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# Analytical Solution for Fluid Flow and Heat Transfer in a Three-Dimensional Inclined Horizontal Channel and Under The Influence of Thermal Radiation

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**Abstract**

In this paper, the analytical solution to the problem of heat transfer and fluid flow was obtained by using the quadruple Laplace transform method. Temperature distribution and fluid flow distribution were shown, temperature and fluid flow increase when the value of z increases, as well as the effect of the radiation parameter $R_d$ shown, it was concluded that the temperature increase with the increase in the value of the radiation coefficient $R_d$. Matlab was used to plot the results.

***Keywords***:
Radiation Heat Transfer, Quadruple Laplace Transform, Navier-Stokes Equations, Energy Equation, Cartesian Coordinate.

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## I. INTRODUCTION

Numerous engineering applications, including transpiration cooling, drag reduction, thrust bearing, and design of radial diffusers, benefit from the study of heat transport. Typically, fluids are utilized to convey heat in industrial and transportation systems for heating and cooling purposes. It is also noted that academics have been interested in the stretching sheet for a long time. The physical phenomena and heat transmissions across a stretching plate have been the subject of several studies by researchers. Numerous significant industrial production processes use it, such as the manufacturing of glass filters, the extrusion of plastic sheets, and the condensation of metallic plates. The quality of the finished product depends heavily on the skin friction coefficient and the rate of surface heat transfer, thus the research of flow and heat transfer is crucial. The study of flow over a stretching sheet has recently been expanded to include many diverse scenarios, making it more intriguing [12]. Anurag, J. P. Maurya and A. K. Singh utilized finite Hankel transform to solve one-dimensional convection heat transfer problem containing the magnetic field [1]. Aziz-Ur-Rehman, Muhammad Bilal Riaz, Syed Tauseef Saeed and Shaowen Yao applied Laplace transformation to solve the magnetohydrodynamic problem [2]. E. N. Macêdo, R. M. Cotta and H. R. B. Orlande also utilized generalized integral transform technique to get the solution of convection and radiation problem [3]. F. A. A. Gomes, J. B. C. Silva and A. J. Diniz also used generalized integral transform technique to solve radiation heat transfer problem [4]. Gilvan do Nascimento Filho, Jakler Nichele and Leonardo Santos de Brito Alves used classical integral transform technique to solve one-dimensional time dependent heat conduction problem in no-ablation time period [5].Laplace transform technique implemented by Muhammad Iftikhar, Zubair Ahmad, Saqib Murtaza, Ibn e Ali and Ilyas Khan to solve radiation heat transfer problem that was formed by Caputo-Fabrizio fractional operator [9]. N.T. Eldabe, M. El-Shahed and M. Shawkey used the Laplace transform and generalized finite Hankel transform to solve the equation of unsteady flow through a concentric annulus [13]. In this paper we will solve three-dimensional radiation heat transfer problem in Cartesian coordinate by using quadruple Laplace transform.

**II. The Model and Mathematical Method:**

The governing equations and illustration of problem are [14][8][10][ 11][15]:

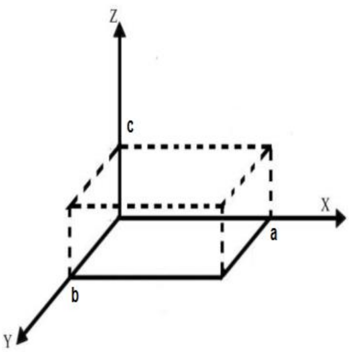


**Figure (1-1)** shows model of heat transfer and fluid flow

1- Continuity equation:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (1)$$

2- Navier-Stokes equations:

(x-direction)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=\frac{-\partial P}{\rho\,\partial x}+\frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\beta g\sin(\alpha)\,(T-T_o) \quad (2)$$

(y-direction)

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=\frac{-\partial P}{\rho\,\partial y}+\frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)-\beta g\cos(\alpha)\,(T-T_o) \quad (3)$$

(z-direction)

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=\frac{-\partial P}{\rho\,\partial z}+\frac{\mu}{\rho}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \quad (4)$$

3- Energy equation:

$$\left[\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right]=\frac{k}{\rho c_p}\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]+\frac{4\,k_f}{3\rho\,c_p}R_d\left[\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right]+q'''+\phi \quad (5)$$

With boundary and initial conditions:

$T(0,y,z,t)=0 \quad , \quad T(a,y,z,t)=c_1$

$T(x,0,z,t)=0 \quad , \quad T(x,b,z,t)=c_2$

$T(x,y,0,t)=0 \quad , \quad T(x,y,c,t)=c_3$

$T(x,y,z,0)=xyz$

$c_1\,,c_2\,,c_3\,:\,constants$

$q'''$ : *Internal Heat Generation Rate*

$\phi$ : *Dissipation Function*

$\rho$ : *Density of Fluid*

$k$ : *Thermal Conductivity of Fluid*

$c_p$ : *Specific Heat of Fluid*

$u, v, w$ : *Velocity Components*

$\mu$ : *Viscosity of Fluid*

$P$ : *Pressure*

$R_d$ : *Radiation Parameter*

$k_f$ : *Thermal Conductivity of Base Fluid*

$\alpha$: *Incline Angle*

$g$: *Gravitational Acceleration*

$\beta$ : *Thermal Expansion Coefficient*

$T$ : *Absolute Temperature*

Quadruple Laplace transform methods is defined as [6]:

$$L_{xyzt}f(x,y,z,t) = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty e^{-px}\, e^{-qy}\, e^{-rz}\, e^{-st}\, f(x,y,z,t)dx\, dy\, dz\, dt$$

*where* $p, q, r, s$ : *Parameters.*

firstly we will solve energy equation by quadruple Laplace transform as follows:

$$\left[\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z}\right] = \frac{k}{\rho c_p}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \frac{4\,k_f}{3\rho\, c_p}R_d\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + q''' + \phi$$

Now apply quadruple Laplace transform to energy equation(6):

$$\left[L_{xyzt}\left[\frac{\partial T}{\partial t}\right] + L_{xyzt}\left[u\frac{\partial T}{\partial x}\right] + L_{xyzt}\left[v\frac{\partial T}{\partial y}\right] + L_{xyzt}\left[w\frac{\partial T}{\partial z}\right]\right] = \frac{k}{\rho c_p}\left[L_{xyzt}\left[\frac{\partial^2 T}{\partial x^2}\right] + L_{xyzt}\left[\frac{\partial^2 T}{\partial y^2}\right] + L_{xyzt}\left[\frac{\partial^2 T}{\partial z^2}\right]\right] + \frac{4\,k_f}{3\rho\, c_p}R_d\left[L_{xyzt}\left[\frac{\partial^2 T}{\partial x^2}\right] + L_{xyzt}\left[\frac{\partial^2 T}{\partial y^2}\right] + L_{xyzt}\left[\frac{\partial^2 T}{\partial z^2}\right]\right] + L_{xyzt}[q'''] + L_{xyzt}[\phi]$$

Let $w_1 = \frac{\rho\, c_p}{k}$ and apply boundary and initial conditions:

$$\left(s + up + vq + rw - \frac{1}{w_1}(p^2+q^2+r^2)\right)L_{xyzt}\,T(x,y,z,t) - \frac{4\,k_f}{3\,\rho\, c_p}R_d\left[(p^2+q^2+r^2)L_{xyzt}\,T(x,y,z,t)\right] = \frac{1}{p^2\,q^2\,r^2} + \frac{1}{\rho\, c_p}\left(\frac{q'''+\phi}{pqrs}\right)$$

After simplicity:

$$L_{xyzt}\,T(x,y,z,t) = \frac{1}{p^2\,q^2\,r^2 s^2\left[\left(\frac{s+up+vq+rw}{s^2} - \frac{(p^2+q^2+r^2)}{w_1 s^2}\right) - \frac{4\,k_f}{3\,\rho\, c_p}R_d\,\frac{[(p^2+q^2+r^2)]}{s^2}\right]} + \frac{q'''+\phi}{pqrs\,\rho\, c_p\left[\left(s+up+vq+rw-\frac{1}{w_1}(p^2+q^2+r^2)\right) - \frac{4\,k_f}{3\,\rho\, c_p}R_d\,[(p^2+q^2+r^2)]\right]}$$

Take $L^{-1}_{xyzt}$ :

$$T(x,y,z,t) = \frac{xyzt}{\left[\left(\frac{s+up+vq+rw}{s^2} - \frac{1}{w_1}\frac{(p^2+q^2+r^2)}{s^2}\right) - \frac{4\,k_f}{3\,\rho\, c_p}R_d\,\frac{[(p^2+q^2+r^2)]}{s^2}\right]} + \frac{q'''+\phi}{\rho\, c_p\left[\left(s+up+vq+rw-\frac{1}{w_1}(p^2+q^2+r^2)\right) - \frac{4\,k_f}{3\,\rho\, c_p}R_d\,[(p^2+q^2+r^2)]\right]}$$

Now after combine navier-stokes equations(2),(3)and(4) and simplicity we get[7]:

(6)

$$\frac{\partial^2\zeta}{\partial x\partial t} - \frac{\partial^2\zeta}{\partial y\partial t} = \frac{-\mu}{\rho}\left[3\frac{\partial\zeta}{\partial x} - 3\frac{\partial\zeta}{\partial y}\right] - \frac{1}{3}\beta g\frac{\partial}{\partial z}\left[\sin(\alpha)\frac{\partial T}{\partial y} - \cos(\alpha)\frac{\partial T}{\partial x}\right] \qquad (9)$$

With boundary and initial conditions:

$\zeta(0,y,z,t) = 0 \quad , \quad \zeta(a,y,z,t) = 0$

$\zeta(x,0,z,t) = 0 \quad , \quad \zeta(x,b,z,t) = 0$

$\zeta(x,y,0,t) = 0 \quad , \quad \zeta(x,y,c,t) = 0$ (7)

$\zeta(x,y,z,0) = V_o$

$\zeta$ : *Fluid Flow*

$V_o$ : *constant*

Apply quadruple Laplace transform to equation(9):

$$p\, s\, L_{xyzt}[\zeta(x,y,z,t)] - pL_{xyz}[\zeta(x,y,z,0)] - q\, sL_{xyzt}[\zeta(x,y,z,t)] + qL_{xyz}[\zeta(x,y,z,0)] = \frac{-3\mu}{\rho}\, p\, L_{xyzt}[\zeta(x,y,z,t)] + \frac{3\mu}{\rho}\, q\ L_{xyzt}[\zeta(x,y,z,t)] - \frac{1}{3}\, \beta g \sin(\alpha) L_{xyzt}[\frac{\partial^2 T}{\partial y\, \partial z}] + \frac{1}{3}\, \beta g \cos(\alpha) L_{xyzt}[\frac{\partial^2 T}{\partial x\, \partial z}]$$

Then we get:

$$L_{xyzt}[\zeta(x,y,z,t)] = \frac{V_o}{s+\frac{3\mu}{\rho}} + \frac{\frac{1}{3}\beta g \cos(\alpha)\,(r\,p) - \frac{1}{3}\beta g \sin(\alpha)\,(r\,q)}{(p-q)(s+\frac{3\mu}{\rho})} L_{xyzt}[T(x,y,z,t)]$$

Take $L_{xyzt}^{-1}$ :

$$\zeta(x,y,z,t) = \frac{pqrs}{s+\frac{3\mu}{\rho}} V_o + \frac{\frac{1}{3}\beta g \cos(\alpha)\,(r\,p) - \frac{1}{3}\beta g \sin(\alpha)\,(r\,q)}{(p-q)(s+\frac{3\mu}{\rho})}\ T(x,y,z,t)$$

From energy equation solution(8):

$$T = T(x,y,z,t) = \frac{xyzt}{\left[\left(\frac{\mathrm{s+up+vq+rw}}{s^2} - \frac{1}{\mathrm{w_1}}\frac{(\mathrm{p^2+q^2+r^2})}{s^2}\right) - \frac{4\,\mathrm{k_f}}{3\,\rho\,\mathrm{c_p}}\,\mathrm{R_d}\,\frac{[(\mathrm{p^2+q^2+r^2})]}{s^2}\right]} + \frac{q''' + \phi}{\rho\,\mathrm{c_p}\left[\left(\mathrm{s+up+vq+rw} - \frac{1}{\mathrm{w_1}}(\mathrm{p^2+q^2+r^2})\right) - \frac{4\,\mathrm{k_f}}{3\,\rho\,\mathrm{c_p}}\,\mathrm{R_d}\,[(\mathrm{p^2+q^2+r^2})]\right]}$$

Then:

$$\zeta(x,y,z,t) = \frac{pqrs}{s+\frac{3\mu}{\rho}} V_o + \frac{\frac{1}{3}\beta g \cos(\alpha)\,(r\,p) - \frac{1}{3}\beta g \sin(\alpha)\,(r\,q)}{(p-q)\left(s+\frac{3\mu}{\rho}\right)}$$

$$\left[\frac{xyzt}{\left[\left(\frac{\mathrm{s+up+vq+rw}}{s^2} - \frac{1}{\mathrm{w_1}}\frac{(\mathrm{p^2+q^2+r^2})}{s^2}\right) - \frac{4\,\mathrm{k_f}}{3\,\rho\,\mathrm{c_p}}\,\mathrm{R_d}\,\frac{[(\mathrm{p^2+q^2+r^2})]}{s^2}\right]} + \frac{q''' + \phi}{\rho\,\mathrm{c_p}\left[\left(\mathrm{s+up+vq+rw} - \frac{1}{\mathrm{w_1}}(\mathrm{p^2+q^2+r^2})\right) - \frac{4\,\mathrm{k_f}}{3\,\rho\,\mathrm{c_p}}\,\mathrm{R_d}\,[(\mathrm{p^2+q^2+r^2})]\right]}\right]$$

## III. Results:

By use Matlab we get results represented by the following figures:

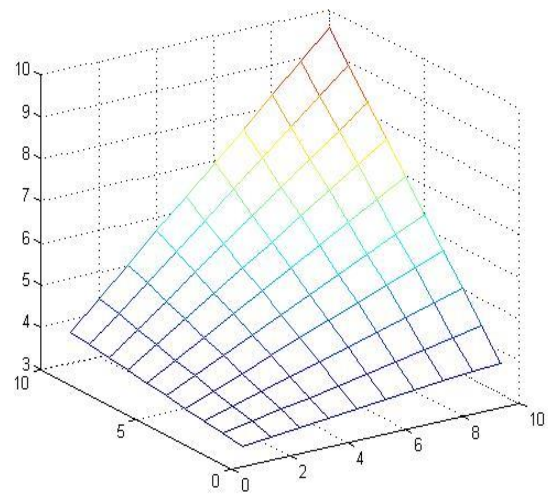


**Figure (1-2)** shows distribution of temperature for $T(x,y,z,t)$ , $x = 1:10$ , $y = 1:10$ , $z = 1$ , $t = 1$.

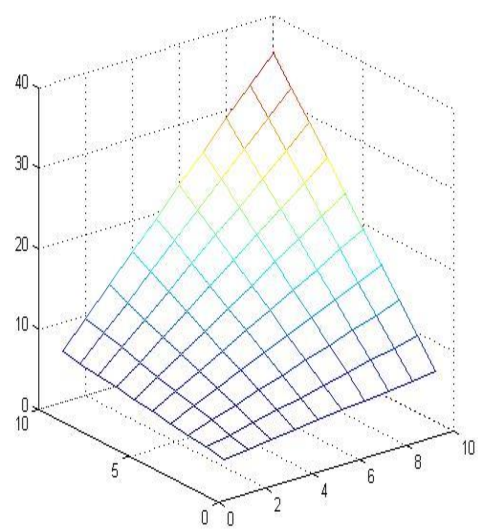


**Figure (1-3)** shows distribution of temperature for $(x,y,z,t)$ , $x = 1:10$ , $y = 1:10$ , $z = 5$ , $t = 1$.

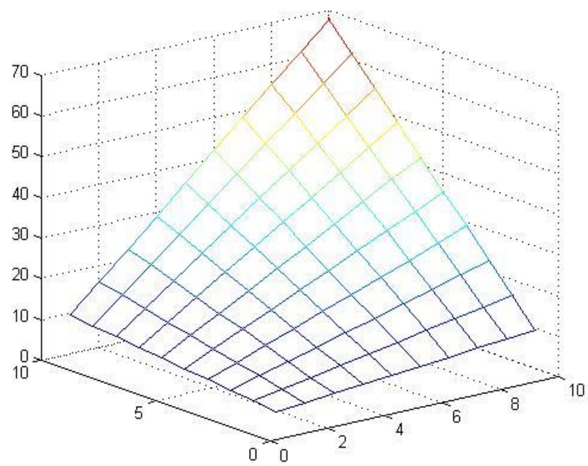


**Figure (1-4)** shows distribution of temperature for $T(x, y, z, t)$ , $x = 1:10$ , $y = 1:10$ , $z = 10$ , $t = 1$.

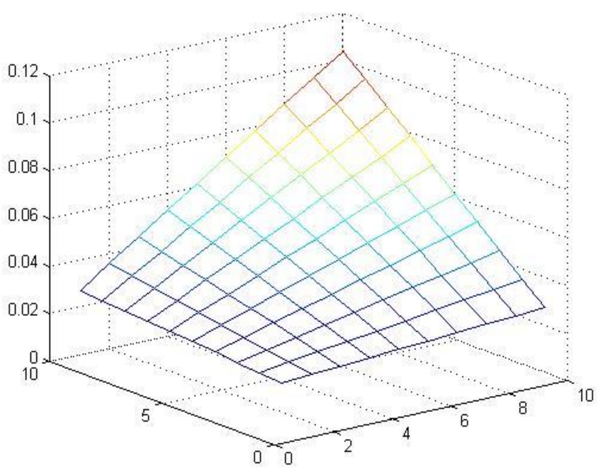


**Figure (1-5)** shows distribution of fluid flow for $(x, y, z, t)$ , $x = 1:10$ , $y = 1:10$ , $z = 1$ , $t = 1$ , $\alpha = 0$.

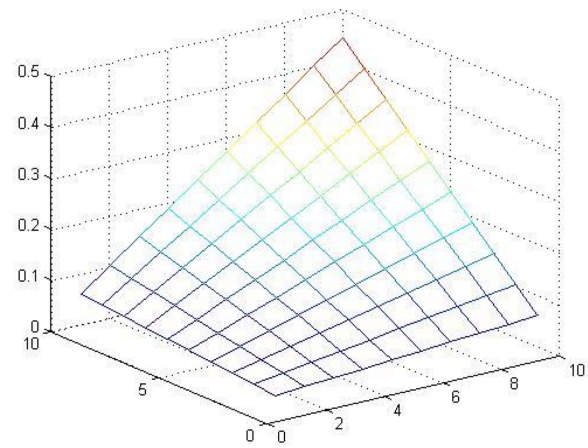


**Figure (1-6)** shows distribution of fluid flow for $\zeta(x, y, z, t)$ , $x = 1:10$ , $y = 1:10$ , $z = 5$ , $t = 1$ $\alpha = 0$.

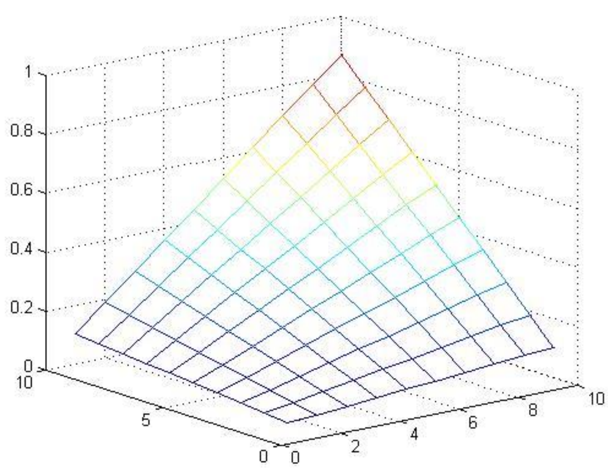


**Figure (1-7)** shows distribution of fluid flow for $(x, y, z, t)$ , $x = 1:10$ , $y = 1:10$ , $z = 10$ , $t = 1$ $\alpha = 0$.

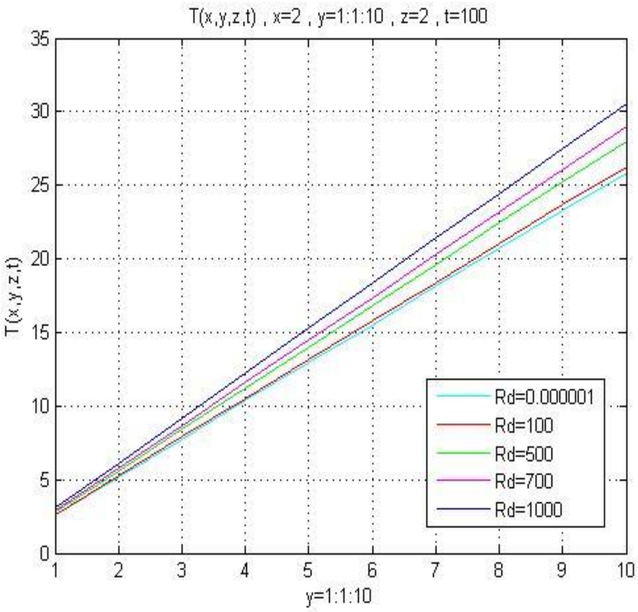


**Figure (1-8)** shows radiation parameter $R_d$ effect for $T(x, y, z, t)$ , $x = 2$ , $y = 1:1:10$ , $z = 2$ , $t = 100$ $for$ $R_d = 0.000001$ , 100,500,700,1000.

# IIII. Conclusions:

We notice the temperature distribution is gradual and in the form of a plate and as the value of z increases the value of the temperature increases, that is clear from the figures (1-2),(1-3) and (1-4) while $z = 10$ temperature is maximum, when $z = 5$ temperature is decrease and at $z = 1$ temperature is minimum. It was also shown that the fluid flow is gradually and in the form of a plate and as the value of z increases the value of the fluid flow increases with incline angle $\alpha = 0$, that is clear from the figures (1-5),(1-6)and(1-7) while $z = 10$ fluid flow is maximum, when $z = 5$ fluid flow is decrease and at $z = 1$ fluid flow is minimum, That means the increases in fluid flow is related to the increases in temperature and vice versa. Figure (1-8) shows effect of radiation parameter $R_d$ which show that, when value of $R_d$ increase then temperature increase gradually.

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### References

[1] Anurag, J. P. Maurya and A. K. Singh,(2021), Significance of time-dependent magnetohydrodynamic transient free convective flow in vertical annuli: An analytical approach with the finite Hankel transform, Heat Transfer Journal ,Vol.50,Issue 7,pp. 6719-6736.

[2] Aziz-Ur-Rehman, Muhammad Bilal Riaz, Syed Tauseef Saeed and Shaowen Yao,(2021), Dynamical Analysis of Radiation and Heat Transfer on MHD Second Grade Fluid, Computer Modeling in Engineering & Sciences Journal,Vol. 129,No. 2,pp. 689-703.

[3] E. N. Macêdo, R. M. Cotta and H. R. B. Orlande,(2000), A solution via generalised intergral transform technique for the simultaneous transport processes during combustion of wood cylinders, INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN ENGINEERING ,Vol.49,Issue 11,pp. 1455-1477.

[4] F. A. A. Gomes, J. B. C. Silva and A. J. Diniz,(2005), RADIATION HEAT TRANSFER WITH ABLATION IN A FINITE PLATE, Journal of Thermal Engineering ,Vol.4,No. 2,pp. 190-196.

[5] Gilvan do Nascimento Filho, Jakler Nichele and Leonardo Santos de Brito Alves ,(2020), A Detailed Approach for the Classical Integral Transform Technique in Ablation Phenomenon with Moving Boundaries, 18th Brazilian Congress of Thermal Sciences and Engineering, Brazilian Association of Engineering and Mechanical Sciences(ABCM), ENC-2020-0350.

[6] Hamood Ur Rehman, Muzammal Iftikhar, Shoaib Saleem, Muhammad Younis ,Abdul Mueed ,(2014), A Computational Quadruple Laplace Transform for the Solution of Partial Differential Equations ,Applied Mathematics,Vol.5,No.21.

[7] John C. Tannehill, Dale A. Anderson and Richard H. Pletcher ,(2013), COMPUTATIONAL FLUID MECHANICS AND HEAT TRANSFER, Second Edition , Taylor & Francis copyright .

[8] Latif M.Jiji ,(2009), Heat Conduction , Third Edition , Springer-Verlag Berlin Heidelberg.

[9] Muhammad Iftikhar, Zubair Ahmad, Saqib Murtaza, Ibn e Ali and Ilyas Khan,(2020),Caputo-Fabrizio Fractional Model of Electro-Osmotic Flow of Walters'-B Fluid in the Presence of Diffusion-Thermo: Exact Solution via Integral Transform, International Journal of Computational Analysis,Vol. 4,No. 2,pp. 01-18.

[10] Nurul Amira Zainal , Roslinda Nazar , Kohilavani Naganthran and Ioan Pop ,(2022), The Impact of Thermal Radiation on Maxwell Hybrid Nanofluids in the Stagnation Region, Mathematics Journal – MDPI,12(7),1109.

[11] Natalia C. Rosca, Alin V. Rosca and Ioan Pop,(2020), Axisymmetric flow of hybrid nanofluid due to a permeable non-linearly stretching/shrinking sheet with radiation effect, International Journal of Numerical Methods for Heat & Fluid Flow ,Vol.31,No.7,pp. 2330-2346.

[12] Noor Farizza Haniem Mohd Sohut , Siti Khuzaimah Soid , Sakhinah Abu Bakar and Anuar Ishak ,(2022), Unsteady Three-Dimensional Flow in a Rotating Hybrid Nanofluid over a Stretching Sheet, Mathematics Journal – MDPI,10(3),348.

[13] N.T. Eldabe, M. El-Shahed and M. Shawkey,(2004), An extension of the finite Hankel transform, Applied Mathematics and Computation Journal,Vol.151,Issue 3,pp. 713-717.

[14] Ronald L. Panton,(2013), Incompressible Flow, Fourth Edition, John Wiley & Sons, Inc.

[15] William F. Hughes and John A. Brighton,(1999),FLUID DYNAMICS, Third Edition, Schaum's Outline, McGraw-Hill Companies, Inc .

# الحل التحليلي لجريان مائع وانتقال الحرارة في قناة افقية مائلة بثلاثة ابعاد وتحت تأثير الاشعاع الحراري

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**الملخص**

في هذا البحث, تم الحصول على الحل التحليلي لمسالة انتقال الحرارة و جريان المائع باستخدام طريقة لابلاس الرباعي. تم اظهار توزيع درجات الحراة و توزيع جريان المائع, درجة الحرارة و جريان المائع تزداد عندما تزداد قيمة z, كما تم اظهار تأثير معلمة الاشعاع $R_d$, تم استنتاج ان درجة حرارة تزداد مع ازدياد في قيمة معلمة الاشعاع $R_d$. تم استخدام Matlab في رسم النتائج.

**الكلمات المفتاحية:** انتقال الحرارة بالإشعاع, تحويل لابلاس الرباعي, معادلات نافيير- ستوكس, معادلة الحرارة, الاحداثيات الكارتيزية.