{-1}(1/\alpha)$  and  $x = \pi - \sin^{-1}(1/\alpha)$  are not defined, thus  $x = 0$  is the unique fixed point for  $\alpha < 1$ , (see Figure 1), below:



**Figure 1:** The graph of  $S_\alpha(x) = \alpha x \sin(x)$ ,  $\alpha < 1$

2. Let  $\alpha = 1$

Then,  $x = \sin^{-1}(1) = \pi - \sin^{-1}(1)$

Thus,  $x = \pi/2 + 2n\pi$ ,  $n = 0,1,2,3,\dots\dots\dots$

Hence, the function has infinite number of fixed points.

To study the types of these points :

Taking the derivative of  $S_\alpha(x)$ , we get  $S'_\alpha(x) = \alpha(\sin(x) + x \cos(x))$ .

Now

$$|S'_1((\pi/2) + 2n\pi)| = |\sin((\pi/2) + 2n\pi) + ((\pi/2) + 2n\pi) * \sin((\pi/2) + 2n\pi)|$$

$$= |1 + ((\pi/2) + 2n\pi) * 0|$$

$$\begin{aligned}
 &=|1+0| \\
 &=|1| \\
 &=1
 \end{aligned}$$

Thus, the fixed points  $x = (\pi/2) + 2n\pi$ ,  $n = 0,1,2,3, \dots$  are not hyperbolic fixed points, (see Figure 2).

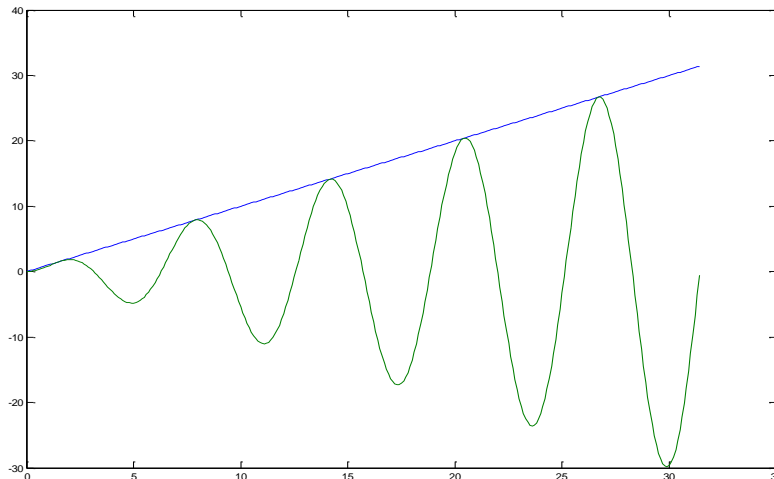


Figure 2: The graph of  $S_\alpha(x) = \alpha x \sin(x)$ ,  $\alpha = 1$

3. Let  $\alpha > 1$ , thus  $(1/\alpha) < 1$  Therefore  $x = \sin^{-1}(1/\alpha) < \pi/2$ .

Assume that  $x = \pi/2 - \varepsilon$  is a fixed point, then

$$S_\alpha((\pi/2) - \varepsilon) = \alpha((\pi/2) - \varepsilon) \sin((\pi/2) - \varepsilon) = ((\pi/2) - \varepsilon)$$

$$\text{Thus, } \alpha \sin((\pi/2) - \varepsilon) = 1.$$

Since  $\sin(\pi/2 - \theta) = \sin(\pi/2 + \theta)$  for each  $\theta \leq \pi/2$  then  $\alpha \sin((\pi/2) + \varepsilon) = 1$ , then  $\alpha((\pi/2) + \varepsilon) \sin((\pi/2) + \varepsilon) = ((\pi/2) + \varepsilon)$ .

$$\text{Thus, } S_\alpha((\pi/2) + \varepsilon) = ((\pi/2) + \varepsilon),$$

Then  $(\pi/2) + \varepsilon$  is fixed point,

Therefore,  $(\pi/2) + \varepsilon$  is fixed point iff  $(\pi/2) - \varepsilon$  is fixed point, hence the number of fixed points is doubled (see Figure 3).

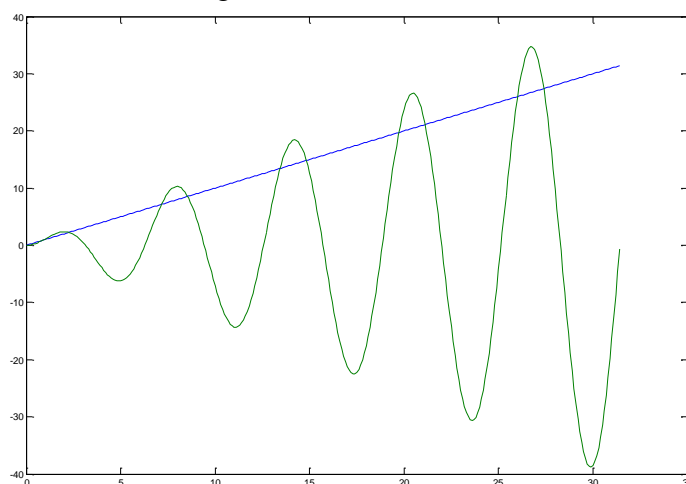


Figure 3: The graph of  $S_\alpha(x) = \alpha x \sin(x)$ ,  $\alpha > 1$

4. Let  $x$  be a fixed point for  $S_\alpha$ . Then

$$x = \sin^{-1}(1/\alpha), \text{ and}$$

$$\begin{aligned} \sin(x) &= 1/\alpha, \text{ then} \\ \sin(x + 2n\pi) &= 1/\alpha, \quad n = 0,1,2,3,\dots \\ \text{Thus, } x + 2n\pi &= \sin^{-1}(1/\alpha). \text{ Hence} \\ x_{1n} &= \sin^{-1}(1/\alpha) + 2n\pi \end{aligned} \quad \dots(1)$$

But,  $x = \sin^{-1}(1/\alpha)$  is also implies that

$$\sin(\pi - x) = 1/\alpha \quad (\text{ since } \sin(\pi - \theta) = \sin(\theta) )$$

Then,  $\sin(2n\pi + \pi - x) = 1/\alpha$ . Hence,

$$\sin((2n+1)\pi - x) = 1/\alpha, \text{ thus}$$

$$(2n+1)\pi - x = \sin^{-1}(1/\alpha), \text{ therefore}$$

$$x_{2n} = (2n+1)\pi - \sin^{-1}(1/\alpha) \quad \dots(2)$$

Hence,(1)and (2) give the general form of the fixed point of  $S_\alpha$ .

**5.** Now to study the nature of the fixed points of  $S_\alpha$

Let  $x$  be attracting fixed points .This implies that

$$\begin{aligned} |S'_\alpha(x)| &< 1, \\ -1 &< \alpha \sin(x) + \alpha x \cos(x) < 1, \\ -1 &< 1 + \alpha x \cos(x) < 1, \\ -2 &< \alpha x \cos(x) < 0, \\ -2/\alpha &< x \cos(x) < 0. \end{aligned}$$

Now we have two cases :

Case 1: If the fixed point of the form  $x_{1n}$  is

$$\begin{aligned} x &= \sin^{-1}(1/\alpha) + 2n\pi \\ \text{Hence, } -2/\alpha \cos(\sin^{-1}(1/\alpha)) &< x < 0. \end{aligned}$$

Then, all the fixed points that have the form  $x_{1n}$  will be attracting on the interval  $(-2/\alpha \cos(\sin^{-1}(1/\alpha)), 0)$ , and repelling out of this interval .

Case 2: If  $x$  is of the form  $x_{2n}$ . Then,

$$0 < x < -2/(-\alpha \cos(\sin^{-1}(1/\alpha))).$$

Thus, all fixed points which have the form  $x_{2n}$  will be attracting on the interval  $(0, 2/\alpha \cos(\sin^{-1}(1/\alpha)))$  and repelling out of this interval.

From (Case1) and (Case2), we conclude that if the fixed point of the form  $x_{1n}$  is attracting ,then the fixed point of the form  $x_{2n}$  is surely repelling and vise versa .

**4. Bifurcation Analysis and Chaotic Behavior for the Semi-Triangular Family  $S$**

**4.1:Theorem**

If  $S = \{ S_\alpha(x) = \alpha x \sin(x), \alpha > 0, x > 0 \}$  then, this family has a saddle node bifurcation at  $\alpha = 1$ .

**Proof**

According to the theorem 3.3  $S_\alpha$  has no fixed point when  $\alpha < 1$ ,and  $S_\alpha$  has one fixed point when  $\alpha = 1$  .

By the same theorem for  $\alpha > 1$  ,two fixed points were born at each interval ,one is attracting and the other is repelling ;this is exactly a saddle-node bifurcation .

The following theorem studies the period doubling bifurcation of the family  $S$  :

**4.2: Theorem**

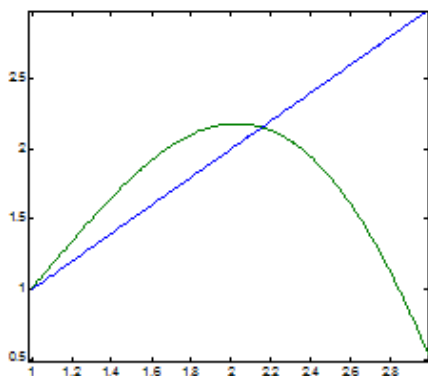
Let  $S_\alpha(x) = \alpha x \sin(x)$ ,  $\alpha > 0, x > 0$  be a family of maps ,then this family has period doubling bifurcation at  $\alpha \approx 1.327295$ .

**Proof**

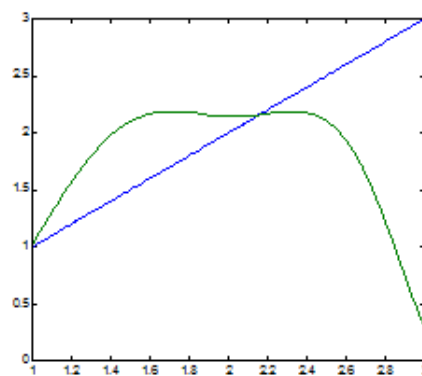
Our earlier experiments showed that the value  $\alpha \approx 1.327295$  is a bifurcation value. If  $\alpha < 1.327295$ , then the family has attracting fixed point and has one periodic point of period 2 in the interval  $[0,2\pi]$  (see Figure 4).

If  $\alpha \approx 1.327295$  ,Then the family has not hyperbolic fixed point and has one periodic point of period 2 in the interval  $[0,2\pi]$  (see Figure 5).

If  $\alpha > 1.327295$ ,then the family has a repelling fixed point and has two attracting periodic points of period 2 in the interval  $[0,2\pi]$  (see Figure 6).Thus,  $\alpha \approx 1.327295$  is a period doubling bifurcation value for the functions in  $S$  .

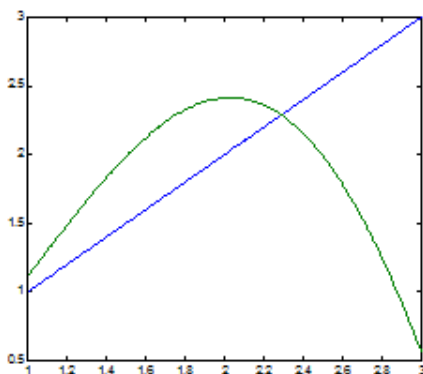


A:  $S_\alpha$  has only one attracting fixed point in the interval  $(0,2\pi)$ ,  $\alpha < 1.327295$

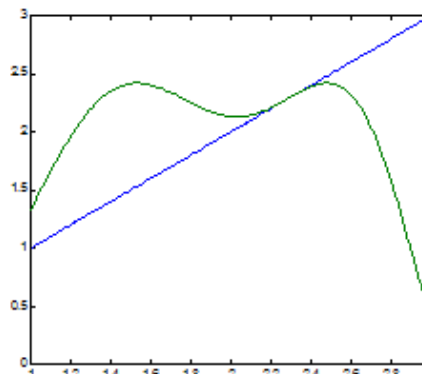


B:  $S_\alpha$  has a periodic point of period 2 in the interval  $(0,2\pi)$ ,  $\alpha < 1.327295$

**Figure 4**

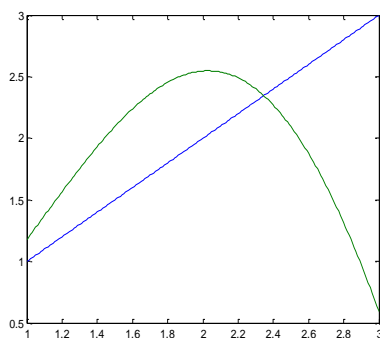


A:  $S_\alpha$  has not hyperbolic fixed point in the interval  $(0,2\pi)$ ,  $\alpha \approx 1.327295$

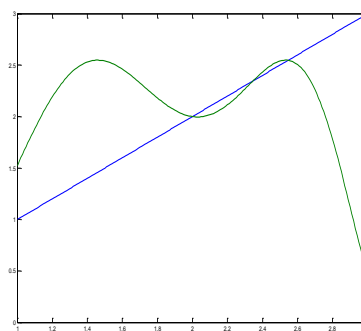


B:  $S_\alpha$  has one periodic point of period 2 in the interval  $(0,2\pi)$ ,  $\alpha \approx 1.327295$

**Figure 5**



A:  $S_\alpha$  has repelling fixed point in the interval  $(0, 2\pi)$ ,  $\alpha > 1.327295$



B:  $S_\alpha$  has two attracting periodic point of period 2 in the interval  $(0, 2\pi)$ ,  $\alpha > 1.327295$

Figure 6

### 4.3:Theorem

Functions of the family  $S$  are sensitive dependence on initial condition in the interval  $\{x : x > 0\}$  for all  $\alpha > 1$ .

#### Proof

**First:** In the interval  $\left(0, \frac{\pi}{2}\right]$

Let  $\delta > 0$ , the theorem can be divided into two parts:

1. Let  $x \in \left(0, \frac{\pi}{2}\right)$

Choose  $y = x + \beta$  such that  $\beta$  is small positive number and  $\beta < \delta$ ,  $\beta < x$

Now  $|x - y| = |\beta| < \delta$

Notice that  $\sin(x + \beta) > \sin x$

Then  $(x + \beta)\sin(x + \beta) > (x + \beta)\sin x$

And by multiplying both sides by  $\alpha$  the following is obtained

$$\alpha(x + \beta)\sin(x + \beta) > \alpha x \sin x + \alpha \beta \sin x$$

Therefore

$$\begin{aligned} \alpha(x + \beta)\sin(x + \beta) - \alpha x \sin x &> \alpha \beta \sin x \\ &> \alpha \beta \sin \beta \\ &> 0 \end{aligned}$$

By taking the absolute value of both sides the following is obtained

$$|\alpha(x + \beta)\sin(x + \beta) - \alpha x \sin x| > \alpha \beta \sin \beta$$

Then  $|S_\alpha(y) - S_\alpha(x)| > \alpha \beta \sin \beta$

Now choose  $\varepsilon = \alpha \beta \sin \beta$ ,  $n = 1$ , and that proves  $S_\alpha$  is sensitive dependence on initial condition in the interval  $\left(0, \frac{\pi}{2}\right)$ .

2. Let  $x = \frac{\pi}{2}$

Choose  $y = x - \beta$  such that  $\beta$  is small positive number and  $\beta < \delta$ ,  $\beta < y$



Now  $|x - y| = |\beta| < \delta$

Notice that  $\sin x > \sin(x - \beta)$

Then  $(x - \beta)\sin x > (x - \beta)\sin(x - \beta)$

Multiplying both sides by  $\alpha$  the following is obtained

$$\alpha(x - \beta)\sin x > \alpha(x - \beta)\sin(x - \beta)$$

$$\alpha x \sin x - \alpha \beta \sin x > \alpha(x - \beta)\sin(x - \beta)$$

Therefore

$$\begin{aligned} \alpha x \sin x - \alpha(x - \beta)\sin(x - \beta) &> \alpha \beta \sin x \\ &> \alpha \beta \sin \beta \\ &> 0 \end{aligned}$$

By taking the absolute value of both sides the following is obtained

$$|\alpha x \sin x - \alpha(x - \beta)\sin(x - \beta)| > \alpha \beta \sin \beta$$

Then  $|S_\alpha(y) - S_\alpha(x)| > \alpha \beta \sin \beta$

Now choose  $\varepsilon = \alpha \beta \sin \beta$ ,  $n = 1$ , and that proves  $S_\alpha$  is sensitive dependence on initial condition in the interval  $x = \frac{\pi}{2}$ .

From (1) and (2) it can be obtained the function  $S_\alpha$  is sensitive dependence on initial condition in the interval  $\left(0, \frac{\pi}{2}\right]$ .

**Second:** In the interval  $\left[\pi, \frac{3\pi}{2}\right]$

Let  $\delta > 0$ , the theorem can be divided into two parts:

1. Let  $x \in \left[\pi, \frac{3\pi}{2}\right)$

Choose  $y = x + \beta$  such that  $\beta$  is small positive number and  $\beta < \delta$ ,  $\beta < x - \pi$

Now  $|x - y| = |\beta| < \delta$

Notice that  $\sin x > \sin(x + \beta)$

Then  $(x + \beta)\sin x > (x + \beta)\sin(x + \beta)$

Multiplying both sides by  $\alpha$  the following is obtained

$$\alpha x \sin x + \alpha \beta \sin x > \alpha(x + \beta)\sin(x + \beta)$$

Therefore

$$\begin{aligned} \alpha x \sin x - \alpha(x + \beta)\sin(x + \beta) &> -\alpha \beta \sin x \\ &> \alpha \beta \sin \beta \\ &> 0 \end{aligned}$$

By taking the absolute value of both sides the following is obtained

$$|\alpha x \sin x - \alpha(x + \beta)\sin(x + \beta)| > \alpha \beta \sin \beta$$

Then  $|S_\alpha(y) - S_\alpha(x)| > \alpha \beta \sin \beta$

Now choose  $\varepsilon = \alpha \beta \sin \beta$ ,  $n = 1$ , and that proves  $S_\alpha$  is sensitive dependence on initial condition in the interval  $\left[\pi, \frac{3\pi}{2}\right)$ .

2. Let  $x = \frac{3\pi}{2}$

Choose  $y = x - \beta$  such that  $\beta$  is small positive number and  $\beta < \delta, \beta < x - \pi$

Now  $|x - y| = |\beta| < \delta$

Notice that  $\sin(x - \beta) > \sin x$

Then  $(x - \beta)\sin(x - \beta) > (x - \beta)\sin x$

Multiplying both sides by  $\alpha$  the following is obtained

$$\alpha(x - \beta)\sin(x - \beta) > \alpha x \sin x - \alpha \beta \sin x$$

Therefore

$$\begin{aligned} \alpha(x - \beta)\sin(x - \beta) - \alpha x \sin x &> -\alpha \beta \sin x \\ &> \alpha \beta \sin \beta \\ &> 0 \end{aligned}$$

By taking the absolute value of both sides the following is obtained

$$|\alpha(x - \beta)\sin(x - \beta) - \alpha x \sin x| > \alpha \beta \sin \beta$$

Then  $|S_\alpha(y) - S_\alpha(x)| > \alpha \beta \sin \beta$

Now choose  $\varepsilon = \alpha \beta \sin \beta, n = 1$ , and that proves  $S_\alpha$  is sensitive dependence on initial condition in the interval  $x = \frac{3\pi}{2}$ .

From (1) and (2) it can be obtained the function  $S_\alpha$  is sensitive dependence on initial condition in the interval  $\left[\pi, \frac{3\pi}{2}\right]$ .

From first and second it can be obtained the function  $S_\alpha$  is sensitive dependence on initial condition in any subinterval, in which the function  $S_\alpha$  is positive and increasing or negative and decreasing, of the interval  $(0, 2\pi)$ .

The intervals within the function  $S_\alpha$  is positive and increasing or negative and decreasing are increasing for all  $\alpha > 1, n > 1$  by increasing  $n$ .

For example if  $n = 2$  and  $\alpha = 2$ , the function  $S_2^2$  is positive and increasing in the following intervals:

$$[0, 1.2], [2.44, 2.78], [3.15, 3.45], [4.04, 4.32], [5.1, 5.46], [6.02, 6.12].$$

Also, it is negative and decreasing in the following intervals:

$$[1.57, 2.05], [3.59, 3.85], [4.71, 4.92], [5.7, 5.85], \text{ (see Figure 8).}$$

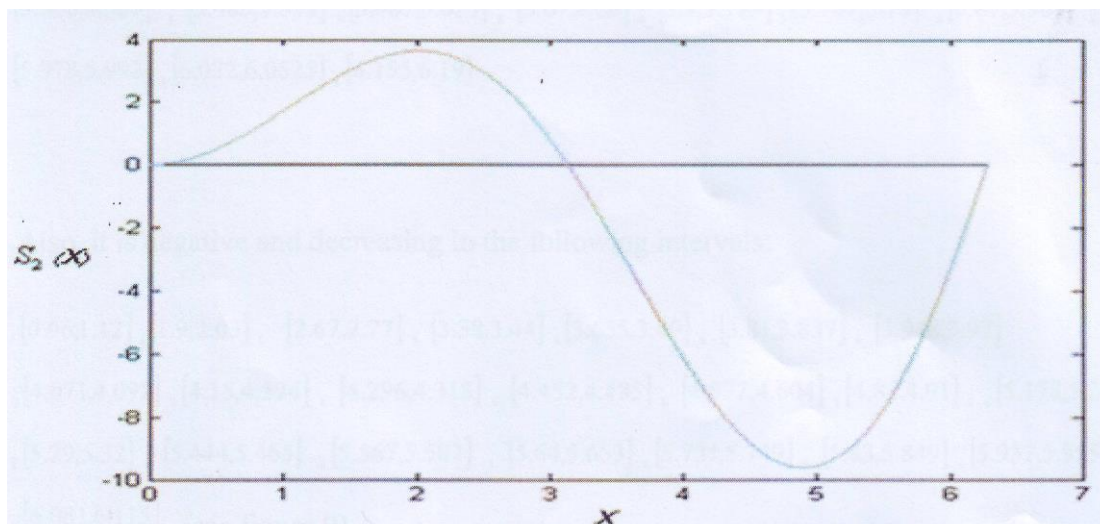


Figure 7: The Graph of Function  $S_2$  in the Interval  $(0, 2\pi)$

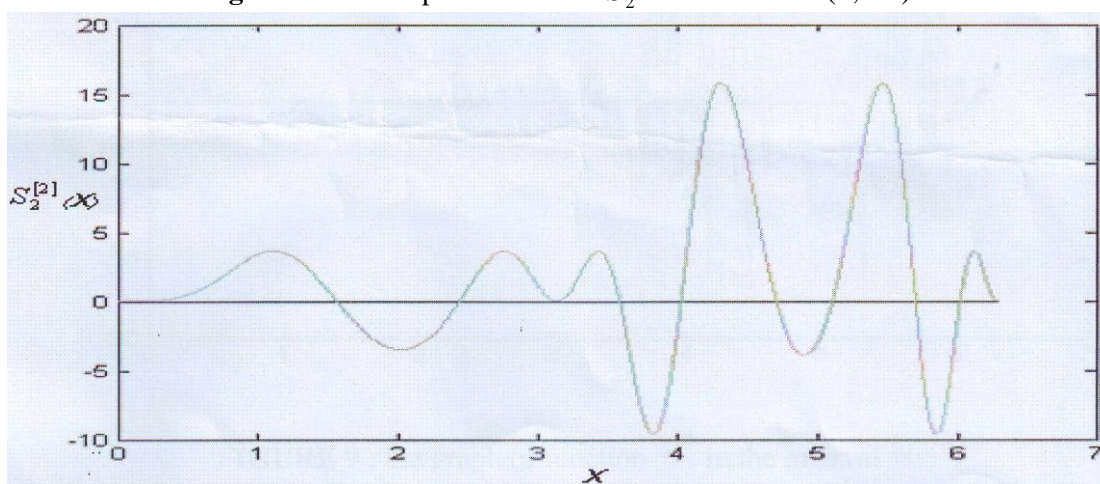


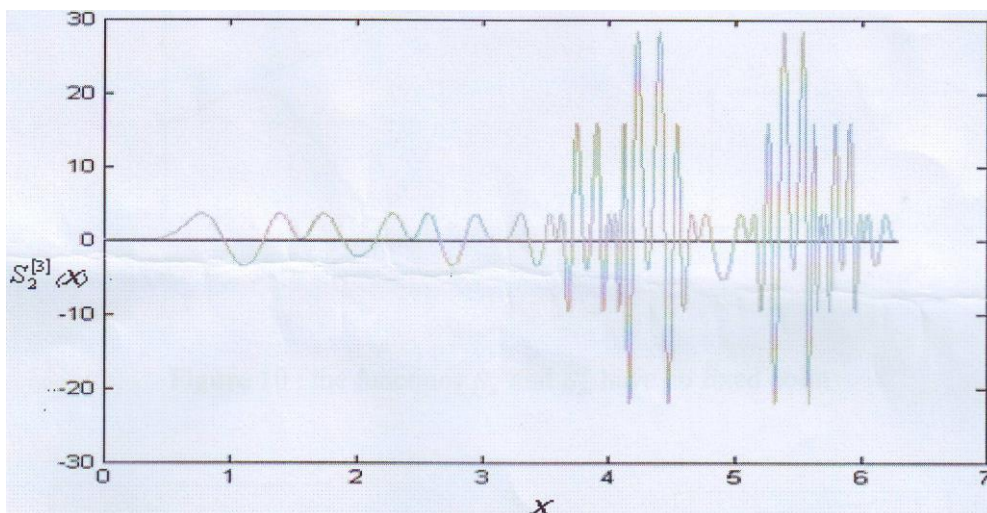
Figure 8: The Graph of Function  $S_2^{[2]}$  in the Interval  $(0, 2\pi)$

But if  $n = 3$  and  $\alpha = 2$ , the function  $S_2^3$  is positive and increasing in the following intervals:

[0, 0.8], [1.28, 1.4], [1.57, 1.75], [2.157, 2.293], [2.44, 2.575], [2.86, 2.95], [3.14, 3.31], [3.5, 3.54], [3.595, 3.635], [3.715, 3.755], [3.865, 3.915], [3.995, 4.01], [4.035, 4.13], [4.2, 4.235], [4.345, 4.41], [4.517, 4.545], [4.64, 4.663], [4.71, 4.775], [5, 5.05], [5.11, 5.155], [5.236, 5.265], [5.348, 5.386], [5.485, 5.535], [5.607, 5.623], [5.67, 5.68], [5.7, 5.719], [5.767, 5.79], [5.87, 5.909], [5.978, 5.992], [6.022, 6.0525], [6.155, 6.19].

Also, it is negative and decreasing in the following intervals:

[0.96, 1.12], [1.9, 2.03], [2.67, 2.77], [3.38, 3.44], [3.655, 3.69], [3.81, 3.837], [3.948, 3.97], [4.071, 4.092], [4.15, 4.174], [4.296, 4.318], [4.452, 4.485], [4.577, 4.604], [4.83, 4.91], [5.178, 5.211], [5.29, 5.32], [5.444, 5.465], [5.567, 5.587], [5.64, 5.653], [5.731, 5.749], [5.83, 5.849], [5.937, 5.955], [6.081, 6.115], (see Figure 9).



**Figure 9:** The Graph of Function  $S_2^{[3]}$  in the Interval  $(0, 2\pi)$

So the intervals within the function  $S_\alpha$  is sensitive dependence on initial condition for all  $\alpha > 1, n > 1$  by increasing  $n$ .

If chosen  $n$  is large enough then the interval  $(0, 2\pi)$  is covered completely.

In general, for all the interval  $[2n\pi, 2(n+1)\pi)$  for  $n=1,2,\dots$ , it can be obtained that the function  $S_\alpha$  is sensitive dependence on initial condition in  $\{x : x > 0\}$ .

**4.4:Theorem**

Let  $S_\alpha : \mathbb{R} \rightarrow \mathbb{R}$  be defined by  $S_\alpha(x) = \alpha x \sin(x), \alpha > 0$ , then the function  $S_\alpha$  is chaotic on  $\mathbb{R}$ .

**Proof**

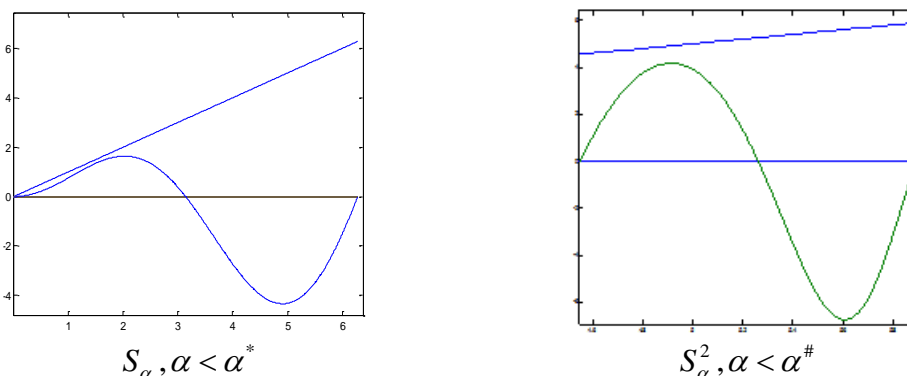
From theorem 4.3  $S_\alpha$  is sensitive dependence on initial conditions on  $\mathbb{R}$  then, by definition 8  $S_\alpha$  is chaotic on  $\mathbb{R}$ .

Now we try to answer the following question : how the family  $S_\alpha(x) = \alpha x \sin(x), \alpha > 0$  becomes chaotic .

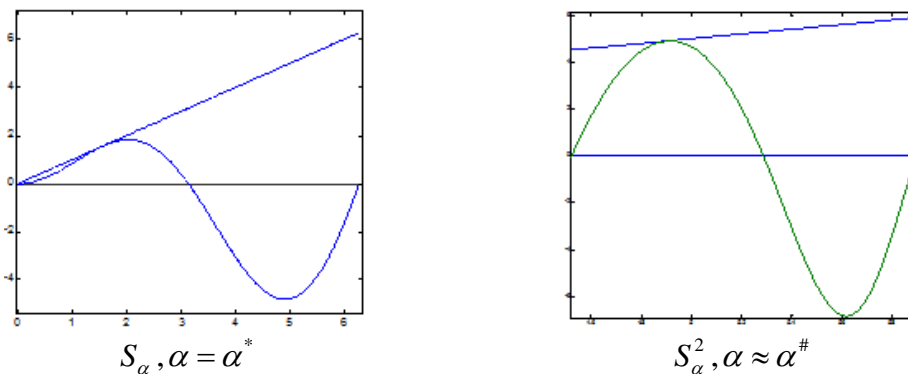
We will show that the mapping  $S_\alpha$  route to chaos by a series of saddle-node bifurcations .In fact, this is a typical route to chaos . We will show that  $S_\alpha^2$  has the same " behavior

" inside certain box .We conclude that  $S_\alpha^2$  has a saddle node bifurcation in this box .

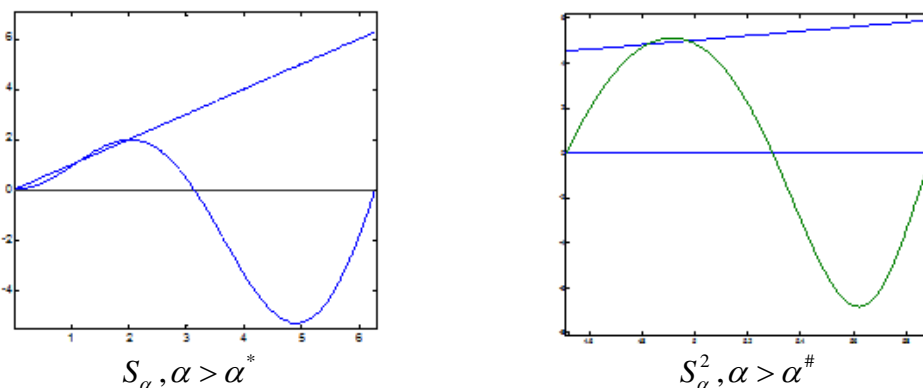
Experimentally ,we choose the interval  $(4.5, 5.9)$ .Consider the Figures 10,11,12 and 13.



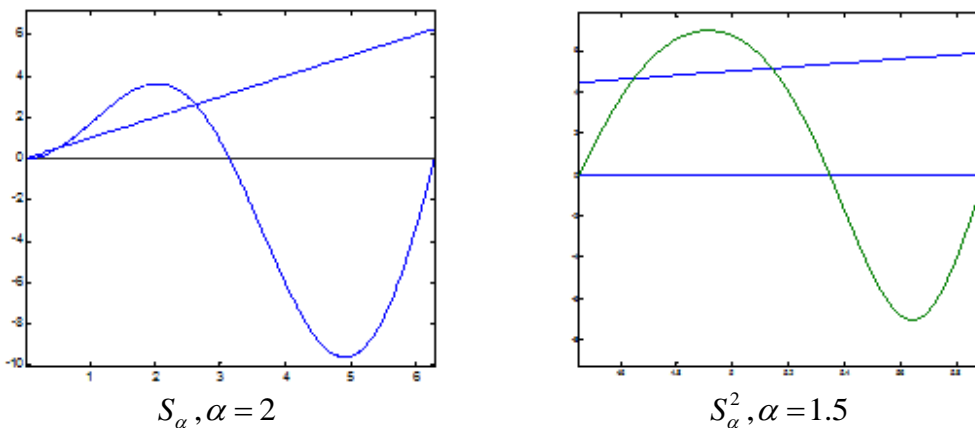
**Figure 10 :** The functions  $S_\alpha$  and  $S_\alpha^2$  have no fixed point



**Figure 11:** The functions  $S_\alpha$  and  $S_\alpha^2$  have only one fixed point



**Figure 12:** The functions  $S_\alpha$  and  $S_\alpha^2$  have two fixed points one of them attracting and another repelling



**Figure 13:** The functions  $S_\alpha$  and  $S_\alpha^2$  have two repelling fixed points

The graphs of  $S_\alpha$  and  $S_\alpha^2$  have the same patterns around the parameter value  $\alpha^* = 1$  and  $\alpha^\# \approx 1.4158182$  respectively. In fact, we have the following cases:

1. When  $\alpha < \alpha^\#$  then  $S_\alpha^2$  has no fixed point in the interval  $(4.5, 5.9)$ .
2. When  $\alpha \approx \alpha^\#$  then  $S_\alpha^2$  has only one fixed point in the interval  $(4.5, 5.9)$ .
3. When  $\alpha > \alpha^\#$ , for example  $\alpha = 1.416$ , two fixed points for  $S_\alpha^2$  are born; one of them is attracting and the other is repelling.
4. Moreover, for  $\alpha = 2$ , the two fixed points in (3) will be repelling, and

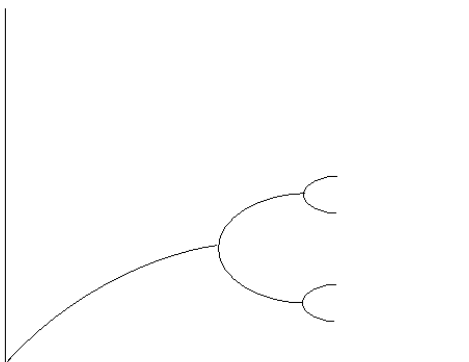
$S_\alpha^2$  has the same critical points  $\{4.913, 5.6172\}$  in the box  $(4.5, 5.9)$  for each  $\alpha$  in cases 1, 2, 3 and 4.

The above observations show that the behavior of  $S_\alpha^2$  in the interval  $(4.5, 5.9)$  is similar to that of  $S_\alpha$  in  $(0, 2\pi]$ . The remarks 1, 2 and 3 above show that  $S_\alpha^2$  has a saddle node bifurcation in the interval  $(4.5, 5.9)$  at  $\alpha = \alpha^\#$ .

Continuing this process, we have a series of saddle-node bifurcation for  $S_\alpha$  as  $\alpha$  increases.

Therefore, the bifurcation diagram for  $S_\alpha$  must be as in Figure 14.

The above discussion shows, experimentally, that this family encountered with chaotic dynamics for certain values of  $\alpha$  namely  $\alpha = 2$ . This is called a saddle node bifurcation route to chaos.



**Figure 14:** The bifurcation diagram of  $S_\alpha, S_\alpha^2, \alpha \in IR$

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- [2] D. Gulick, Encountered with Chaos Mc grow –Hill, Inc, 1992.