On Conjugate Space of 2-Fuzzy Generalized 2-Normed Space

<div dir="rtl">

حول الفضاء المرافق للفضاء 2-المعياري المعمم 2-الضبابي

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Abstract:

The main goal of this paper is to prove the extension theorem for 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional in 2-fuzzy generalized 2-normed space. Also, the definition of the 2-fuzzy adjoint operator of 2-fuzzy generalized 2-fuzzy bounded 2-fuzzy 2-linear operator defined on 2-fuzzy generalized 2-normed space is introduced.

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الخلاصة:

ان الهدف الرئيسي من هذا البحث هو برهنة مبرهنة التوسع للدالي 2-الخطي 2-الضبابي 2- المقيد 2-الضبابي المعمم 2-الضبابي في الفضاء 2- المعياري المعمم 2-الضبابي. ايضا قدم تعريف المؤثر المجاور2- الضبابي للمؤثر2-الخطي 2-الضبابي 2-المقيد 2-الضبابي المعمم 2-الضبابي المعرف على الفضاء 2-المعياري المعمم 2-الضبابي.

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1. Introduction:

The theory of 2-norm on a linear space has introduced and developed by Gahler in [1]. In 2006 Lewandowska and et.al. [2] introduced the notation of Hahn-Banach extension theorem in generalized 2-normed space. Somasundaram and Beaula [3] defined the notion of 2-fuzzy 2-normed linear space. Later, Thangaraj and Angeline [4] introduced Hahn-Banach theorem in the realm of 2-fuzzy 2-normed linear spaces. Faria and Rasha [5], introduced and proved the form of Hahn-Banach theorem in generalized 2-normed spaces and gave the definition of adjoint operator for generalized 2-bounded 2-linear operator. In this paper we redefined in a general setting the idea of generalized 2-normed space, that appeared in ([2],[5]) and give 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator and studies extension theorem for 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy linear functional. Also, we give definition of 2-fuzzy adjoint operator of 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator. Moreover, we prove that the 2-fuzzy adjoint operator has the same norm as the 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator itself.

2. 2-Fuzzy Adjoint Operator of 2-Fuzzy Generalized 2-Fuzzy 2-Bounded 2-Fuzzy 2-Linear Operator:

In this section, we give a definition of 2-fuzzy generalized 2-normed space which based the idea that appeared in [2]. Also, some facts that appeared in [2] are generalized to 2-fuzzy setting.

Definition (2.1), [2]:

Let X be a real linear space of dimension greater than one. A function $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ is said to be a generalized 2-norm on X in case for each x, y and $z \in X$ and for each $\alpha \in R$,

$(N_1) \|x, \alpha y\| = |\alpha| \cdot \|x, y\| = \|\alpha x, y\|$

$(N_2) \|x, y + z\| \leq \|x, y\| + \|x, z\|$

$(N_3) \|x + y, z\| \leq \|x, z\| + \|y, z\|$

The pair $(X \times X, \|., .\|)$ will be referred to as a generalized 2-normed space on $X \times X$.

Definition (2.2), [4]:

Let X be real linear space and F(X) be the set of all fuzzy sets on X. For $U, V \in F(X)$ and $k \in R$, define

$$U + V = \{(x + y, \lambda \wedge \mu) \mid (x, \lambda) \in U, (y, \mu) \in V\} \text{ and}$$

$$kU = \{(kx, \lambda) \mid (x, \lambda) \in U\}$$

Definition (2.3), [4]:

A fuzzy linear space $\widetilde{X} = X \times (0,1]$ over the real field R where the addition and scalar multiplication operation on $\widetilde{X}$ are defined by

$$(x, \lambda) + (y, \mu) = (x + y, \lambda \wedge \mu)$$

$$k(x, \lambda) = (kx, \lambda)$$

Is a fuzzy normed space in case for each $(x, \lambda) \in \widetilde{X}$

(1) $\|(x, \lambda)\| = 0$ if and only if $x = 0$, $\lambda \in (0,1]$

(2) $\|k(x, \lambda)\| = |k| \|(x, \lambda)\|$ for each $(x, \lambda) \in \widetilde{X}$ and for each $k \in R$

(3) $\|(x, \lambda) + (y, \mu)\| \le \|(x, \lambda \wedge \mu)\| + \|(y, \lambda \wedge \mu)\|$ for each $(x, \lambda), (y, \mu) \in \widetilde{X}$

(4) $\left\| (x, \underset{t}{\vee} \lambda_t) \right\| = \underset{t}{\wedge} \|(x, \lambda_t)\|$ for each $\lambda_t \in (0,1]$.

Definition (2.4), [4]:

Let X be a real linear space and F(X) be the set of all fuzzy sets in X the addition and scalar multiplication are defined by

$$f + g = \{(x + y, \lambda \wedge \mu) \mid (x, \lambda) \in f, (y, \mu) \in g\} \text{ and}$$

$$kf = \{(kx, \lambda) \mid (x, \lambda) \in f, k \in R\}$$

Definition (2.5),[4]:

Let X be a real linear space. A function $\|.\| : F(X) \rightarrow [0, \infty)$ is said to be norm on a F(X) in case for each $f, f_1, f_2 \in F(X)$ and $k \in R$, the following conditions hold

(1) $\|f\| = 0$ if and only if $f = 0$

(2) $\|kf\| = |k| \|f\|$

(3) $\|f_1 + f_2\| \le \|f_1\| + \|f_2\|$

The pair $(F(X), \|.\|)$ will be refered to as a fuzzy normed space.

Definition (2.6):

A 2-fuzzy generalized 2-normed space is a generalized 2-normed space on $F(X) \times F(X)$.

In order to make definition (2.6) as clear as possible we will consider the following example.

Example (2.7):

Let $(F(X), \|.\|)$ be a fuzzy normed space. For each $f_1, f_2 \in F(X)$ and $k \in R$ define

$$\|f_1, f_2\| = \|f_1\| \|f_2\|$$

It is easy to check that $(F(X) \times F(X), \|., .\|)$ is a 2-fuzzy generalized 2-normed space.

Definition (2.8), [4]:

Let X and Y be real linear spaces. A function T from $F(X) \times F(X)$ into $F(Y)$ is said to be 2-fuzzy 2-linear operator in case satisfies the following conditions: for all $f_1, f_2, f_3, f_4 \in F(X)$.

(1) $T(f_1 + f_2, f_3 + f_4) = T(f_1, f_3) + T(f_1, f_4) + T(f_2, f_3) +$
$$T(f_2, f_4).$$

(2) $T(\alpha f_1, \beta f_2) = \alpha . \beta \, T(f_1, f_2)$, for all scalars α, β.

Definition (2.9), [4]:

Let X be real linear space. A 2-fuzzy 2-linear functional is a real valued function on $F(X) \times F(X)$ satisfies the following conditions: for all $f_1, f_2, f_3, f_4 \in F(X)$.

$(1)\, T(f_1 + f_2, f_3 + f_4) = T(f_1, f_3) + T(f_1, f_4) + T(f_2, f_3) +$
$\qquad\qquad T(f_2, f_4).$

$(2)\, T(\alpha f_1, \beta f_2) = \alpha.\beta\; T(f_1, f_2)$, for all scalars α, β .

Definition (2.10):

Let $(F(X) \times F(X), \|.,.\|)$ be a 2-fuzzy generalized 2-normed space and $(F(Y), \|.\|)$ be a fuzzy normed space a 2-fuzzy 2-linear operator $T : (F(X) \times F(X), \|.,.\|) \to (F(Y), \|.\|)$ is said to be 2-fuzzy generalized 2-fuzzy 2-bounded in case there is a constant $k > 0$ such that $\|T(f_1, f_2)\| \le k \|f_1, f_2\|$, for each $f_1, f_2 \in F(X)$.

Remark (2.11)

Let $(F(X) \times F(X), \|.,.\|)$ be 2-fuzzy generalized 2-normed space and $(F(Y), \|.\|)$ be fuzzy normed space. We denote the set of all 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator from $(F(X) \times F(X), \|.,.\|)$ by $B(F(X) \times F(X), F(Y))$.

Proposition (2.12):

Let $T : (F(X) \times F(X), \|.,.\|) \to (F(Y), \|.\|)$ be a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator. Then

$\|T\| = \text{Inf} \left\{ k : \|T(f_1, f_2)\| \le k \|f_1, f_2\| ; (f_1, f_2) \in F(X) \times F(X) \right\}$ is the norm on the set of all 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator T.

Theorem (2.13):

Let $(F(X) \times F(X), \|.,.\|)$ be 2-fuzzy generalized 2-normed space and M be a linear subspace of $F(X) \times F(X)$. If T is a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional defined on M then T can be extended to a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional T_0 defined on the whole space $F(X) \times F(X)$ such that $\|T\| = \|T_0\|$.

Proof

If $M = F(X) \times F(X)$ or $\|T\| = 0$ then take $T = T_0$. Otherwise without lose the generality assume that $\|T\| = 1$ consider the family Ǎ of all possible extentions of T of norm one. Partially order Ǎ with $\le$ as follows given $(G_1, L_1), (G_2, L_2) \in$ Ǎ, put $(G_1, L_1) \le (G_2, L_2)$ if and only if G_2 is an extension of G_1 that is $L_1 \subseteq L_2$, $G_2(f_1, f_2) = G_1(f_1, f_2)$ for each $(f_1, f_2) \in L_1$ and $\|G_2\| = \|G_1\|$.

The family Ǎ is non-empty, because $(T, M) \in$ Ǎ. Let $\Im$ be chain of Ǎ. Define $\tilde{L} = \bigcup_{(G,L) \in \Im} L$. Clearly $\tilde{L}$ is a real linear subspace of $F(X) \times F(X)$ and contains M. Define

$\tilde{G} : \tilde{L} \to R$ by $\tilde{G}(f_1, f_2) = G(f_1, f_2)$ where G is associated with some L, $(G, L) \in \Im$, which contains (f_1, f_2). Then $\tilde{G}$ is a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional on $\tilde{L}$ that is an extinction of every G and $\|\tilde{G}\| = 1$. So the constructed pair $(\tilde{G}, \tilde{L})$ is hence an upper bound for the chain $\Im$. By using Zorn's lemma there exists a maximal element $(T_0, L_m) \in$ Ǎ. To complete the proof it is enough to show that $L_m = F(X) \times F(X)$. Suppose by contrary that there exists (f_0, g_0) in $F(X) \times F(X) \setminus L_m$. Then consider the linear space

$L' = L_m + R(f_0, g_0) = \{(f + \alpha f_0, g + \mu g_0); (f,g) \in L_m\}$. Define
$T' : L' \to R$

$T'(f + \alpha f_0, g + \mu g_0) = T_0(f,g) + \alpha\mu\gamma$ where $(f,g) \in L_m$ and $\gamma \in R$ will be chosen in such away

that $\|T'\| = 1$. But $\|T'\| = 1$ provided that

$$\left|T_0(f,g) + \alpha\mu\gamma\right| \leq \left\|f + \alpha f_0, g + \mu g_0\right\| \dots\dots\dots\dots\dots\dots\dots\dots(1)$$

For each $(f,g) \in M$ and $\gamma \in R$. Replace (f,g) by $(-\alpha f, -\mu g)$, and divide both sides of (1) by $|\alpha\mu|$.

Then the requirement is that

$$\left|T_0(f,g) - \gamma\right| \leq \left\|f - f_0, g - g_0\right\| \dots\dots\dots\dots\dots\dots\dots\dots\dots(2)$$

For each $(f,g) \in M$ and $\gamma \in R$. Since T_0 is 2-fuzzy 2-linear by choosing γ in such a way that

$T_0(f,g) - \|f - f_0, g - g_0\| \leq \gamma \leq T_0(f,g) - \|f - f_0, g - g_0\|$. Then (2) and therefore (1) holds. So we

have proved that $(T', L') \in \check{A}$, $(T_0, L_m) \neq (T', L')$ and $(T_0, L) \leq (T', L')$

which is a contradiction.

Theorem (2.14):

Let (f_0, g_0) be a vector in the 2-fuzzy generalized 2-normed space $(F(X) \times F(X), \|\cdot, \cdot\|)$ such that

$\|f_0, g_0\| \neq 0$. Then there exists a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear

functional T_0, defined on the whole space, such that $T_0(f_0, g_0) = \|f_0, g_0\|$ and $\|T_0\| = 1$.

Proof:

Consider the linear space

$M = \{(\alpha f_0, \mu g_0)\}$ and consider the functional T, defined on M as follows

$T(\alpha f_0, \mu g_0) = \alpha\mu\|f_0, g_0\|$

Clearly, T is a 2-fuzzy 2-linear functional with the property that $T(f_0, g_0) = \|f_0, g_0\|$.

Further, since for any $(f,g) \in M$

$$\left|T(f,g)\right| = |\alpha\mu|\|f_0, g_0\| = \|\alpha f_0, \mu g_0\| = \|f, g\|$$

We see that T is a 2-fuzzy 2-bounded 2-fuzzy 2-functional. Moreover $\|T\| = 1$. It now remains only

to apply theorem to assert the existence of a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional defined on the whole space, extending T and having the same norm as T, that

$\|T_0\| = 1$.

Notation (2.15):

Let us denote the set of all 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear functional

defined on 2-fuzzy generalized 2-normed space $(F(X) \times F(X), \|., .\|)$ by $(F(X) \times F(X))^*$ and we call

conjugate space of 2-fuzzy generalized 2-normed space, and the set of all bounded linear functional

defined on $F(Y)$ by $F(Y)^*$.

Proposition (2.16):-

Let $(F(X) \times F(X), \|., .\|)$ be 2-fuzzy generalized 2-normed space. Then $((F(X) \times F(X))^*, \|.\|)$ is a

complete normed linear space with norm defined by

$\|T\| = \text{Inf}\left\{k : |T(f_1, f_2)| \leq k\|f_1, f_2\| : (f_1, f_2) \in F(X) \times F(X)\right\}$.

Proof:-

It is easy to see $((F(X) \times F(X))^*, \|.\|)$ is a normed linear space. In order to prove $(F(X) \times F(X))^*$ complete let $\{T_k\}_{k \in N}$ be a Cauchy sequence in $(F(X) \times F(X))^*$ thus $\lim_{k \to \infty} \|T_k - T_{k+p}\| = 0, \forall\, p = 1,2,....$ Also,

$$\left|(T_k - T_{k+p})(f_1, f_2)\right| \le \|T_k - T_{k+p}\| \|f_1, f_2\|.\text{Then}$$

$$\left|(T_k - T_{k+p})(f_1, f_2)\right| \longrightarrow 0 \text{ as } k \longrightarrow \infty, \forall f_1, f_2 \in F(X).$$ Thus $\{T_k(f_1, f_2)\}$ is a Cauchy sequence in R. Since $(R, \|.\|)$ is complete then $\lim_{k \to \infty} T_k(f_1, f_2) = y$ exists in $(R, \|.\|)$. Define $T : F(X) \times F(X) \to R$ by $T(f_1, f_2) = y$ then it can be easily verified that T is 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2- linear functional. Hence

$$\left|T_k(f_1,f_2) - T_{k+p}(f_1,f_2)\right| \le \|T_k - T_{k+p}\| \|f_1,f_2\| \le \varepsilon \|f_1,f_2\| \;\forall k \ge N(\varepsilon),\, f_1, f_2 \in F(X),\, p = 1,2,....\text{ L}$$

etting $p \longrightarrow \infty$ we get

$$\left|T_k(f_1,f_2) - T(f_1,f_2)\right| \le \varepsilon \|f_1,f_2\|, \;\forall\, k \ge N(\varepsilon) \text{ and } \forall f_1, f_2 \in F(X). \text{ Thus}$$

$$\|T_k - T\| \le \varepsilon, \;\forall k \ge N(\varepsilon). \text{Then } \|T_k - T\| \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Hence $(F(X) \times F(X))^*$ is complete.

Definition (2.17):

Let $T : (F(X) \times F(X), \|.,.\|) \to (F(Y), \|.\|)$ be a 2-fuzzy generalized 2-fuzzy 2-bounded 2-fuzzy 2-linear operator from a 2-fuzzy generalized 2-normed space $F(X) \times F(X)$ to a normed space $F(Y)$. The operator $T^X : F(Y)^* \to (F(X) \times F(X))^*$ is defined by

$$(T^X g)(f_1, f_2) = g(T(f_1, f_2)) = h(f_1, f_2), \; g \in F(Y),\, f_1, f_2 \in F(X),$$ is called the 2-fuzzy adjoint operator of T.

Next, we give the following theorem which is based on the idea that appeared in [5].

Theorem (2.18):

Let $(F(X) \times F(X) , \|\cdot,\cdot\|)$ be a 2-fuzzy generalized 2-normed space and $(F(Y), \|.\|)$ be a normed space. If $T : (F(X) \times F(X), \|.,.\|) \to (F(Y), \|.\|)$ is a 2-fuzzy generalized 2-fuzzy 2-bouned 2-fuzzy 2-linear operator. Then $T^X : F(Y)^* \to (F(X) \times F(X))^*$ is a bounded linear operator and $\|T^X\| = \|T\|$.

Proof:-

Since the operator T^X with its domain $F(Y)^*$ is a linear space then

$$T^X(\alpha_1 g_1 + \alpha_2 g_2)(f_1, f_2) = (\alpha_1 g_1 + \alpha_2 g_2)T(f_1, f_2)$$
$$= \alpha_1 g_1 T(f_1, f_2) + \alpha_2 g_2 T(f_1, f_2)$$
$$= \alpha_1 (T^X g_1)(f_1, f_2) + \alpha_2 (T^X g_2)(f_1, f_2)$$

Also, $\|T^X g\| = \|h\| \le \|g\| \|T\|$

Moreover, $\|T^X\| = \inf\{K : \|T^X g\| \le K \|g\| \}$

Then, $\left\|T^x\right\| \le \|T\|$

For every vector (f_1, f_2) in $F(X) \times F(X)$ such that $\|f_1, f_2\| \ne 0$, then there is $g_0 \in F(Y)^*$ such that $\|g_0\| = 1$ and $g_0(T(f_1, f_2)) = \|T(f_1, f_2)\|$.

Writing $h_0 = T^x g_0$, we obtain

$$\|T(f_1, f_2)\| = g_0(T(f_1, f_2)) = h_0(f_1, f_2) \le \|h_0\| \|f_1, f_2\| = \left\|T^x g_0\right\| \|f_1, f_2\|$$

$$\le \left\|T^x\right\| \|g_0\| \|f_1, f_2\|$$

since, $\|g_0\| = 1$, we have

$$\|T(f_1, f_2)\| \le \left\|T^x\right\| \|f_1, f_2\|$$

But, $\|T(f_1, f_2)\| \le \|T\| \|f_1, f_2\|$

where, $k = \|T\|$ is the smallest constant k such that $\|T(f_1, f_2)\| \le k \|f_1, f_2\|$

Hence, $\left\|T^x\right\| \ge \|T\|$. Therefore $\left\|T^x\right\| = \|T\|$.

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