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Restricted Detour Polynomial of a Straight Chain of Wheel Graphs

Herish Omer Abdullah[1] and Ivan Dler Ali [2]

Salahaddin University-Erbil [1] , Salahaddin University-Erbil [2]
**Corresponding author. Email: herish.abdullah@su.edu.krd [1]*

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Abstract

Restricted detour distance between two vertices u and v of a graph G is the length of a longest $u-v$ path P for the induced condition $\langle V(P)\rangle = P$. The restricted detour polynomial depends on restricted detour distance and is denoted by $D^*(G,x)$ and defined by $D^*(G,x) = \sum_{\{u,v\}} x^{D^*(u,v)}$ where the summation is taken over all unordered pairs (u,v) of distinct vertices of u and v of $V(G)$. In this paper, we investigate the restricted detour polynomial of the straight chain of wheel graphs $\varphi(W_k^{\beta})$ and compute the restricted detour index of $\varphi(W_k^{\beta})$.

Keywords:
Restricted Detour Distance, Restricted Detour Polynomial, Straight chain, Wheel graph.

Correspondence:
Author : Herish Omer Abdullah
Email: herish.abdullah@su.edu.krd

1. INTRODUCTION

In this research paper, we consider finite simple connected graphs. For undefined concepts and notations on the theory of graphs, we refer the reader to [5].

Topological indices are graph invariants that play important roles for studying and analyzing the physicochemical properties of molecules. Some of the most worthy types of topological indices of graphs are distance-based topological indices, degree-based topological indices, and spectrum-based topological indices. The Wiener index is the first studied topological index. For more details on the concept of topological indices of graphs, we refer the reader to [1, 4, 6, 7, 8, 11, 12].

The concept of restricted detour distance was first proposed in 1993 by Chartrand, Johns and Tian [6]. For standard graph theory and acquired the restricted detour polynomial and restricted detour index of certain graphs see [2, 3, 9, 10]. Specifically, let u and v be two distinct vertices in a connected graph G. The (standard) distance $d_G(u,v)$ from a vertex u to a vertex v in a graph G is the smallest length of a $u-v$ path in G[5]. An induced $u-v$ path of length $D^*(u,v)$ is called a restricted detour path. The restricted detour distance between two vertices u and v of a graph G is the length of a longest $u-v$ path P for the induced condition $\langle V(P)\rangle = P$ and indicated $D^*(u,v)$. The restricted detour polynomial depends on restricted detour distance and is denoted by $D^*(G,x)$ and defined by $D^*(G,x) = \sum_{\{u,v\}} x^{D^*(u,v)}$, where the summation is taken over all unordered pairs (u,v) of a distinct vertices u and v of G. The index is also based on the restricted detour distance and denoted by $dd^*(G)$ and is defined by $dd^*(G) = \sum_{\{u,v\}} D_G^*(u,v)$, where the summation is taken over all unordered pairs u,v of vertices of G[10].

Also, $dd^*(G) = \frac{d}{dx} D^*(G;x)|_{x=1}$.

A wheel graph W_n of order n, is a graph that contains a cycle C_{n-1} of order $n-1$, and for which every vertex in the cycle is connected to one other vertex which is known as the hub (or the center).

In 2012, Ali and Gashaw [1] obtained the restricted detour polynomials of chain a hexagonal ladder graph. In 2017, Ali, I. D. and Herish computed the restricted detour polynomial of edge-Identification of two wheel graphs[3]. Ali, I. D, obtained restricted detour polynomial of some cycle related graphs[2]. In this study, we obtained the restricted detour polynomial and restricted detour index of chain k-wheels consisting of one row

of β wheels.

2. Identified edges of a Straight chain of wheels

Let G_1 and G_2 be two disjoint graphs, let $e_1 = u_1v_1 \in E(G_1)$ and $e_2 = u_2v_2 \in E(G_2)$, an edge identification of G_1 and G_2 is denoted by $G_1:G_2$ obtained from identifying e_1 with e_2 where u_1 identifying with u_2 and $v_1$with v_2 to get the new edge e [3].

Now, a **straight chain of wheels** indicated by $\varphi(W_k^{\beta})$, $\beta \geq 3$, is a graph established on one row of β copies of wheels of orders $k \geq 6$ such that every two consecutive wheels have exactly one common cycle-edge proportion forming a straight chain as illustrated in the Figures 1 and 2 for even and odd k respectively. The hub vertices of each copy of the k-wheels are denoted by $w_1, w_2, \ldots, w_\beta$.

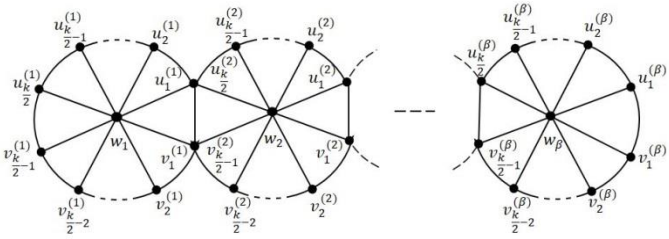

Figure 1: The graph $\varphi(w_k^{\beta})$ for even k.

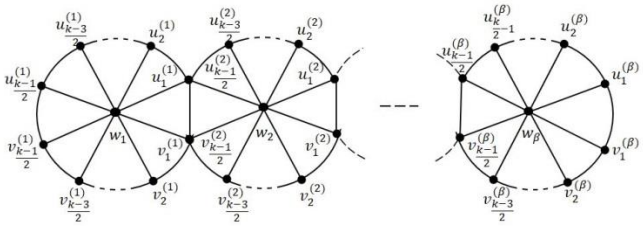

Figure 2: The graph $\varphi(w_k^{\beta})$ for odd k.

The order of the straight chain of wheels $\varphi(W_k^{\beta})$ is provided by $p(\varphi(W_k^{\beta})) = (k-2)\beta + 2$; while the size of $\varphi(W_k^{\beta})$ is provided by $q(\varphi(W_k^{\beta})) = (2k-3)\beta + 1$.

Proposition 2.1.[10] For $k \geq 4$, we have

$$D^*(\varphi(w_k^1); x) = D^*(w_k; x) =$$

$$\begin{cases} k + 2(k-1)x + (k-1)\left[\frac{1}{2}x^{\frac{k-1}{2}} + \sum_{i=\frac{k+1}{2}}^{k-3} x^i\right], & \text{when } k \text{ is odd,} \\ k + 2(k-1)x + (k-1)[\sum_{i=\frac{k}{2}}^{k-3} x^i], & \text{when } k \text{ is even.} \end{cases} \blacksquare$$

Theorem 2.2.[3] For odd $k \geq 7$, we have

$$D^*(\varphi(W_k^2); x) = 2D^*(W_k; x) - x - 2$$
$$+x^2 + 2\sum_{i=2}^{\frac{k-1}{2}}\sum_{j=2}^{\frac{k-1}{2}} x^{2k-2-(i+j)} + 4\sum_{j=2}^{\frac{k-1}{2}} x^{k-j}$$
$$+2\sum_{i=2}^{\frac{k-1}{2}} x^{k-4+i} + 2\sum_{i=3}^{\frac{k-1}{2}}\sum_{j=3}^{\frac{k-1}{2}} x^{2k-1-(i+j)} +$$
$$2\sum_{j=3}^{\frac{k-1}{2}} x^{k-4+j}. \blacksquare$$

Theorem 2.3.[3] For even $k \geq 6$, we have

$$D^*(\varphi(W_k^2); x) = 2D^*(W_k; x) - x - 2 + x^2 + x^{k-1}$$
$$-2x^{k-2} + 2x^{\frac{k}{2}} + 4x^{\frac{3k-8}{2}} + 4\sum_{i=3}^{\frac{k-2}{2}} x^{\frac{3k-2}{2}-i}$$
$$+2\sum_{i=3}^{\frac{k-2}{2}}\sum_{j=3}^{\frac{k-2}{2}} x^{2k-1-(i+j)} + 2\sum_{i=2}^{\frac{k-2}{2}}\sum_{j=2}^{\frac{k-2}{2}} x^{2k-2-(i+j)}$$
$$+4\sum_{i=2}^{\frac{k-2}{2}} x^{k-i} + 4\sum_{j=2}^{\frac{k-2}{2}} x^{k-4+j}. \blacksquare$$

Let us indicate the *j-th* copy of W_k in $\varphi(W_k^{\beta})$ by $W_{k,j}$, for $j = 1, \ldots, \beta$. Then the two consecutive wheels $W_{k,1}$ and $W_{k,2}$ will have the same proportion of the common edge $e_{1=}(u_1^{(1)}, v_1^{(1)}) = (u_1^{(2)}, v_1^{(2)})$. Now, for $j = 2, \ldots, \beta - 1$; and if k is even, then the two consecutive wheels $W_{k,j}$ and $W_{k,j+1}$ will share the common edge $e_{i=}(u_{\frac{k}{2}}^{(j)}, v_{\frac{k}{2}}^{(j)}) = (u_1^{(j+1)}, v_1^{(j+1)})$ as illustrated in Figure1; and if k is odd then the two consecutive wheels $W_{k,j}$ and $W_{k,j+1}$ will have the same proportion of the common edge $e_{i=}(u_{\frac{k+1}{2}}^{(j)}, v_{\frac{k-1}{2}}^{(j)}) = (u_1^{(j+1)}, v_1^{(j+1)})$ as demonstrated in Figure2.

We mention Figures1 and 2 and indicate $M_1 = V(W_{k,1}) - \{u_1^{(1)}, v_1^{(1)}\}$ and $M_2 = V(W_{k,\beta}) - \{u_1^{(\beta)}, v_1^{(\beta)}\}$. Let u and v be any two vertices of $V(\varphi(W_k^{\beta}))$. Hence, for all possibilities of $\in V(\varphi(W_k^{\beta}) - M_1$ {or $u, v \in V(\varphi(W_k^{\beta}) - M_2\}$, we get the corresponding polynomial $D^*(\varphi(W_k^{\beta-1}; x))$; and for all possibilities of $u, v \in V(\varphi(W_k^{\beta}) - \{M_1 \cup M_2\}$, we obtain the corresponding polynomial $D^*(\varphi(W_k^{\beta-2}); x)$.

Consequently, we reach the following reduction formula

$$D^*(\varphi(W_k^{\beta}); x) = 2D^*(\varphi(W_k^{\beta-1}); x) - D^*(\varphi(W_k^{\beta-2}); x) + \rho_{k,\beta}(x),$$

in which $\rho_{k,\beta}(x)$ is the polynomial corresponding to all possibilities of u, v for which $u \in M_1$ and $v \in M_2$.

Now, after some calculation having carried out, we readily

$D^*(\varphi(W_k^{\beta}); x) = (\beta-1)D^*(\varphi(W_k^2); x)$

$-(\beta-2)D^*(\varphi(W_k^1); x) + \sum_{m=3}^{\beta}(\beta-m+1)\rho_{k,m}(x).$

3. Restricted Detour Polynomial of Straight Chain of Wheels $\varphi(W_k^{\beta})$, for $\beta = 3$ and $\beta = 4$

In this section, we will discover the restricted detour polynomials and the restricted detour indices of straight chains of wheels $\varphi(W_k^{\beta})$ for $\beta = 3$ and $\beta = 4$.

The restricted detour polynomial of $\varphi(W_k^{\beta})$ for $\beta = 3$ is given in the next two propositions.

Proposition 3.1. For odd $k \geq 7$, the restricted detour polynomial of $\varphi(W_k^3)$ is given by

$D^*(\varphi(W_k^3); x) = 2D^*(\varphi(W_k^2); x) - D^*(\varphi(W_k^1); x) +$

$\rho_{k,3}(x)$, in which

$$\rho_{k,3}(x) = 4\sum_{j=3}^{\frac{k-1}{2}} x^{\frac{1}{2}(3k-1)-j} + x^{\frac{1}{2}(k+1)} + 2x^{2k-4} + 4x^{\frac{1}{2}(3k-7)}$$

$$+2x^{\frac{5}{2}[k-3]}[1+(1+x)(2\sum_{i=3}^{\frac{k-1}{2}} x^{2-i} + \sum_{i=3}^{\frac{k-1}{2}}\sum_{j=3}^{\frac{k-1}{2}} x^{5-(i+j)})].$$

Proof. We refer to Figure2, with $\beta = 3$, and denote

$U = \{u_2^{(1)}, u_3^{(1)}, \dots, u_{\frac{k-1}{2}}^{(1)}\}$, $\acute{U} = \{v_2^{(1)}, v_3^{(1)}, \dots, v_{\frac{k-1}{2}}^{(1)}\}$,

$V = \{u_2^{(3)}, u_3^{(3)}, \dots, u_{\frac{k-1}{2}}^{(3)}\}$ and $\acute{V} = \{v_2^{(3)}, v_3^{(3)}, \dots, v_{\frac{k-1}{2}}^{(3)}\}$.

Let w_1, w_2 and w_3 are the hub vertices of $W_{k,1}, W_{k,2}$ and $W_{k,3}$ respectively. Let u and v be any two distinct vertices of $\varphi(W_k^3)$ with $u \in U \cup \acute{U} \cup \{w_1\}$ and $v \in V \cup \acute{V} \cup \{w_3\}$.

Evidently, from the proof of Theorem 3.4.1 in [2, page71]; the restricted detour polynomial of all vertices on the cycles except for $(u = u_2^{(1)}, v = v_2^{(3)})$ and $(u = v_2^{(1)}, v = u_2^{(3)})$ is given by

$\rho_1(x) =$

$$2x^{\frac{5}{2}[k-3]}[1+(1+x)(2\sum_{i=3}^{\frac{k-1}{2}} x^{2-i} + \sum_{i=3}^{\frac{k-1}{2}}\sum_{j=3}^{\frac{k-1}{2}} x^{5-(i+j)})].$$

Now, it remains to compute the restricted detour polynomial of the other vertices in the graph $\varphi(w_k^3)$. To do so, we consider the following cases

(1) If $u = u_2^{(1)}$ and $v = v_2^{(3)}$, then the restricted detour path P_1 between u and v;

$P_1 = u_2^{(1)}, u_3^{(1)}, u_4^{(1)}, \dots, u_{\frac{k-1}{2}}^{(1)}, v_{\frac{k-1}{2}}^{(1)}, \dots$

$\dots, v_3^{(1)}, v_2^{(1)}, v_1^{(2)}, w^{(2)}, u_1^{(3)}, u_2^{(3)}, \dots, u_{\frac{k-1}{2}}^{(3)}, v_{\frac{k-1}{2}}^{(3)}, \dots, v_3^{(3)}, v_2^{(3)}$ is a longest $u-v$ path of length $2(k-2)$. If $u = v_2^{(1)}$ and $v = u_2^{(3)}$, then the path $\acute{P}_1$ between u and v is

$\acute{P}_1 = v_2^{(1)}, v_3^{(1)}, v_4^{(1)}, \dots, v_{\frac{k-1}{2}}^{(1)}, u_{\frac{k-1}{2}}^{(1)}, \dots, u_3^{(1)}, u_2^{(1)}, u_1^{(2)}, w^{(2)},$

$v_1^{(3)}, v_2^{(3)}, \dots, v_{\frac{k-1}{2}}^{(3)}, u_{\frac{k-1}{2}}^{(3)}, \dots, u_3^{(3)}, u_2^{(3)}$ is a longest $u-v$ path of length $2(k-2)$. Then the polynomial in this case is given by $\rho_2(x) = 2x^{2(k-2)}$.

(2) If $u = w_1$ and $v = u_2^{(3)}$ (or $u = u_2^{(1)}$ and $v = w_3$), then the path P_2 between u and v is

$P_2 = v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k-1}{2}-1}^{(2)},$

$v_1^{(3)}, v_2^{(3)}, v_3^{(3)}, \dots, v_{\frac{k-1}{2}}^{(3)}, u_{\frac{k-1}{2}}^{(3)}, \dots, u_3^{(3)}, u_2^{(3)}$ is a longest $u-v$ path of length $\frac{1}{2}(3k-7)$. If $u = w_1$ and $v = v_2^{(3)}$(or $u = v_2^{(1)}, v = w_3$), then the path $\acute{P}_2$ between u and v is

$\acute{P}_2 =$

$u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, u_2^{(3)}, u_3^{(3)}, \dots, u_{\frac{k-1}{2}}^{(3)}, v_{\frac{k-1}{2}}^{(3)}, \dots, v_3^{(3)}, v_2^{(3)}$

is a longest $u-v$ path of length $\frac{1}{2}(3k-7)$. Then the polynomial in this case is given by $\rho_3(x) = 4x^{\frac{1}{2}(3k-7)}$.

(3) If $u = w_1$ and $v = u_i^{(3)}$ $(or\ u = u_i^{(1)}, v = w_3)$, then the path P_3 between u and v is

$P_3 = u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots, v_{\frac{k-1}{2}}^{(3)}, u_{\frac{k-1}{2}}^{(3)}, u_i^{(3)}$ is a longest $u-v$ path of length $\frac{1}{2}(3k-1)-i$ where $i = 3,4,\dots,\frac{k-1}{2}$. If $u = w_1, v = v_j^{(3)}$ (or $u = v_j^{(1)}, v = w_3$), then the path $\acute{P}_3$ between u and v is

$\acute{P}_3 =$

$v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k-1}{2}-1}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots, u_{\frac{k-1}{2}-1}^{(3)}, v_{\frac{k-1}{2}-1}^{(3)}, v_j^{(3)}$

is a longest $u-v$ path of length $\frac{1}{2}(3k-1)-j$, for $j = 3,4,\dots,\frac{k-1}{2}$.

Then we get the polynomial $\rho_4(x) = 4\sum_{j=3}^{\frac{k-1}{2}} x^{\frac{1}{2}(3k-1)-j}$.

(4) If $u = w_1$ and $v = w_3$, then the path P_4 between u and v is

$P_4 = u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, w_3$ is a longest $u - v$ path of length $\frac{1}{2}(k+1)$. Then we get the polynomial $\rho_5(x) = x^{\frac{1}{2}(k+1)}$.

Now, we add the polynomials $\rho_1(x), \rho_2(x), \rho_3(x), \rho_4(x)$ and $\rho_5(x)$ and then simplifying we get the result as given in the statement of the proposition.■

Proposition 3.2. For even $k \geq 6$, the restricted detour polynomial of $\varphi(w_k^3)$ is given by

$D^*(\varphi(W_k^3); x) = 2D^*(\varphi(W_k^2); x) - D^*(\varphi(W_k^1); x) + \rho_{k,3}(x)$ in which

$$\rho_{k,3}(x) = x^{\frac{1}{2}(5k-16)}[1 + x + 2(x+1)^2 \sum_{i=3}^{\frac{k}{2}-1} x^{2-i} + (x+3)\sum_{i=3}^{\frac{k}{2}-1}\sum_{j=3}^{\frac{k}{2}-1} x^{6-(i+j)}] + 2x^{2(k-2)} + x^{\frac{3}{2}k-1} + 2(1+x)x^{2k-5}[1 + \sum_{i=3}^{\frac{k}{2}-1} x^{3-i}] + x^{\frac{1}{2}k+1} + 2x^{\frac{3}{2}k-10}[x^6 + x^7 + \sum_{i=3}^{\frac{k}{2}-1} x^{9-i} + \sum_{j=3}^{\frac{k}{2}-1} x^{10-j}] + 2x^k.$$

Proof. We can proof the proposition by using similar techniques and steps followed in the proof of Proposition 3.1.■

The restricted detour polynomial of $\varphi(W_k^{\beta})$ for $\beta = 4$ is given in the next two proposition.

Proposition 3.3. For odd $k \geq 7$, the restricted detour polynomial of $\varphi(W_k^4)$ is given by

$D^*(\varphi(W_k^4); x) = 2D^*(\varphi(W_k^3); x) - D^*(\varphi(W_k^2); x) + \rho_{k,4}(x)$, in which

$$\rho_{k,4}(x) = x^{2k-2}[x^{2-k} + 4\sum_{i=3}^{\frac{k-1}{2}} x^{1-i} + 4x^{-2}] + 2(1+x)x^{3k-9}[1 + 2\sum_{j=3}^{\frac{k-1}{2}} x^{3-j} + \sum_{i=3}^{\frac{k-1}{2}}\sum_{j=3}^{\frac{k-1}{2}} x^{6-(i+j)}].$$

Proof. We refer to Figure2, with $\beta = 4$, and denote $U = \{u_2^{(1)}, u_3^{(1)}, \dots, u_{\frac{k-1}{2}}^{(1)}\}$, $\acute{U} = \{v_2^{(1)}, v_3^{(1)}, \dots, v_{\frac{k-1}{2}}^{(1)}\}$, $V = \{u_2^{(4)}, u_3^{(4)}, \dots, u_{\frac{k-1}{2}}^{(4)}\}$ and $\acute{V} = \{v_2^{(4)}, v_3^{(4)}, \dots, v_{\frac{k-1}{2}}^{(4)}\}$.

Let w_1, w_2, w_3 and w_4 are the hub vertices of $W_{k,1}, W_{k,2}, W_{k,3}$ and $W_{k,4}$ respectively.

Let u and v be any two distinct vertices of $\varphi(W_k^4)$ with $u \in U \cup \acute{U} \cup \{w_1\}$ and $v \in V \cup \acute{V} \cup \{w_4\}$.

Evidently, by Theorem 3.4.2 in [2, page72]; the restricted detour polynomial of all vertices on the cycles is given by

$$\rho_1(x) = 2(1+x)x^{3k-9}[1 + 2\sum_{j=3}^{\frac{k-1}{2}} x^{3-j} + \sum_{i=3}^{\frac{k-1}{2}}\sum_{j=3}^{\frac{k-1}{2}} x^{6-(i+j)}].$$

Now, it remains to compute the restricted detour polynomial of the other vertices in the graph $\varphi(w_k^4)$, to do so, we consider the following cases

(1) If $u = w_1$ and $v = u_2^{(4)}$ (or $u = u_2^{(1)}$ and $v = w_4$), then the restricted detour $u - v$ path P_1

$P_1 = u_1^{(1)}, u_2^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots,$

$v_{\frac{k-1}{2}-1}^{(3)}, v_1^{(4)}, v_2^{(4)}, \dots, v_{\frac{k-1}{2}-1}^{(4)}, v_{\frac{k-1}{2}}^{(4)}, u_{\frac{k-1}{2}}^{(4)}, u_{\frac{k-1}{2}-1}^{(4)}, \dots, u_2^{(4)}$

is a longest $u - v$ path of length $(2k-4)$.

If $u = w_1$ and $v = v_2^{(4)}$(or $u = v_2^{(1)}, v = w_4$), then the restricted detour $u - v$ path $\acute{P}_1$;

$\acute{P}_1 = v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k-1}{2}-1}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots$

$, u_{\frac{k-1}{2}-1}^{(3)}, u_1^{(4)}, u_2^{(4)}, \dots, u_{\frac{k-1}{2}-1}^{(4)}, u_{\frac{k-1}{2}}^{(4)}, v_{\frac{k-1}{2}}^{(4)}, v_{\frac{k-1}{2}-1}^{(4)}, \dots, v_2^{(4)}$ is a longest $u - v$ path of length $(2k-4)$. Then the polynomial in this case is $\rho_2(x) = 4x^{(2k-4)}$.

(2) If $u = w_1$ and $v = u_i^{(4)}$ $(or\ u = u_i^{(1)}, v = w_4)$ for $i = 3,4, \dots, \frac{k-1}{2}$, then the restricted detour $u - v$ path P_2

$P_2 = v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k-1}{2}-1}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots$

$, u_{\frac{k-1}{2}-1}^{(3)}, u_1^{(4)}, v_1^{(4)}, v_2^{(4)}, \dots, v_{\frac{k-1}{2}-1}^{(4)}, v_{\frac{k-1}{2}}^{(4)}, u_i^{(4)}$

is a longest $u - v$ path of length $2k - 1 - i$, where $i = 3,4, \dots, \frac{k-1}{2}$.

If $u = w_1, v = v_j^{(4)}$ (or $u = v_j^{(1)}, v = w_4$) for $j = 3,4, \dots, \frac{k-1}{2}$, then the restricted detour $u - v$ path $\acute{P}_2$

$\acute{P}_2 = u_1^{(1)}, u_2^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots$

$, v_{\frac{k-1}{2}-1}^{(3)}, v_1^{(4)}, u_1^{(4)}, u_2^{(4)}, \dots, u_{\frac{k-1}{2}-1}^{(4)}, u_{\frac{k-1}{2}}^{(4)}, v_j^{(4)}$ is the longest $u-v$ path of length $2k-1-j$, where $j = 3,4, \dots, \frac{k-1}{2}$. Then we get the polynomial $\rho_3(x) = 4\sum_{j=3}^{\frac{k-1}{2}} x^{2k-1-j}$.

(3) If $u = w_1$ and $v = w_4$ then the restricted detour $u-v$ path

$P_3 =$

$v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k-1}{2}-1}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots, u_{\frac{k-1}{2}-1}^{(3)}, u_1^{(4)}, w_4,$

is a longest $u-v$ path of length k.

Then the polynomial in this case is given by $\rho_4(x) = x^k$.

Now, we add the polynomials $\rho_1(x), \rho_2(x), \rho_3(x)$ and $\rho_4(x)$ and simplifying we get the result as given in the statement of the proposition.■

Proposition 3.4. For even $k \geq 6$, the restricted detour polynomial of $\varphi(w_k^4)$ is given by

$D^*(\varphi(w_k^4); x) = 2D^*(\varphi(w_k^3); x) - D^*(\varphi(w_k^2); x) + \rho_{k,4}(x)$

in which

$\rho_{k,4}(x) = x^{3k-10}[1 + 3x^2 + (3x+1)(2\sum_{i=3}^{\frac{k}{2}-1} x^{4-i} + \sum_{i=3}^{\frac{k}{2}-1}\sum_{j=3}^{\frac{k}{2}-1} x^{7-(i+j)})] + x^{2(k-1)} + 4x^{2(k-2)}$

$+4x^{\frac{1}{2}(5k-10)}[1 + \sum_{i=3}^{\frac{k}{2}-1} x^{3-i}] + 4\sum_{j=3}^{\frac{k}{2}-1} x^{2k-1-j} + 2x^{\frac{3}{2}k-1} + x^k.$

Proof. We can proof the proposition by using similar techniques and steps followed in the proof of Proposition 3.1.■

Next, we obtain the restricted detour index of $\varphi(W_k^3)$, by taking the derivative of $D^*(\varphi(W_k^3); x)$ at $x = 1$ obtained from Propositions 3.1. and 3.2.

Corollary 3.5. For $k \geq 6$, the restricted detour index of $\varphi(W_k^3)$ is given as follow

(1) For even $k \geq 6$ we have $dd^*(\varphi(W_k^3)) = \frac{49}{8}k^3 - \frac{349}{8}k^2 + \frac{431}{4}k - 81$.

(2) For odd $k \geq 7$ we have $dd^*(\varphi(W_k^3)) = \frac{1}{8}(49k^3 - 351k^2 + 871k - 657)$.

Proof.

(1) Taking the derivative of $D^*(\varphi(W_k^3); x)$ given in Proposition 3.2., at $x = 1$, we get

$dd^*(\varphi(W_k^3)) = 2dd^*(\varphi(W_k^2)) - dd^*\varphi(W_k^1) + \frac{d}{dx}\rho_{k,3}|_{x=1}$,

where $\frac{d}{dx}\rho_{k,3}|_{x=1} = 2k^3 - 13k^2 + \frac{49}{2}k - \frac{13}{2}$.

Now, simplifying the results above, we get $dd^*(\varphi(W_k^3))$ as given in the statement of the proposition.

(2) Obvious.■

The restricted detour index of $\varphi(W_k^4)$ is obtained in the next corollary by taking the derivative of $D^*(\varphi(W_k^4); x)$ at $x = 1$ obtained in Propositions 3.3. and 3.4.

Corollary 3.6. For $k \geq 6$, the restricted detour index of $\varphi(W_k^4)$ is given as follows

(1) For even $k \geq 6$, we have $dd^*(\varphi(W_k^4)) = \frac{25}{2}k^3 - \frac{343}{4}k^2 + 197k - 135$.

(2) For odd $k \geq 7$, we have $dd^*(\varphi(W_k^4)) = \frac{1}{2}(25k^3 - 173k^2 + 403k - 283)$.

Proof. Obvious.■

4. Restricted Detour Polynomial of Straight Chains of Wheels $\varphi(W_k^\beta)$ for $\beta \geq 5$

In this section, we will discover the restricted detour polynomials and the restricted detour indices of the straight chain of wheels $\varphi(W_k^\beta)$, where $\beta \geq 5$.

In the next theorem, we shall find the restricted detour polynomial of $\varphi(W_k^\beta)$ odd $k \geq 7$ and $\beta \geq 5$ (for even and odd β).

Theorem 4.1. For odd $k \geq 7$ and $\beta \geq 5$, the restricted detour polynomial of $\varphi(w_k^\beta)$ is given by

$D^*(\varphi(W_k^\beta); x) = 2D^*(\varphi(W_k^3); x) - D^*(\varphi(W_k^2); x) + \rho_{k,\beta}(x)$, in which

$\rho_{k,\beta}(x) = 2(1+x)x^{(k-1)\left(1+\frac{1}{2}\beta\right)-6}[1 + 2\sum_{i=3}^{\frac{k-1}{2}} x^{3-i} +$

$\sum_{i=3}^{\frac{k-1}{2}}\sum_{j=3}^{\frac{k-1}{2}} x^{6-(i+j)}]+x^{\frac{1}{2}\beta(k-1)}[x^{2-k}+4\sum_{j=3}^{\frac{k-1}{2}} x^{1-j}$

$+4x^{-2}]$.

Proof. We refer to Figure2 for odd $k \geq 7$ and $\beta \geq 5$; and denote $U=\{u_2^{(1)}, u_3^{(1)}, \dots, u_{\frac{k-1}{2}}^{(1)}\}$, $\acute{U}=\{v_2^{(1)}, v_3^{(1)}, \dots, v_{\frac{k-1}{2}}^{(1)}\}$, $V=\{u_2^{(\beta)}, u_3^{(\beta)}, \dots, u_{\frac{k-1}{2}}^{(\beta)}\}$, and $\acute{V}=\{v_2^{(\beta)}, v_3^{(\beta)}, \dots, v_{\frac{k-1}{2}}^{(\beta)}\}$. Let $w_1, w_2, \dots$, and w_β be the hub vertices of $W_{k,1}, W_{k,2}, \dots$, and $W_{k,\beta}$ respectively. We consider two main cases for β in which it is either an odd or an even number.

Let $\beta \geq 5$ be an odd positive integer, and let u and v be any two distinct vertices of $\varphi(W_k^\beta)$ with $u \in U \cup \acute{U} \cup \{w_1\}$ and $v \in V \cup \acute{V} \cup \{w_\beta\}$.

By Theorem 3.4.3 in [2, page74]; it is obvious that the restricted detour polynomial of all vertices on the cycles is given by

$\rho_1(x)=2(1+x)x^{(k-1)\left(1+\frac{1}{2}\beta\right)-6}[1+2\sum_{i=3}^{\frac{k-1}{2}} x^{3-i}+$

$\sum_{i=3}^{\frac{k-1}{2}}\sum_{j=3}^{\frac{k-1}{2}} x^{6-(i+j)}]$.

Now, it remains to compute the restricted detour polynomial of the other vertices in the graph $\varphi\left(W_k^\beta\right)$. To do so, we consider the following cases

(1) If $u=w_1$ and $v=u_2^{(\beta)}$ (or $u=u_2^{(1)}$ and $v=w_\beta$), then the restricted detour $u-v$ path P_1 is

$P_1=v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k-1}{2}-1}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots$

$, u_{\frac{k-1}{2}-1}^{(3)}, u_1^{(4)}, v_1^{(4)}, \dots, u_1^{(\beta-1)}, v_1^{(\beta-1)}, v_2^{(\beta-1)}, \dots, v_{\frac{k-1}{2}-1}^{(\beta-1)},$

$v_1^{(\beta)}, v_2^{(\beta)}, \dots, v_{\frac{k-1}{2}}^{(\beta)}, u_{\frac{k-1}{2}}^{(\beta)}, \dots, u_3^{(\beta)}, u_2^{(\beta)}$

is a longest $u-v$ path of length $\frac{1}{2}\beta k-\frac{1}{2}\beta-2$.

If $u=w_1$ and $v=v_2^{(\beta)}$(or $u=v_2^{(1)}, v=w_\beta$), then the restricted detour $u-v$ path $\acute{P}_1$ is

$\acute{P}_1=u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots$

$, v_{\frac{k-1}{2}-1}^{(3)}, v_1^{(4)}, u_1^{(4)}, \dots, v_1^{(\beta-1)}, u_1^{(\beta-1)}, u_2^{(\beta-1)}, \dots, u_{\frac{k-1}{2}-1}^{(\beta-1)},$

$u_1^{(\beta)}, u_2^{(\beta)}, \dots, u_{\frac{k-1}{2}}^{(\beta)}, v_{\frac{k-1}{2}}^{(\beta)}, \dots, v_3^{(\beta)}, v_2^{(\beta)}$

is a longest $u-v$ path of length $\frac{1}{2}\beta k-\frac{1}{2}\beta-2$. The polynomial in this case is given by $\rho_2(x)=4x^{\frac{1}{2}\beta k-\frac{1}{2}\beta-2}$.

(2) If $u=w_1$ and $v=u_i^{(\beta)}$ $(or\ u=u_i^{(1)}, v=w_\beta)$

for $i=3,4,\dots,\frac{k-1}{2}$, then the restricted detour $u-v$ path P_2 is

$P_2=u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots,$

$v_1^{(\beta-1)}, u_1^{(\beta-1)}, u_2^{(\beta-1)}, \dots, u_{\frac{k-1}{2}-1}^{(\beta-1)}, u_1^{(\beta)}, v_1^{(\beta)}, v_2^{(\beta)}, \dots, v_{\frac{k-1}{2}}^{(\beta)}, u_{\frac{k-1}{2}}^{(\beta)}, u_i^{(\beta)}$

is a longest $u-v$ path of length $\frac{1}{2}\beta k-\frac{1}{2}\beta+1-i$ where $i=3,4,\dots,\frac{k-1}{2}$.

If $u=w_1, v=v_j^{(4)}$ (or $u=v_j^{(1)}, v=w_\beta$), for $j=3,4,\dots,\frac{k-1}{2}$, then the restricted detour $u-v$ path $\acute{P}_2$ is

$\acute{P}_2=v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k-1}{2}-1}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots$

$, u_1^{(\beta-1)}, v_1^{(\beta-1)}, v_2^{(\beta-1)}, \dots, v_{\frac{k-1}{2}-1}^{(\beta-1)}, v_1^{(\beta)}, u_1^{(\beta)}, u_2^{(\beta)}, \dots, u_{\frac{k-1}{2}}^{(\beta)}, v_{\frac{k-1}{2}}^{(\beta)}, v_j^{(\beta)}$

is a longest $u-v$ path of length $\frac{1}{2}\beta k-\frac{1}{2}\beta+1-j$, where $j=3,4,\dots,\frac{k-1}{2}$. The polynomial in this case is given by $\rho_3(x)=4\sum_{j=3}^{\frac{k-1}{2}} x^{\frac{1}{2}\beta k-\frac{1}{2}\beta+1-j}$.

(3) If $u=w_1$ and $v=w_\beta$ then the restricted detour $u-v$ path P_3 is $P_3=u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k-1}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots$

$, v_1^{(\beta-1)}, u_1^{(\beta-1)}, u_2^{(\beta-1)}, \dots, u_{\frac{k-1}{2}-1}^{(\beta-1)}, u_1^{(\beta)}, w_\beta$ is a longest $u-v$ path of length $\frac{1}{2}\beta k-\frac{1}{2}\beta+2-k$. The polynomial in this case is given by

$\rho_4(x)=x^{\frac{1}{2}\beta k-\frac{1}{2}\beta+2-k}$.

Now, adding the polynomials $\rho_1(x), \rho_2(x), \rho_3(x)$ and$\rho_4(x)$ and simplifying we get the result as given in the statement of the theorem.

Similarly if β is even.

This completes the proof.X

In the next two theorems, we shall find the restricted detour polynomial of $\varphi(W_k^\beta)$ even $m \geq 6$ and $\beta \geq 5$.

Theorem 4.2. For even $k \geq 6$ and odd $\beta \geq 5$, the restricted

detour polynomial of $\varphi(W_k^{\beta})$ is given by

$D^*(\varphi(w_k^{\beta});x) = 2D^*(\varphi(w_k^3);x) - D^*(\varphi(w_k^2);x) + \rho_{k,\beta}(x)$

in which

$$\rho_{k,\beta}(x) = (3+x)x^{(k-1)\left(1+\frac{1}{2}\beta\right)-\frac{11}{2}}[1+2\sum_{i=3}^{\frac{k}{2}-1}x^{3-i}+\sum_{i=3}^{\frac{k}{2}-1}\sum_{j=3}^{\frac{k}{2}-1}x^{6-(i+j)}]+2(1+x)x^{\frac{1}{2}(k-1)(1+\beta)-3}[1+\sum_{i=3}^{\frac{k}{2}-1}x^{3-i}]+2x^{\frac{1}{2}\beta(k-1)-\frac{5}{2}}[1+x+\sum_{i=3}^{\frac{k}{2}-1}x^{4-i}+\sum_{j=3}^{\frac{k}{2}-1}x^{3-j}]+x^{\frac{1}{2}(\beta(k-1)+1)}+2x^{\frac{1}{2}k(\beta-1)-\frac{1}{2}(\beta-3)}+x^{\frac{1}{2}k(\beta-2)-\frac{1}{2}(\beta-5)}.$$

Proof. We can proof the theorem by using similar techniques and steps followed in the proof of Theorem 4.1.■

Theorem 4.3. For even $k \geq 6$ and even $\beta \geq 6$, the restricted detour polynomial of $\varphi(W_k^{\beta})$ is given by

$D^*(\varphi(W_k^{\beta});x) = 2D^*(\varphi(W_k^3);x) - D^*(\varphi(W_k^2);x) + \rho_{k,\beta}(x)$ in which

$$\rho_{k,\beta}(x) = (3x+1)x^{(k-1)\left(1+\frac{1}{2}\beta\right)-6}[1+2\sum_{i=3}^{\frac{k}{2}-1}x^{3-i}+\sum_{i=3}^{\frac{k}{2}-1}\sum_{j=3}^{\frac{k}{2}-1}x^{6-(i+j)}]+x^{\frac{1}{2}\beta(k-1)}+4x^{\frac{1}{2}(k-1)(1+\beta)-\frac{5}{2}}[1+\sum_{i=3}^{\frac{k}{2}-1}x^{3-i}]+2x^{\frac{1}{2}\beta(k-1)-2}[2+2\sum_{i=3}^{\frac{k}{2}-1}x^{3-i}]+x^{\frac{1}{2}\beta(k-1)}[2x^{1-\frac{1}{2}k}+x^{2-k}].$$

Proof. We refer to Figure1, and denote $U = \{u_2^{(1)}, u_3^{(1)}, \dots, u_{\frac{k}{2}-1}^{(1)}\}$, $\acute{U} = \{v_2^{(1)}, v_3^{(1)}, \dots, v_{\frac{k}{2}-1}^{(1)}\}$, $V = \{u_2^{(\beta)}, u_3^{(\beta)}, \dots, u_{\frac{k}{2}-1}^{(\beta)}\}$ and $\acute{V} = \{v_2^{(\beta)}, v_3^{(\beta)}, \dots, v_{\frac{k}{2}-1}^{(\beta)}\}$.

Let u and v be any two distinct vertices of $\varphi(W_k^{\beta})$ with $u \in U \cup \acute{U} \cup \{u_{\frac{k}{2}}^{(1)}, w_1\}$ and $v \in V \cup \acute{V} \cup \{u_{\frac{k}{2}}^{(\beta)}, w_\beta\}$.

Evidently, from the proof of Theorem 3.3.4 in [2, page76]; the restricted detour polynomial of all vertices on the cycles is given by

$$\rho_1(x) = (3x+1)x^{(k-1)\left(1+\frac{1}{2}\beta\right)-6}[1+2\sum_{i=3}^{\frac{k}{2}-1}x^{3-i}+\sum_{i=3}^{\frac{k}{2}-1}\sum_{j=3}^{\frac{k}{2}-1}x^{6-(i+j)}]+x^{\frac{1}{2}\beta(k-1)}+4x^{\frac{1}{2}(k-1)(1+\beta)-\frac{5}{2}}[1+\sum_{i=3}^{\frac{k}{2}-1}x^{3-i}].$$

Now, it remains to compute the restricted detour polynomial of the other vertices in the graph $\varphi(W_k^{\beta})$, to do so, we consider the following cases:

(1) If $u = w_1$ and $v = u_2^{(\beta)}$ (or $u = u_2^{(1)}$ and $v = w_\beta$), then the $u-v$ path P_1 is

$$P_1 = v_1^{(1)}, v_2^{(2)}, \dots, v_{\frac{k}{2}-2}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots, u_{\frac{k}{2}-1}^{(3)}, u_1^{(4)}, v_1^{(4)}, \dots,$$
$$v_1^{(\beta-1)}, v_2^{(\beta-1)}, \dots, v_{\frac{k}{2}-2}^{(\beta-1)}, v_1^{(\beta)}, v_2^{(\beta)}, \dots, v_{\frac{k}{2}-1}^{(\beta)},$$
$$u_{\frac{k}{2}}^{(\beta)}, u_{\frac{k}{2}-1}^{(\beta)}, \dots, u_3^{(\beta)}, u_2^{(\beta)}$$

is a longest $u-v$ path of length $\frac{1}{2}\beta k - \frac{1}{2}\beta - 2$. If $u = w_1$ and $v = v_2^{(\beta)}$(or $u = v_2^{(1)}, v = w_\beta$)), then the $u-v$ path $\acute{P}_1$ is

$$\acute{P}_1 = u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots,$$
$$v_1^{(\beta-1)}, u_1^{(\beta-1)}, u_2^{(\beta-1)}, \dots, u_{\frac{k}{2}-1}^{(\beta-1)}, u_1^{(\beta)}, u_2^{(\beta)}, \dots, u_{\frac{k}{2}-1}^{(\beta)},$$
$$u_{\frac{k}{2}}^{(\beta)}, v_{\frac{k}{2}-1}^{(\beta)}, \dots, v_3^{(\beta)}, v_2^{(\beta)}$$

is the longest $u-v$ path of length $\frac{1}{2}\beta k - \frac{1}{2}\beta - 2$.

Then the polynomial is $\rho_2(x) = 4x^{\frac{1}{2}\beta k - \frac{1}{2}\beta - 2}$

(2) If $u = w_1$ and $v = u_i^{(\beta)}$ (or $u = u_i^{(1)}, v = w_\beta$) then the path P_2 between u and v is

$$P_2 = u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots,$$
$$v_1^{(\beta-1)}, u_1^{(\beta-1)}, u_2^{(\beta-1)}, \dots, u_{\frac{k}{2}-1}^{(\beta-1)}, u_1^{(\beta)}, v_1^{(\beta)}, v_2^{(\beta)}, \dots, v_{\frac{k}{2}-1}^{(\beta)}, u_{\frac{k}{2}}^{(\beta)}, u_i^{(\beta)}$$

is a longest $u-v$ path of length $\frac{1}{2}\beta k - \frac{1}{2}\beta + 1 - i$ where $i = 3,4,\dots,\frac{k}{2}-1$. If $u = w_1$ and $v = v_j^{(\beta)}$ (or $u = v_j^{(1)}, v = w_\beta$) then the path $\acute{P}_2$ between u and v is given by

$$\acute{P}_2 = v_1^{(1)}, v_2^{(2)}, v_3^{(2)}, \dots, v_{\frac{k}{2}-2}^{(2)}, v_1^{(3)}, u_1^{(3)}, u_2^{(3)}, \dots,$$
$$u_1^{(\beta-1)}, v_1^{(\beta-1)}, v_2^{(\beta-1)}, \dots, v_{\frac{k}{2}-2}^{(\beta-1)}, v_1^{(\beta)}, u_1^{(\beta)}, u_2^{(\beta)}, \dots, u_{\frac{k}{2}-1}^{(\beta)}, u_{\frac{k}{2}}^{(\beta)}, v_j^{(\beta)}$$

is a longest $u-v$ path of length $\frac{1}{2}\beta k - \frac{1}{2}\beta + 1 - j$, where $j = 3,4,\dots,\frac{k}{2}-1$.

Then we get the polynomial $\rho_3(x) = 4\sum_{j=3}^{\frac{k}{2}-1} x^{\frac{1}{2}\beta k - \frac{1}{2}\beta + 1 - j}$.

(3) If $u = w_1$ and $v = w_\beta$ then the path P_3 between u and v is

$P_3 = u_1^{(1)}, u_2^{(2)}, u_3^{(2)}, \dots, u_{\frac{k}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots,$

$v_1^{(\beta-1)}, u_1^{(\beta-1)}, u_2^{(\beta-1)}, \dots, u_{\frac{k}{2}-2}^{(\beta-1)}, u_1^{(\beta)}, w_\beta$ is a longest $u-v$ path of length $\frac{1}{2}\beta(k-1)-(k-2)$. Then the polynomial is $\rho_4(x) = x^{\frac{1}{2}\beta(k-1)-(k-2)}$.

(4) If $u = w_1$ and $v = u_{\frac{k}{2}}^{(\beta)}$ (or $u = u_{\frac{k}{2}}^{(1)}, v = w_\beta$) then the path P_4 between u and v is

$P_4 = u_1^{(1)}, u_2^{(2)}, \quad \dots, u_{\frac{k}{2}-1}^{(2)}, u_1^{(3)}, v_1^{(3)}, v_2^{(3)}, \dots,$

$v_1^{(\beta-1)}, u_1^{(\beta-1)}, u_2^{(\beta-1)}, \dots, u_{\frac{k}{2}-1}^{(\beta-1)}, u_1^{(\beta)}, v_1^{(\beta)}, v_2^{(\beta)}, \dots, u_{\frac{k}{2}-1}^{(\beta)}, u_{\frac{k}{2}}^{(\beta)}$

is a longest $u-v$ path of length $\frac{1}{2}\beta(k-1)-\frac{1}{2}k+1$. Then the polynomial is $\rho_5(x) = 2x^{\frac{1}{2}\beta(k-1)-\frac{1}{2}k+1}$. Now, we add the polynomials $\rho_1(x), \rho_2(x), \rho_3(x)$ and $\rho_4(x)$ and then simplifying we get the result.X

Finally, we obtain the restricted detour index of $\varphi(W_k^\beta)$ in the next corollary, by taking the derivative of $D^*(\varphi(W_k^\beta); x)$ at $x = 1$ obtained in the Theorems, 4.1, 4.2, and 4.3.

Corollary 4.4. The restricted detour index of $\varphi(W_k^\beta)$ is given as follows

(1) For odd $k \geq 7$ and $\beta \geq 5$, we have

$dd^*(\varphi(W_k^\beta)) = (\frac{1}{12}\beta^3 + \frac{1}{2}\beta^2 + \frac{5}{24}\beta)k^3 - (\frac{5}{12}\beta^3 + 4\beta^2 - \frac{19}{24}\beta - 1)k^2 + (\frac{2}{3}\beta^3 + \frac{35}{4}\beta^2 + \frac{161}{24}\beta - 8)k - (\frac{1}{3}\beta^2 + \frac{23}{4}\beta + \frac{163}{24})\beta - 1.$

(2) For even $m \geq 6$ and $\beta \geq 5$, we have

$$dd^*(\varphi(W_k^\beta)) = (\frac{1}{12}\beta^3 + \frac{1}{2}\beta^2 - \frac{5}{24}\beta)k^3 - (\frac{5}{12}\beta^3 + \frac{31}{8}\beta^2 - \frac{5}{12}\beta - \frac{5}{4})k^2 + (\frac{2}{3}\beta^3 + 8\beta^2 + \frac{103}{12}\beta - 8)k - (\frac{1}{3}\beta^2 + \frac{9}{2}\beta + \frac{29}{3})\beta - \frac{5}{2}.$$

(3) For even $k \geq 6$ and $\beta \geq 6$, we have

$dd^*(\varphi(W_k^\beta)) = (\frac{1}{12}\beta^3 + \frac{1}{2}\beta^2 - \frac{5}{24}\beta)k^3 - (\frac{5}{12}\beta^3 + \frac{31}{8}\beta^2 - \frac{5}{12}\beta - \frac{5}{4})k^2 + (\frac{2}{3}\beta^3 + 8\beta^2 + \frac{103}{12}\beta - 8)k - (\frac{1}{3}\beta^2 + \frac{9}{2}\beta + \frac{29}{3})\beta - 3.$

Proof. Obvious.■

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متعددة حدود ألألتفاف ألمقيدة لسلسلة مستقيمة من ألعجلات

هيرش عمر عبدألله — **ايفان دلير علي**

evan.ali1@su.edu.krd herish.abdullah@su.edu.krd

كلية ألعلوم — **كلية ألتربية**

جامعة صلاح ألدين-أربيل — **جامعة صلاح ألدين-أربيل**

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الملخص

تعرف مسافة ألألتفاف المقيدة بين الحافتين u و v في البيان G على أنها ألطول لأطول درب P بين u و v و ألذي يحقق ألشرط $\langle V(P) \rangle = P$. تعرف متعددة حدود ألألتفاف ألمقيدة و ألتي تعتمد على مسافة ألألتفاف المقيدة بالصيغة $D^*(G,x) = \sum_{\{u,v\}} x^{D^*(u,v)}$ حيث أن ألمجموع تأخذ حول جميع ألأزواج ألغير مرتبة ألمختلفة u,v في ألبيان G . في هذه ألمقالة، نبحث متعددة حدود ألألتفاف ألمقيدة لسلسلة مستقيمة من ألعجلات $\varphi(W_k^\beta)$ و كذلك مسافة ألألتفاف المقيدة و أيضا نحسب دليل ألألتفاف المقيدة ل $\varphi(W_k^\beta)$.

الكلمات المفتاحية: مسافة ألألتفاف المقيدة، متعددة حدود ألألتفاف ألمقيدة، دليل ألألتفاف المقيدة.