Al-Rafidain Journal of Computer Sciences and Mathematics (RJCM)

www.csmj.mosuljournals.com

The Analytic solution for some non-linear stochastic differential equation by linearization (Linear-transform)

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Article information

Article history:
Received: 8 /1 /2023
Accepted: 28 /2 /2023
Available online:

Abstract

In this paper, we study a reducible method which is called linearization(Linear-transform) for some non-linear stochastic differential equations (SDEs) to linear by using the Ito-integrated formula. And then finding their analytic solution, we compare the obtained solution for the nonlinear SDEs with the approximate solution by using numerical (Euler - Maruyama and Milstein) Methods.

Keywords:
Stochastic differential equation, Ito formula, Reducible, Euler-Maruyama & Milstein.

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1. Inroduction:

The stochastic differential equations are increasingly used in many scientific and industrial fields which demonstrate the importance of stochastic modeling, since they can be used to any address problem caused by noise, accidents, etc. [1]

Sunday Fadugba and Ado Ekiti (2013) [2], study and explain the convergence of the Euler-Maruyama method and Milstein scheme for the solution of stochastic differential equations, Elborala M.M. and M.I. Youssef (2019) [3] extends some results about the existences of solution for a functional nonlocal SDEs under suitable conditions, Xu, J., et.al (2012) [4] explain and study Uniqueness and explosion time of solutions of stochastic differential equations driven by fractional Brownian motion, D. Kim and D. Stanescu, (2008) [5] used the Runge-Kutta methods for Low-storage stochastic differential equations.

In this paper, we study the linearization to the non-linear SDEs in order to find the analytic solution to the obtained linear stochastic differential equations, there are several examples with numerical solution are used to validate the results.

Let X(t) satisfied the following stochastic differential equation

$$dX(t) = f\big(t, X(t)\big)dt + g\big(t, X(t)\big)dW(t) \qquad \text{….(1)}$$

Where W(t) denote wiener process , $f\big(t, X(t)\big)$ and $g\big(t, X(t)\big)$ are deterministic non-linear functions and $dW(t)$ represent the differential form of $W(t)$, equation(1) is called nonlinear stochastic differential equation other wise it is called linear.

let the real-valued function F (t, x(t)) has continuous partial derivatives

$\frac{\partial F}{\partial t}, \frac{\partial F}{\partial x}, \frac{\partial^2 F}{\partial x^2}$, t ϵ [0, T] and x ϵ R, then F (t, x(t)) satisfies the following equation:

$$\boldsymbol{dF(t,x(t)) = \left[\frac{\partial F}{\partial t} + f\ \frac{\partial F}{\partial x} + \frac{1}{2}\ g^2 \frac{\partial^2 F}{\partial x^2}\right](t,X(t))dt + \left[\frac{\partial F}{\partial x} g\right](t,X(t))dW(t)} \quad \text{... (2)}$$

Equation (2) is called (Ito-formula). [6]

2: Preliminaries and Results:

We introduce some fundamental and necessary concept steps for the transformation of non-linear SDEs to reduce to linear and then find the analytic solution**.**

Definition2.1. (**Wiener Process) [1][7]**

The continuous-time stochastic process $\{W_t : t \geq 0\}$ is called wiener process (Brownian motion) over the interval $[0, \mathcal{T}]$ which satisfies the following conditions:

1. W (0) = 0
2. $If\ t,s\ \geq\ 0, then\ W(t) -\ W(s)$ is normally distributed with zero mean and variance $|t - s|$.
3. For $0 \leq s < t < k < j \leq \mathcal{T}$, W(t)- W(s)and $W(j) - W(k)$ are independent increments.

Definition 2.2. (Stochastic integral) [8]

A stochastic integral is an integral which is defined as a sum more than integration and it is increased by the rise in time on the Wiener process trajectory. That is:

$$\int_c^b c(t)dW(t) = \sum_{i=0}^{k-1} \delta_i\left(W_{t_{i+1}} - W_{t_i}\right) \quad \text{…(3)}$$

Definition2.3. (The analytic solution of linear SDE) [9]

Let we have the general linear stochastic differential equation given by

$$dX_t = (a_1(t)X_t + a_2(t))dt + (b_1(t)X_t + b_2(t))dW_t \ldots \text{ (4)}$$

Then the solution will be in the form:

$$X_t = \emptyset_{t,t_0}(X_{t_0} + \int_{t_0}^t {\emptyset_{s,t_0}}^{-1}(a_2(s) - b_1(s)b_2(s))\text{ds} + \int_{t_0}^t {\emptyset_{s,t_0}}^{-1} b_2(s)\, dW_t\ ) \quad \text{…(5)}$$

Where

$$\emptyset_{t,t_0} = \exp(\int_{t_0}^t \left(a_1(s) - \tfrac{1}{2}{b_1}^2(s)\right)d\,t + \int_{t_0}^t b_1(s)d\,W_s) \quad \text{… (6)}$$

3. Linearization method for non-linear (SDEs): [10]

The linearization method means to find a suitable function which reduce such nonlinear stochastic differential equation to linear in order to find their analytic solution, we explain it by the following main steps:

Step 1.

Suppose we have the following nonlinear stochastic differential equation:

$$dY(t) = f\big(t,Y(t)\big)dt + g\big(t,Y(t)\big)dW(t)\ \ ;\ \text{y}(\boldsymbol{t_0}) = \boldsymbol{y_0} \ldots\text{(7)}$$

Here f(t, y) and g(t, y) are real non-linear functions, Suppose that the function U(t, y(t)) be smooth and has continuous derivatives $\frac{\partial U}{\partial t}, \frac{\partial U}{\partial Y}, \frac{\partial^2 U}{\partial Y^2}$, that reduce equation (7) to some linear stochastic differential equation. Which we want to find it.

By using Ito formula to U(t, y(t)) , we get

$$dU(t,x(t)) = \left[\frac{\partial U}{\partial t} + f\ \frac{\partial U}{\partial Y} + \frac{1}{2}\ g^2 \frac{\partial^2 U}{\partial Y^2}\right](t,Y(t))dt + \left[\frac{\partial U}{\partial Y} g\right](t,Y(t))dW(t) \quad \text{…(8)}$$

Step 2. suppose that equation (8) transformed to linear stochastic differential equation of the form:

$$dx(t) = (a_1(t)X(t) + a_2)dt + (b_1(t)X(t) + b_2)dw(t) \quad \text{..(9)}$$

Step3. Determined the parameters (a_1, a_2, b_1, b_2) for the linearity by using the comparison of eq. (8) and eq. (9), then we have

$$\left[\frac{\partial U}{\partial t} + f\ \frac{\partial U}{\partial Y} + \frac{1}{2}\ g^2 \frac{\partial^2 U}{\partial Y^2}\right](t,Y) = (\text{a}_1(t)U(Y) + a_2) \text{ ...(10.1)}$$

$$\left[\frac{\partial U}{\partial Y} g\right](t,Y) = (b_1(t)U(Y) + b_2) \quad \text{…(10.2)}$$

By solving equation (10.1) and equation (10.2), we get the values of the parameter of the linear equation(9)

Step4. The analytic solution for the obtained linear stochastic differential equation has the form:

$$X_t = \Phi_t\left\{X_0 + (a_2 - b_1(t)b_2)\int_0^t {\emptyset_t}^{-1}ds + b_2\int_0^t {\emptyset_t}^{-1}dW_s\right\}$$

where $\emptyset_t = \exp(a_1(t)t - \frac{1}{2}{b_1}^2(t)t + b_1(t)W_t)$

By using transformation

$Y_t =\ U^{-1}(t,X_t)$

we obtain the solution of eq. (7).

3.1.The method of finding the analytic solution: In this paragraph, we explain the main step of finding the suitable function which transformed the nonlinear stochastic differential equation to linear stochastic differential equation.

From eq(10.2), let $g(Y) \neq 0$ and $b_1(t) = b_1 \neq 0$, then we get

$$\frac{\partial U}{\partial Y} = \frac{b_1}{g(Y)}U(Y) + \frac{b_2}{g(Y)} \quad \text{…..(10.3)}$$

which is a first order differential equation,and the integrated factor is:

$M = e^{\int -\frac{b_1}{g(Y)}dy}$, let $B(y) = \int_{y_0}^y \frac{ds}{g(s)}$,then

$$U(Y) = e^{b_1B(y)}\int e^{-b_1B(y)}\frac{b_2}{g(y)}dy + ke^{b_1B(y)}$$

$$U(Y) = \frac{-b_2}{b_1}e^{b_1B(y)}\int e^{-b_1B(y)}\frac{-b_1}{g(y)}dy + ke^{b_1B(y)}$$

$$U(y) =\ ke^{b_1B(y)} - \frac{b_2}{b_1}$$

Then equation (10.1) becomes:

$$ke^{b_1B(y)}\left[\frac{f(y)}{g(y)}b_1 + \frac{1}{2}b_1^2 - \frac{1}{2}b_1\frac{dU}{dy}\right] = a_1\left[ke^{b_1B(y)} - \frac{b_2}{b_1}\right] + a_2$$

$$ke^{b_1B(y)}\left[\frac{f(y)}{g(y)}b_1 + \frac{1}{2}b_1^2 - \frac{1}{2}b_1\frac{dU}{dy}\right] = a_1ke^{b_1B(y)} - \frac{a_1b_2}{b_1} + a_2$$

$$\left[\frac{f(y)}{g(y)}b_1 + \frac{1}{2}b_1^2 - \frac{1}{2}b_1\frac{dU}{dy} - a_1\right]ke^{b_1B(y)} = a_2 - \frac{a_1b_2}{b_1}$$

$let\ A(y) = \frac{f(y)}{g(y)} - \frac{1}{2}\frac{dU}{dy}$, then

$$\left[b_1 A(y) + \frac{1}{2}b_1^2 - a_1\right] k e^{b_1 B(y)} = a_2 - \frac{a_1 b_2}{b_1} \quad ... (11)$$

The derivative of equation (11) is equal to zero, i.e.

$$[\left(b_1 A(u) + \frac{1}{2}b_1^2 - a_1\right) k e^{b_1 B(y)}]' = 0$$

$$b_1 \frac{dA}{dy}[k e^{b_1 B(u)}] + k b_1 A(u) \frac{d}{dy}(e^{b_1 B(u)})$$

$$k b_1 e^{b_1 B(y)} \frac{dA}{dy} + k b_1 A(y) \frac{b_1}{g(y)} e^{b_1 B(y)}$$

$$k b_1 e^{b_1 B(y)} [\frac{dA}{dy} + \frac{b_1}{g(y)} A(y)]$$

Multiplying by $\frac{g(y)e^{-b_1 B(y)}}{b_1}$, we get

$$kb(y)\left[\frac{dA}{dy} + \frac{b_1}{g(y)}A(y)\right]$$

$= g(y)\frac{dA}{dy} + b_1 A(y)$, then taking the derivative with respect to y we get

$$= b_1 A_y + \frac{d}{dy}(b_1\, A_y) = 0$$

then we have the relation

$$b_1 \frac{dA}{dy} + \frac{d}{dy}\left(g\frac{dA}{dy}\right) = 0 \quad ... \quad (12)$$

Which is satisfied if

$$\frac{dA}{dy} = 0 \quad or \quad \frac{d}{dy}\left(\frac{\frac{d}{dy}\left(g\frac{dA}{dy}\right)}{\frac{dA}{dy}}\right) = 0 \quad ....(13)$$

Provided that $b_1 = -\frac{\frac{d}{dy}(g\frac{dA}{dy})}{\frac{dA}{dy}}$

Result. The suitable linearization of a non-linear stochastic differential equation is:

1.If $b_1 \neq 0$ then the best transformation is
$U(y) = k e^{b_1 B(y)} - \frac{b_2}{b_1}$

2. if $b_1 = 0$, then the transformation is $U(y) = b_1 B(y) + \mathrm{k}$

by using a suitable value for b_1 and obtaining $U_y and\ U_{yy}$,from equations (10.1) and (10.2) we obtain the parameters a_1, a_2 , b_1, and b_2 for the linear stochastic differential equation (which is given in eq.(9)).

From equation (4) we obtain the general solution of the transformed linear stochastic differential equation by the transformed function rewrite y(t)=$U(x)^{-1}$.[11]

4.Numerical methods for solving nonlinear SDEs:

Since many stochastic differential equations have unknown solution, so it is necessary to derive numerical methods to generate approximations to the exact solution. we used (Euler-Maruyama and Milstein's) method.

4.1. Euler -Maruyama method: [12] [13]

Euler-Maruyama method is similar to Euler method for solving ordinary differential equations that are presented from the point of view of Taylor's algorithm which greatly simplifies accurate analysis.

Euler approximation is one of the simplest discrete time estimates for Ito-Taylor expansion .let X_t be an Ito process on $t \in [t_0, T]$ satisfying the stochastic differential equation

$$\left.\begin{matrix} dX_t = f(t, X_t)dt + g(t, X_t)dW_t \\ X_{t_0} = X_0 \end{matrix}\right\} \quad .... (14)$$

For a given estimate $t_0 < t_1 < t_2 < \cdots < t_N = T$, Euler approximation is a random process in a continuous time X_t. Now we write

$$X_{t_{n+1}} = X_{t_n} + \int_{t_n}^{t_{n+1}} f(t,x)dt + \int_{t_n}^{t_{n+1}} g(t,x)dW_t \quad ..(15)$$

By replacing the integral of (15) with Taylor's chain expansion for functions $f(t,x)$ and $g(t,x)$ around the point (t_n, x_n) and $X_n = X_{t_n}$ terms containing $(x - x_n)$ that arise in the expansion above ,we get

$$f(t,x) = f(t_n + \delta t, x_n + \delta t) = f(t_n, x_n) + \frac{\partial f}{\partial t}\delta t + \frac{\partial f}{\partial x}\delta x + \cdots$$

$$= f(t_n, x_n) + \frac{\partial f}{\partial t}\bigg|_{(t_n - x_n)}^{t - t_n} + \frac{\partial f}{\partial x}\bigg|_{(t_n - x_n)}^{x - x_n} + \ldots \quad .. (16)$$

$$g(t,x) = g(t_n + \delta t, x_n + \delta t) = g(t_n, x_n) + \frac{\partial g}{\partial t}\delta t + \frac{\partial g}{\partial x}\delta x + \cdots$$

$$= g(t_n, x_n) + \frac{\partial g}{\partial t}\bigg|_{(t_n - x_n)}^{t - t_n} + \frac{\partial g}{\partial x}\bigg|_{(t_n - x_n)}^{x - x_n} + \ldots \quad ... (17)$$

Substituting equations (16) and (17) into (15)

$$X_{t_{n+1}} = X_{t_n} + \int_{t_n}^{t_{n+1}} \left(f(t_n, x_n) + \frac{\partial f}{\partial t}\bigg|_{(t_n - x_n)}^{t - t_n} + \frac{\partial f}{\partial x}\bigg|_{(t_n - x_n)}^{x - x_n}\right) dt$$

$$+\int_{t_n}^{t_{n+1}}\left(f(t_n,x_n)+\frac{\partial g}{\partial t}\Big|\frac{t-t_n}{(t_n-x_n)}\right.$$
$$\left.+\frac{\partial g}{\partial x}\Big|\frac{x-x_n}{(t_n-x_n)}\right)dW_t \qquad \dots(18)$$

$$X_{t_{n+1}}=X_{t_n}+f_n(t_{n+1}-t_n)+g_n(W_{t_{n+1}}-W_{t_n}) \qquad ..(19)$$

This implies $X_{t_{n+1}}=X_{t_n}+f_n(\Delta t)+g_n(\Delta W)$

Which is the Euler-Maruyama formula.

4.2. Milstein's method: [12] [13]

The Milstein's method obtained by add the following second order term for Ito integral to the Euler-Maruyama scheme

$$g(X_{t_n})g'(X_{t_n})\int_{t_0}^{t}\int_{t_0}^{s}dW_z ds=g(X_{t_n})g'(X_{t_n})$$
$$=\frac{1}{2}g(X_{t_n})g'(X_{t_n})(\,(dW_t)^2-dt)$$

From the Ito-Taylor expansion, we obtain (Milstein formula) given by

$$X_{t_{n+1}}=X_{t_n}+f_n(t_{n+1}-t_n)+g_n(W_{t_{n+1}}-W_{t_n})$$
$$+\frac{1}{2}g(X_{t_n})g'(X_{t_n})[\,(W_{t_{n+1}}-W_{t_n})^2-(t_{n+1}-t_n)] \qquad ...(20)$$

This implies

$$X_{t_{n+1}}=X_{t_n}+f(X_{t_n})\Delta t+g(X_{t_n})\Delta W$$
$$+\frac{1}{2}g(X_{t_n})g'(X_{t_n})(\,(\Delta W)^2-\Delta t) \qquad ..(21)$$

Eq. (21) is called Milstein's formula.

5. Examples: In this paragraph we give some examples in order to explain the methods (analytically and numerically).

Example 5. 1. Consider the nonlinear SDEs given by

$$dY_t=Y_t\left(\alpha+\beta {Y_t}^{n-1}\right)dt+\gamma Y_t dw_t\,;\,Y(0)=Y_0$$
$$\alpha,\beta \text{ and } \gamma \text{ are constants.}$$

Solve it (analytically and numerically).

The analytic solution:

Here a(y)= $Y_t\left(\alpha+\beta {Y_t}^{n-1}\right)$ and b(y)= γY_t

Since A= $A(y)=\frac{a(y)}{b(y)}-\frac{1}{2}\frac{db(y)}{dy}$

Then $A=(\frac{\alpha}{\gamma}+\frac{\beta}{\gamma}Y^{n-1}-\frac{\gamma}{2})\neq 0$

$A_y=\frac{\beta}{\gamma}(n-1)Y^{n-2}$ and $b(y)A_y=\gamma y[\frac{\beta}{\gamma}(n-1)Y^{n-2}$ $=\beta(n-1)Y^{n-1}$

$\frac{d}{dy}\left(b(y)A_y\right)=\beta(n-1)^2Y^{n-2}$,

then $\frac{\frac{d}{dy}(b(y)A_y)}{A_y}=\gamma(n-1)$ *which is constant*

$\frac{d}{dy}\left(\frac{\frac{d}{dy}(b(y)A_y)}{A_y}\right)=0$, we can take

$b_1=-\frac{\frac{d}{dy}(b(y)A_y)}{A_y}=\gamma(1-n)\neq 0, b_2=0, k=1$

We get $U=ke^{b_1 iB(t)}=e^{b_1B(t)}$

Where $B(t)=\int_{y_0}^{y}\frac{ds}{b(s)}=\frac{1}{\gamma}\ln\left(\frac{y}{y_0}\right);\ y_0>0$

Then $U=[\frac{y}{y_0}]^{1-n}$ .. (22)

so we can assume

X= U(y) = $[\frac{y}{y_0}]^{1-n}$, $b_1=-\gamma(n-1), b_2=0$

To find a_1, a_2 , we need to find $U_y(y), U_{yy}(y)$ to use equation (10.1)

From equation (22), we get

$U_y(y)=(1-n)[\frac{y}{y_0}]^{-n}$ and $U_{yy}(y)=-n(1-n)[\frac{y}{y_0}]^{-n-1}$

So we have

$$a(y)\,U_y(y)+\frac{1}{2}b^2(y)U_{yy}=a_1U+a_2$$
$$y(\alpha+\beta y^{n-1})(1-n)[\frac{y}{y_0}]^{-n}$$
$$-\frac{1}{2}\gamma^2y^2[n(1-n)\left[\frac{y}{y_0}\right]^{-n-1}]$$
$$=a_1[\frac{y}{y_0}]^{1-n}+a_2$$

This implies

$y^{1-n}[(1-n)\alpha y_0+\frac{1}{2}(n^2-n)\gamma^2{y_0}^2-a_1)]+$ $[(1-n)\beta y_0-a_2{y_0}^{1-n}]=0,$

since $y^{n-1}\neq 0$

Then

$a_1=(1-n)\alpha y_0+\frac{1}{2}(n^2-n)\gamma^2{y_0}^2$ and $a_2=\beta(1-n){y_0}^n$

Therefore, the linearization for the given equation is

$dx(t)=[\left\{\left((1-n)\alpha y_0+\frac{1}{2}(n^2-n)\gamma^2{y_0}^2\right\}x(t)+\right.$ $\left.\beta(1-n){y_0}^n\right]dt+\left(\gamma(1-n)x(t)\right)dw(t)$

by using equation (4) , we get

$X_t=\emptyset_{t,t_0}*[X_{t_0}+\beta(1-n){y_0}^n\int_{t_0}^{t}{\emptyset_{s,t_0}}^{-1}ds]$

$\emptyset_{t,t_0}=\exp[\left(\alpha(1-n)\alpha y_0+\frac{1}{2}(n^2-n)\gamma^2{y_0}^2\right)t-$ $\frac{1}{2}\gamma^2(n-1)^2t+\gamma(1-n)w_t$

$=\exp[\{\left(\alpha(1-n)\alpha y_0+\frac{1}{2}(n^2-n)\gamma^2 y_0{}^2\right)-\frac{1}{2}\gamma^2(n-1)^2\}t+\gamma(1-n)w_t]$

$\int_{t_0}^{t}\emptyset_{s,t_0}{}^{-1}ds=\int_{t_0}^{t}\exp\left[-\left\{\left(\alpha(1-n)\alpha y_0+\frac{1}{2}(n^2-n)\gamma^2 y_0{}^2\right)-\frac{1}{2}\gamma^2(n-1)^2\right\}s-\gamma(1-n)w_s\right]ds$

From the transformation $Y_t=U^{-1}(X_t)$,

We obtain the general solution of the original equation

$$y(t)=\left[\frac{\exp[\frac{1}{2}\gamma^2(n-1)^2-\left(\alpha(1-n)\alpha y_0+\frac{1}{2}(n^2-n)\gamma^2 y_0{}^2\right)]t-\gamma(1-n)w_t]}{[y_{t_0}{}^{-1}+(\beta(1-n)\int_{t_0}^{t}\emptyset_{s,t_0}{}^{-1}ds]}\right]$$

For special case , let n=2 , $y_{t_0}=1$

$$dy_t=y_t(\alpha+\beta y_t)dt+\gamma y_t dw_t\ ;\ y(y_{t_0})=y_0$$

Then

$b_1=-\gamma\neq 0\ , we\ can\ take$

$U=ce^{b_1 B(t)}$

$$B(t)=\int_{y_0}^{y}\frac{ds}{\gamma y_t}=\frac{1}{\gamma}\ln(y)$$

$$U=ce^{-\gamma\frac{1}{\gamma}\ln(y)}$$

$$=ce^{-\ln(y)}=c\frac{1}{e^{\ln y}}=\frac{c}{y}$$

If we take $c=1\ , b_2=0\ , b_1=-\gamma$

Then $=\frac{1}{Y}; U_Y=\frac{-1}{y^2}; U_{yy}=\frac{-2}{y^3}$

From equation (10.1), we get

$$a(y).U_y+\frac{1}{2}b^2(y)U_{yy}=a_1U+a_2$$

$$y(\alpha+\beta y)\frac{-1}{y^2}+\frac{1}{2}\gamma^2y^2\frac{2}{y^3}=a_1\left[\frac{1}{y}\right]+a_2$$

$a_1=\gamma^2-\alpha\ ;\ a_2=-\beta, b_1=-\gamma\ , b_2=0$

From the linear equation

$X_t=[a_1x_t+a_2]dt+[b_1x_t+b_2]dw_t$, we obtain

$X_t=[(\gamma^2-x)x_t-that\ is\beta]dt+[-\gamma x_t+0]dw_t$

Then the solution is

$$X_t=\varphi_t\{x_0-\beta\int_0^t\varphi_t^{-1}ds\}$$

$$\varphi_t=\exp\{(\gamma^2-\alpha)t-\frac{1}{2}(-\gamma)^2t-\gamma w_t\}$$

$$=\exp\{\gamma^2t-\alpha t+\frac{1}{2}\gamma^2t-\gamma w_t\}$$

$$=\exp\{\frac{3}{2}\gamma^2t-\alpha t-\gamma w_t\}$$

$$=\exp\{-\left[\left(\alpha-\frac{3}{2}\gamma^2\right)t+\gamma w_t\right\}]$$

$$\int_0^t\varphi_t^{-1}ds=\int_0^t\exp\left[\left(\alpha-\frac{3}{2}\gamma^2\right)s+\gamma w_s\right]ds$$

$$\therefore\ x_t=\varphi_t\{x_0-\beta\int_0^t\exp\left[\left(\alpha-\frac{3}{2}\gamma^2\right)s+\gamma w_s\right]ds\}$$

$$=\exp\left\{-\left[\left(\alpha-\frac{3}{2}\gamma^2\right)t+\gamma w_t\right\}\right]x_0-\beta\exp\left\{-\left[\left(\alpha-\frac{3}{2}\gamma^2\right)t+\gamma w_t\right\}\right]\int_0^t\exp\left[\left(\alpha-\frac{3}{2}\gamma^2\right)s+\gamma w_s\right]ds$$

$$X_t=\exp\left\{-\left[\left(\alpha-\frac{3}{2}\gamma^2\right)t+\gamma w_t\right\}\right]x_0-\beta\int_0^t ds$$

Then the solution for this example is

$$Y_t=\frac{\exp[\left(\alpha-\frac{3}{2}\gamma^2\right)t+\gamma w_t]}{y_0{}^{-1}-\beta t[\exp[\left(\alpha-\frac{3}{2}\gamma^2\right)t+\gamma w_t]]}$$

The numerical solution:

Here $f_n=y_{t_n}(\alpha+\beta y_{t_n})$ and $g_n=\gamma y_{t_n}$

$\alpha=1.3;\ \beta=0.05\ ;\gamma=0.1:y_0=0.1$

The Euler -Maruyama method for this example take the form

$Y_{t_{i+1}}=Y_{t_i}+f_n\Delta t+g_n\sqrt{\Delta t}\eta_i$ … (23)

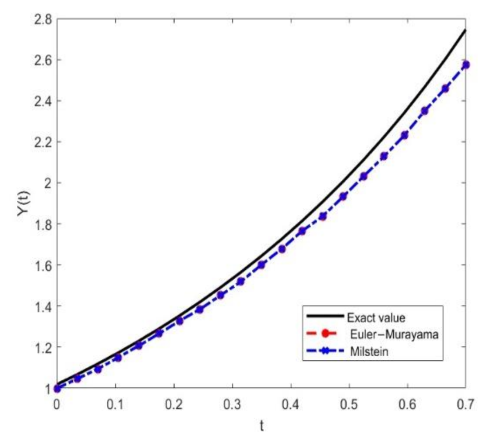
With $=i\Delta t\Rightarrow\Delta t=\frac{1}{N}$, $i\sim(0,1)$, $t\in[0,1]$

And Milstein's equation has the form:

$$y_{t_{n+1}}=y_{t_n}+f_{t_n}\Delta t+g_{t_n}\sqrt{\Delta t}\eta_j+\frac{1}{2}g_{t_n}\frac{\partial g_{t_n}}{\partial y_{t_n}}(\eta_j{}^2-1)\Delta t$$

Where $dW_t=W_t-W_{t-1}$, $dW_t=\sqrt{\Delta t}\eta_j$

The following figure shows the comparison between numerical (Euler -Maruyama & Milstein) with the analytic solution:



Figure(1):comperison between the analytic and numerical solution.

Table (1): The values of the exact and numerical solutions.

No.	exact	Euler-Maruyama	Milstein
1	1	1	1
2	1.051271	1.049880296	1.049875736
3	1.105170918	1.099308549	1.099274569
4	1.161834243	1.156438536	1.156368842
5	1.221402758	1.214723305	1.214618882

6	1.284025417	1.277548016	1.277375117
7	1.349858808	1.342431505	1.34226644
8	1.419067549	1.407831718	1.407647837
9	1.491824698	1.480546686	1.480344879
10	1.568312185	1.551690781	1.551476857
11	1.648721271	1.62957608	1.629403032
12	1.733253018	1.710609004	1.710385131
13	1.8221188	1.793461386	1.793250898
14	1.915540829	1.88440306	1.884199245
15	2.013752707	1.979965129	1.97980565
16	2.117000017	2.083915513	2.083786781
17	2.225540928	2.185768706	2.185550228
18	2.339646852	2.2929589	2.292713618
19	2.459603111	2.407314042	2.407003038
20	2.585709659	2.53117821	2.530715999
21	2.718281828	2.658939597	2.65844557

Example 5.2. consider the following nonlinear SDEs:
$dy(t) = -e^{-4y_t}dt + e^{-2y_t}dw_t$
Reduce it to a linear SDEs and find the solution (analytically and numerically)

The analytic solution: By compare the above equation with eq. (7), we get

$$a = f\big(t, Y(t)\big) = f(y) = -e^{-4y} \quad ,$$

$$b = g\big(t, Y(t)\big) = g(y) = e^{-2y}$$

$$A = \frac{a}{b} - \frac{1}{2}\frac{d}{dy}(e^{-2y})$$

$$\frac{-e^{-4y}}{e^{-2y}} - \frac{1}{2}(-2e^{-2y}) = 0$$

$$-e^{-2y} + e^{-2y} = 0$$

$b_1 = 0$, then we can take

$$U = b_2 B(s) + k\,;\;\; b_2 = 1\,, k = 0\;\; ; B\,(s) = \int \frac{ds}{b(s)}$$

$Then\ U = \frac{1}{2}(e^{2y_0} - 1)\ \ , U_y = e^{2y}\,, U_{yy} = 2e^{2y}$, so from equation (10.1)

$$[-e^{-4y}](e^{2y})] + \frac{1}{2}(e^{-4y}).(2e^{2y}) = a_1\frac{1}{2}(e^{2y_0} - 1) + a_2$$

$$Then\ -2e^{-2y} + 2(e^{-2y}) = a_1 e^{y} + a_2$$

$a_1\frac{1}{2}(e^{2y_0} - 1) + a_2 = 0\ \ if\ and\ only\ if\ \ a_1 = a_2 = 0$,
from eq. (9), we get
$dx_t\ = dw_t$. Then solution is

$$x_t\ = x_0\ + w_t$$

Where $x_0\ = \frac{1}{2}(e^{2y_0} - 1)$
This implies

$$x_t\ = \frac{1}{2}(e^{2y_0} - 1)\quad + w_t$$

Since $x_t\ = U\ = \frac{1}{2}(e^{2y_t} - 1)$,
we get $y_t\ = \frac{1}{2}ln[2x_t\ + 1]$
Then the analytic solution is.

$$\boldsymbol{y_t\ = \frac{1}{2}ln[e^{2y_0} + 2w_t\ ]}$$

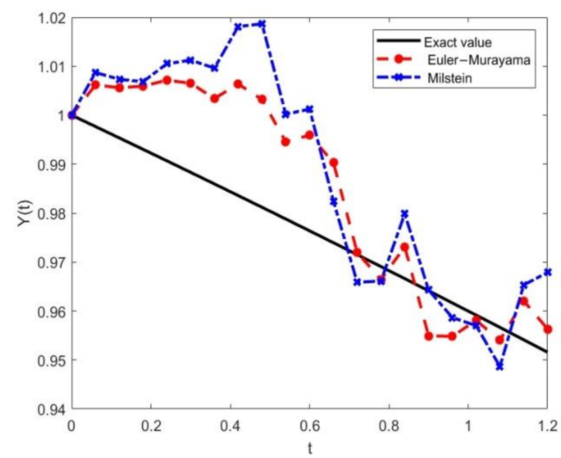
The numerical solution:
$since\ f_n = \boldsymbol{-e^{-4y_{t_n}}}$ **;** $g_n = \boldsymbol{e^{-2y_{t_n}}}$
Then Euler- Maruyama and Milstein' formulas take the form respectively:

$$\mathbf{y}_{t_{n+1}} = \mathbf{y}_{t_n} + (-e^{-4y_{t_n}})\Delta t + e^{-2y_{t_n}}\sqrt{\Delta t}\,\eta_j$$

$$\mathbf{y}_{t_{n+1}} = \mathbf{y}_{t_n} + (-e^{-4y_{t_n}})\Delta t + e^{-2y_{t_n}}\sqrt{\Delta t}\,\eta_j + \frac{1}{2}e^{-2y_{t_n}}\frac{\partial(e^{-2y_{t_n}})}{\partial y_{t_n}}(\eta_j{}^2 - 1)\Delta t$$

The following figure shows the comparison between numerical (Euler -Maruyama & Milstein) with the analytic(exact) solution.

Figure(2): comperison between the analytic and numerical solution.



Table(2): The values of the exact and numerical solutions.

No.	Exact	Euler-Maruyama	Milstein
1	1	1	1
2	0.997480835	1.00630818	1.00890818
3	0.99595704	1.006811734	1.007811734
4	0.993528587	1.0069751917	1.007775191
5	0.991895447	1.0073388	1.0123388
6	0.989357592	1.00699846	1.01199846
7	0.986814892	1.00491386	1.01291386
8	0.984967618	1.00798512	1.01898512
9	0.982715441	1.00495985	1.01985985
10	0.978915843	0.995937143	0.115937143
11	0.977315740	0.996905716	1.196905716
12	0.976596555	0.990979989	0.982979989
13	0.972029787	0.972349501	0.965949501
14	0.971458093	0.966630511	0.966330511
15	0.969881443	0.973519163	0.983519163
16	0.965299806	0.955392517	0.965392517
17	0.96171315	0.955292416	0.959562342
18	0.959521443	0.958562342	0.957252742
19	0.957221401	0.955252342	0.949185642
20	0.955454329	0.962856442	0.966228492
21	0.952332650	0.957985922	0.96894327

5. CONCLUSION AND FUTURE WORKS:

Through our study we applied the linearization method (linear -transform) for the nonlinear stochastic differential equation(SDES) by applying Ito formula. After we obtained a suitable function $X_t = U(t, Y_t)$ for the reducible linear stochastic differential equation , we find the analytic solution. lastly we compare the exact solution with the numerical solution (Euler- Maruyama and milestone) methods by several examples, in the numerical solution we see that the Milstein method better than Euler- Maruyama in convergence with the exact solution for the first example while in the square root example it seems that they are the same.

As a future studies one can study the linearization of some nonlinear(harmonic) stochastic differential equation by using stratonovich formula for their solution and compare it with Ito formula.

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الحل التحليلي لبعض المعادلات التفاضلية التصادفية غير الخطية باستخدام التحويل الخطي.

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تاريخ الاستلام:2023/1/ 8 تاريخ القبول:28/2/2023

الملخص

في هذا البحث تمت دراسة طريقة التحويلات والتي تسمى التحويلات الخطية لبعض المعادلات التفاضلية التصادفية غير الخطية إلى الخطية باستخدام صيغة ايتو التكاملية(Ito-integrated formula)، ومن ثم إيجاد الحل التحليلي(المضبوط) الخاص بها، وتم استخدام الطرق العددية (أويلر- ماريواما وطريقة ميلستاين) (Euler -Maruyama and Milstein)ل كي نجري مقارنة بين الحل الذي تم الحصول عليه للمعادلات التفاضلية التصادفية مع الحل التقريبي.

الكلمات المفتاحية: المعادلة التفاضلية التصادفية,صيغة ايتو التكاملية,التحويل الى الخطية, (طريقتا اويلر-ماريوما ومالستين)