The Sine-Cosine Function Method for Exact Solutions of Nonlinear Partial Differential Equations

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Abstract: *The Sine-Cosine function algorithm is applied for solving nonlinear partial differential equations. The method is used to obtain the exact solutions for different types of nonlinear partial differential equations such as, The K(n + 1, n + 1) equation, Schrödinger-Hirota equation, Gardner equation, the modified KdV equation, perturbed Burgers equation, general Burger's-Fisher equation, and Cubic modified Boussinesq equation which are the important Soliton equations.*

Keywords:* *Nonlinear PDEs, Exact Solutions, Nonlinear Waves, Gardner equation, Sine-Cosine function method, The Schrödinger-Hirota equation.

1. Introduction

Nonlinear evolution equations have a major role in various scientific and engineering fields, such as fluid mechanics, plasma physics, optical fibers, solid state physics, chemical physics and geochemistry. Nonlinear wave phenomena of dispersion, dissipation, diffusion, reaction and convection are very important in nonlinear wave equations [1]. In recent years, quite a few methods for obtaining explicit traveling and solitary wave solutions of nonlinear evolution equations have been proposed. A variety of powerful methods, such as, tanh-sech method [2,3,4], extended tanh method [5,6,7], hyperbolic function method [8,9], Jacobi elliptic function expansion method [10], F-expansion method [11], and the First Integral method [12,13]. The sine-cosine method [14, 15] has been used to solve different types of nonlinear systems of PDEs.

This paper contains two parts. The first part explains the proposed method, while the second part contains the applications. The aim of this paper is to find new exact solutions of the K(n + 1, n + 1) equation, Schrödinger-Hirota equation, Gardner equation, modified KdV equation, perturbed Burgers equation, general Burger's-Fisher equation, and Cubic modified Boussinesq equation by the sine-cosine method.

2. The Sine-Cosine Function Method

Consider the nonlinear partial differential equation in the form

$$F\left(u, u_t, u_x, u_y, u_{tt}, u_{xx}, u_{xy}, u_{yy}, \ldots\ldots\ldots\ldots\right) = 0 \qquad (1)$$

where u(x, y, t) is a traveling wave solution of nonlinear partial differential equation Eq. (1). We use the transformations,

$$u(x, y, t) = f(\xi) \qquad (2)$$

where $\xi = x + y - \lambda t$ This enables us to use the following changes:

$$\frac{\partial}{\partial t}(.) = -\lambda \frac{d}{d\xi}(.)\ ,\ \frac{\partial}{\partial x}(.) = \frac{d}{d\xi}(.)\ ,\ \frac{\partial}{\partial y}(.) = \frac{d}{d\xi}(.) \tag{3}$$

Using Eq. (3) to transfer the nonlinear partial differential equation Eq. (1) to nonlinear ordinary differential equation

$$Q(f, f', f'', f''', \ldots\ldots\ldots\ldots\ldots.) = 0 \tag{4}$$

The ordinary differential equation (4) is then integrated as long as all terms contain derivatives, where we neglect the integration constants. The solutions of many nonlinear equations can be expressed in the form: [16, 17]

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi)\ , \qquad |\xi| \le \frac{\pi}{2\mu}$$

or in the form (5)

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi)\ , \qquad |\xi| \le \frac{\pi}{2\mu}$$

Where α, μ, and β are parameters to be determined, μ and c are the wave number and the wave speed, respectively [15, 18]. We use

$$f(\xi) = \alpha \sin^{\beta}(\mu\xi)$$

$$f'(\xi) = \alpha\, \beta\, \mu\ \sin^{\beta-1}(\mu\xi)\ \cos(\mu\xi) \tag{6}$$

$$f''(\xi) = \alpha\, \beta(\beta-1)\, \mu^2\ \sin^{\beta-2}(\mu\xi) - \alpha\, \beta^2\, \mu^2\ \sin^{\beta}(\mu\xi)$$

and their derivative. Or use

$$f(\xi) = \alpha \cos^{\beta}(\mu\xi)$$

$$f'(\xi) = -\,\alpha\, \beta\, \mu\ \cos^{\beta-1}(\mu\xi)\ \sin(\mu\xi) \tag{7}$$

$$f''(\xi) = \alpha\, \beta(\beta-1)\, \mu^2\ \cos^{\beta-2}(\mu\xi) - \alpha\, \beta^2\, \mu^2\ \cos^{\beta}(\mu\xi)$$

$$f'''(\xi) = -\,\alpha\, \beta(\beta-1)(\beta-2)\, \mu^3\ \cos^{\beta-3}(\mu\xi)\sin(\mu\xi) + \alpha\, \beta^3\, \mu^3\ \cos^{\beta-1}(\mu\xi)\ \sin(\mu\xi)$$

and so on. We substitute Eq.(6) or Eq.(7) into the reduced equation Eq.(4), balance the terms of the sine functions when Eq.(6) are used, or balance the terms of the cosine functions when Eq.(7) are used, and solve the resulting system of algebraic equations by using computerized symbolic packages. We next collect all terms with the same power in $\sin^k(\mu\xi)$ or $\cos^k(\mu\xi)$ and set to zero their coefficients to get a system of algebraic equations among the unknown's α , μ and β, and solve the subsequent system.

3. Applications

3.1 The K(n + 1, n + 1) equation

Let us consider the following K(n + 1, n + 1) equation [19]:

$$u_t + a(u^{n+1})_x + b\,(u(u^n)_{xx})_x = 0 \qquad (8)$$

where a and b are nonzero constants. We introduce the transformation $\xi = k(x - \lambda t)$, where k, and λ are real constants. The traveling wave variable ξ permits us converting Eq. (8) into the following ODE:

$$-\lambda\ u' u^{2-n} + a(n+1)\ u^2 u' + bn\,k^2\,u^2\,u''' + b\,k^2 n\,[3n - 2\,]\,u\,u'\,u'' + bn(n-1)^2\,k^2 u'^3 = 0 \qquad (9)$$

Seeking the solution in Eq.(7)

$$\lambda\,\beta\,\mu\ \alpha^{3-n}\,cos^{(3-n)\beta-1}(\mu\xi)\ \ sin(\mu\xi) - a(n+1)\ \alpha^3\ \beta\,\mu\ \ cos^{3\beta-1}(\mu\xi)\ sin(\mu\xi) - bn\,k^2\ \ \alpha^3\,\beta(\beta - 1)(\beta-2)\mu^3\ \ cos^{3\beta-3}(\mu\xi)\,sin(\mu\xi) + bn\,k^2\,\alpha^3\,\beta^3\,\mu^3\ \ cos^{3\beta-1}(\mu\xi)\ sin(\mu\xi)\ - b\,k^2 n\,(3n - 2)\,\alpha^3\,\beta^2(\beta-1)\,\mu^3\ \ cos^{3\beta-3}(\mu\xi)\,sin(\mu\xi) + b\,k^2 n\,(3n - 2)\,\alpha^3\,\beta^3\,\mu^3\ \ cos^{3\beta-1}(\mu\xi)\,sin(\mu\xi) - bn(n - 1)^2\,k^2\beta^3\,\mu^3\alpha^3\ \ cos^{3\beta-3}(\mu\xi)\ sin(\mu\xi) = 0 \qquad (10)$$

From Eq.(10), equating exponents $(3 - n)\beta - 1$ and $3\beta - 3$ yield

$$(3-\mathrm{n})\beta - 1 = 3\beta - 3 \qquad (11)$$

so that

$$\beta = \frac{2}{\mathrm{n}} \qquad (12)$$

Thus setting coefficients of Eq.(10) to zero yields the following system of equations:

$$\lambda\, \beta\, \mu\, \alpha^{3-n} - bn\, k^2\, \alpha^3\, \beta(\beta-1)(\beta-2)\mu^3 - b\, k^2 n\,(3n-2)\, \alpha^3\, \beta^2(\beta-1)\mu^3 - bn(n-1)^2\, k^2 \beta^3\, \mu^3 \alpha^3 = 0$$

$$-a(n+1)\,\alpha^3\,\beta\,\mu + bn\,k^2\,\alpha^3\,\beta^3\,\mu^3 + b\,k^2 n\,(3n-2)\,\alpha^3\,\beta^3\,\mu^3 = 0 \qquad (13)$$

By solving the algebraic system (13), we get,

$$\alpha = \left\{\frac{2\;(3n-1)}{a\,(n+1)(2n+1)(n-4)}\,\lambda\right\}^{\frac{1}{n}} \quad , \quad \mu = \frac{\sqrt{\frac{a\,n(n+1)}{b\,(3n-1)}}}{2k} \qquad (14)$$

Then by substituting Eq. (14) into Eq. (7) , the exact soliton solution of Eq.(8) can be written in the form

$$u(x,t) = \left[\frac{2\,(3n-1)}{a\,(n+1)(2n+1)(n-4)}\,\lambda\;\; cos^{\,2}\left(\sqrt{\frac{a\,n(n+1)}{4b\,(3n-1)}}(x-\lambda t)\right)\right]^{\frac{1}{n}}$$

(15)

3.2 The Schrödinger-Hirota equation

Consider the nonlinear The Schrödinger-Hirota equation which governs the propagation of optical Soliton in a dispersive optical fiber:

$$i\, q_t + \frac{1}{2}\, q_{xx} + |q|^2\; q + i\,\lambda\, q_{xxx} = 0 \qquad (16)$$

This equation studied by Biswas *et al* [20] by the ansatz method for bright and dark 1-soliton solution. The power law nonlinearity was assumed. The equation was solved also by using the tanh method. Introduce the transformations:

$$q(x,t) = e^{i\,\theta}.u(\xi) \;, \theta = \alpha x + \omega t + \epsilon_0 \;, \;\; \xi = k_0(x - 2\alpha t + \chi) \tag{17}$$

where $\alpha, \omega, \epsilon_0, k_0,$ and χ are real constants. Substituting Eq.(17) into Eq.(16) we obtain that $\alpha = \frac{-1}{3\lambda}$ and $u(\xi)$ satisfy into the ODE:

$$-\left(\frac{5}{54\lambda^2} + \omega\right) u(\xi) + \frac{3}{2}{k_0}^2 u''(\xi) + (u(\xi))^3 = 0 \tag{18}$$

Then we can write the following equation:

$$u'' + k_1 u^3 - k_2\, u = 0 \tag{19}$$

Where

$$k_1 = \frac{1}{\frac{3}{2}{k_0}^2}\,, \;\; k_2 = \frac{\left(\frac{5}{54\lambda^2} + \omega\right)}{\frac{3}{2}{k_0}^2} \tag{20}$$

Seeking solutions of the form Eq.(6) we get:

$$\alpha\,\beta(\beta-1)\,\mu^2 sin^{\beta-2}(\mu\xi) - \alpha\,\beta^2\,\mu^2\,sin^{\beta}(\mu\xi) + k_1\alpha^3 sin^{3\beta}(\mu\xi) - k_2\,\alpha\,sin^{\beta}(\mu\xi) = 0 \tag{21}$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\beta - 2 = 3\beta$$

$$\alpha\,\beta(\beta-1)\,\mu^2 + k_1\alpha^3 = 0$$

$$-\alpha\,\beta^2\,\mu^2 - k_2\,\alpha = 0 \tag{22}$$

By solving the algebraic system (24), we get,

$$\beta = -1\,, \qquad \mu = \pm\, i\sqrt{k_2} \;\;, \;\; \alpha = \pm\sqrt{\frac{2\,k_2}{k_1}} \tag{23}$$

Then by substituting Eq. (23) into Eq. (6) , the exact soliton solution of equation Eq.(19) can be written in the form:

$$u(\xi) = \pm\sqrt{\frac{5}{27\lambda^2} + 2\beta}\ csc\left(\pm\, i\sqrt{k_2}\xi\right) \tag{24-1}$$

or

$$u(\xi) = \mp\sqrt{\frac{5}{27\lambda^2} + 2\beta}\ csch\left(\sqrt{k_2}\xi\right) \tag{24-2}$$

Therefore

$$u(x,y,t) = \pm\sqrt{\left(\frac{5}{27\lambda^2} + 2\omega\right)}\ csch\left(\sqrt{\frac{\left(\frac{5}{54\lambda^2}+\omega\right)}{\frac{3}{2}k_0{}^2}}\ k_0\left(x+\frac{2}{3\lambda}t+\chi\right)\right) e^{i\left(\frac{-1}{3\lambda}x+\omega t+\epsilon_0\right)} \tag{25}$$

for $\alpha = \omega = k_0 = 1$, $\epsilon_0 = \chi = 0$, then $\lambda = \frac{-1}{3}$ (25) become:

$$u(x,y,t) = \pm\sqrt{\frac{11}{3}}\ csch\left(\frac{\sqrt{11}}{3}\ (x-2t)\right) e^{i\,(x+t)} \tag{26}$$

3.3 Gardner equation

Let us consider the Gardner equation [21, 22]

$$u_t - 6\,(u + \varepsilon^2\ u^2)\,u_x + u_{xxx} = 0 \tag{27}$$

This equation known as the mixed KdV-mKdV equation is very widely studied in various areas of Physics that includes Plasma Physics, Fluid Dynamics, Quantum Field Theory, Solid State Physics and others [22].

We introduce the transformation $\xi = k(x - \lambda t)$, where k, and λ are real constants. Equation (27) transforms to the ODE:

$$-k\lambda\, u' - 3k(u^2)' - 2\varepsilon^2 k(u^3)' + k^3\, u''' = 0 \qquad (28)$$

Integrating Eq.(28) once with zero constant to get the following ordinary differential equation:

$$\lambda\, u + 3u^2 + 2\varepsilon^2\, u^3 - k^2\, u'' = 0 \qquad (29)$$

Seeking the solution in Eq.(7)

$$\lambda\, \alpha\, cos^{\beta}(\mu\xi) + 3\alpha^2\, cos^{2\beta}(\mu\xi) + 2\varepsilon^2\, \alpha^3\, cos^{3\beta}(\mu\xi) - \alpha\, \beta(\beta-1)k^2\, \mu^2\, cos^{\beta-2}(\mu\xi) + \alpha\, \beta^2\, \mu^2\, k^2\, cos^{\beta}(\mu\xi) = 0$$

(30)

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\beta(\beta-1)(\beta-2) \neq 0$$

$$3\beta = \beta - 2 \ \rightarrow \beta = -1 \qquad (31)$$

Substituting Eq. (31) into Eq. (30) to get:

$$\lambda\, \alpha\, cos^{-1}(\mu\xi) + 3\alpha^2\, cos^{-2}(\mu\xi) + 2\varepsilon^2\, \alpha^3\, cos^{-3}(\mu\xi) - 2\, \alpha\, k^2\, \mu^2\, cos^{-3}(\mu\xi) + \alpha\, \mu^2\, k^2\, cos^{-1}(\mu\xi) = 0 \qquad (32)$$

Equating the exponents and the coefficients of each pair of the cosine function, we obtain a system of algebraic equations:

$$\cos^{-3}(\mu\xi) \ : 2\varepsilon^2\, \alpha^3 - 2\, \alpha\, k^2\, \mu^2 = 0$$

$$\cos^{-2}(\mu\xi) \ : \ 3\, \alpha^2 = 0$$

$$\cos^{-1}(\mu\xi) \ : \ \lambda\, \alpha + \alpha\, \mu^2\, k^2 = 0 \qquad (33)$$

By solving the algebraic system (34), we get,

$$\beta = -1\, , \ \lambda = -\mu^2\, k^2\, , \qquad \alpha = \mp \frac{k\ \mu}{\varepsilon} \qquad (34)$$

Then by substituting Eq. (34) into Eq. (7) , the exact soliton solution of Eq.(29)be in the form

$$u(x,t) = \mp\frac{k\ \mu}{\varepsilon}\ \ sec\ (\mu\, k(x + \mu^2\, k^2 t)\,)\ ,0 < \mu\, k(x + \mu^2\, k^2 t) < \pi \quad (35)$$

For , $\mu = k = \varepsilon = 1$, then Eq.(35) becomes:

$$u(x,t) = sec\ (x + t) \quad ,0 < (x + t) < \pi \quad (36)$$

3.4 Dispersive equation

Consider the (1+1)-dimensional nonlinear dispersive equation [23]:

$$u_t - \delta\, u^2\, u_x + u_{xxx} = 0 \quad (37)$$

where δ is a nonzero positive constant. This equation is called the modified KdV equation Elsayed et al [23], which arises in the process of understanding the role of nonlinear dispersion and in the formation of structures like liquid drops, and it exhibits compaction solitons with compact support. To find the traveling wave solutions of Eq.(37), He et al [24] used the Exp-function method, and [23] used G'/G expansion Method.

Let us now solve Eq.(37) by the proposed method. We introduce the transformation $\xi = k(x - \lambda t)$, where k, and λ are real constants. Equation (37) transforms to the ODE:

$$-k\lambda\, u' - \frac{\delta}{3} k(u^3)' + k^3\, u''' = 0 \quad (38)$$

Integrating Eq.(38) once with zero constant to get the following ordinary differential equation:

$$\lambda\, u + \frac{\delta}{3} u^3 - k^2\, u'' == 0 \quad (39)$$

Seeking the solution in Eq.(7)

$$\lambda\, \alpha\, cos^{\beta}(\mu\xi) + \frac{\delta}{3}\alpha^3\, cos^{3\beta}(\mu\xi) - \alpha\, \beta(\beta-1)k^2\, \mu^2\;\; cos^{\beta-2}(\mu\xi) + \alpha\, \beta^2\, \mu^2\, k^2\, cos^{\beta}(\mu\xi) = 0 \tag{40}$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$3\beta = \beta - 2 \;\; \rightarrow \beta = -1$$

$$\cos^{-3}(\mu\xi) \;\; : \;\; \frac{\delta}{3}\alpha^3 \; - \; 2\,\alpha\, k^2\, \mu^2 = 0$$

$$\cos^{-1}(\mu\xi) \;\; : \lambda\, \alpha + \alpha\, \mu^2\, k^2 = 0 \tag{41}$$

By solving the algebraic system (41), we get,

$$\beta = -1\,, \;\; \lambda = -\mu^2\, k^2\,, \quad \alpha = \mp\sqrt{\frac{6}{\delta}}\;\, k\;\, \mu \tag{42}$$

Then by substituting Eq. (42) into Eq. (7) , the exact soliton solution of Eq.(37) can be written in the form

$$u(x,t) = \mp\sqrt{\frac{6}{\delta}}\;\, k\;\, \mu\;\;\, sec\,(\mu\, k(x + \mu^2\, k^2 t)\,)\;, \qquad 0 < \mu\, k(x + \mu^2\, k^2 t) < \pi \tag{43}$$

3.5 Perturbed Burgers equation

In this section the study is going to be focused on the perturbed Burgers equation [25]. The solitary wave ansatz method will be adopted to obtain the exact 1-soliton solution of the Burgers equation in (1+1) dimensions. The search is going to be for a topological 1-soliton solution. The perturbed Burgers equation that is given by the following form [25]:

$$u_t + a\, u\, u_x \; + b\, u_{xx} = c\, u^2\, u_x + \beta\, u\, u_{xx} + \gamma\,(\, u_x\,)^2 + \delta\;\, u_{xxx} \tag{44}$$

Eq. (44) appears in the study of gas dynamics and also in free surface motion of waves in heated fluids. The perturbation terms are obtained from long-wave perturbation theory. Eq. (44) shows up in the long-wave small-amplitude limit of extended systems dominated by dissipation, where dispersion is also present at a higher order [25].

To solve Eq.(44) by the proposed method. We introduce the transformation $\xi = k(x - \lambda t)$, where k, and λ are real constants. Equation (44) transforms to the ODE:

$$-\lambda k\, u' + \; a\, k\, u\, u' \; + b\, k^2\, u'' = \; c\, k\, u^2\, u' + \;\; d\;\, k^2\, u\, u'' + \gamma\, k^2\, (u')^2 + \delta\;\, k^3\, u''' \tag{45}$$

Seeking the solution in Eq.(7)

$$\begin{aligned}
&\lambda\, \alpha\, \beta\, \mu\, cos^{\beta-1}(\mu\xi)\, sin(\mu\xi) - \\
&a\, \alpha^2\, \beta\, \mu\;\, cos^{2\beta - 1}(\mu\xi)\;\, sin(\mu\xi) + \\
&b\, k\, \alpha\, \beta(\beta - 1)\, \mu^2\;\, cos^{\beta - 2}(\mu\xi) - b\, k\, \alpha\, \beta^2\, \mu^2\;\, cos^{\beta}(\mu\xi)\; + \\
&c\, \alpha^3\, \beta\, \mu\;\, cos^{3\beta - 1}(\mu\xi)\;\, sin(\mu\xi)\; - \\
&d\, k\, \alpha^2\, \beta(\beta - 1)\mu^2\;\, cos^{2\beta - 2}(\mu\xi) + dk\, \alpha^2\, \beta^2\, \mu^2\;\, cos^{2\beta}(\mu\xi) - \\
&\gamma\, k\, \alpha^2\;\, \beta^2\, \mu^2\;\, cos^{2\beta - 2}(\mu\xi) + \gamma\, k\, \alpha^2\;\, \beta^2\, \mu^2\;\, cos^{2\beta}\;(\mu\xi) + \\
&\alpha\, \beta(\beta - 1)(\beta - 2)\mu^3\, \delta\;\, k^2\, cos^{\beta - 3}(\mu\xi)\, sin(\mu\xi) - \\
&\alpha\, \beta^3\, \mu^3\, \delta\;\, k^2\, cos^{\beta - 1}(\mu\xi)\;\, sin(\mu\xi)\, = 0
\end{aligned} \tag{46}$$

From (46), equating exponents $2\beta - 2$ and $3\beta - 1$ yield

$$2\beta - 2 = 3\beta - 1 \tag{47}$$

so that

$$\beta = -1 \tag{48}$$

It needs to be noted that the same value of β is obtained when the exponent pairs $\beta - 2 = 2\beta - 1$, $2\beta - 2 = \beta - 3$ are equated , Thus setting their coefficients to zero yields:

$$-d\, k\, \alpha^2\, \beta(\beta-1)\mu^2 - \gamma\, k\, \alpha^2\, \beta^2\, \mu^2 \quad + \ \alpha\, \beta(\beta-1)(\beta - 2)\mu^3\, \delta\ \ k^2 = 0$$

$$b\, k\, \alpha\, \beta(\beta-1)\, \mu^2 \quad - \ a\, \alpha^2\, \beta\, \mu = 0 \tag{49}$$

$$(dk + \gamma\, k\,)\alpha\, \beta\, \mu + \ \lambda - \beta^2\, \mu^2\, \delta\ \ k^2 \quad = 0$$

By solving the algebraic system (49), we get,

$$\delta = \frac{(2d\ +\gamma)\mathrm{b}}{3\ \mathrm{a}} \quad , \quad \alpha = -\frac{2\, b\, k}{a} \quad \mu \quad ,$$
$$\lambda = [4d\ -5\,\gamma\,]\frac{\mathrm{b}}{3\mathrm{a}} \quad k^2\mu^2$$

Then by substituting Eq. (49) into Eq. (7) , the exact soliton solution of equation (44) can be written in the form

$$u(x,t) = -\frac{2\, b\, k}{a} \quad \mu \quad sec\,[\,\mu k(x - [4d\ -5\,\gamma\,]\frac{b}{3a} \quad k^2\mu^2 t)] \tag{50}$$

3.6 The general Burgers-Fisher equation

Consider the following general Burger’s-Fisher equation [26]:

$$u_t - a\, u^n\, u_x \ + b\, u_{xx} + c\, u\,(1 - \ u^n) = 0 \tag{51}$$

where a, b and c are nonzero constants. We introduce the transformation $\xi = k(x - \lambda t)$, where k, and λ are real constants. The traveling wave variable ξ permits us converting Eq. (51) into the following ODE:

$$-\lambda k u' + a\, k\, u^n\, u' \ + b\, k^2 u'' + c\, u - c\, u^{n+1} = 0 \tag{52}$$

Seeking the solution in Eq.(7)

$$\lambda k\ \ \alpha\, \beta\, \mu\ \ cos^{\beta\, -\, 1}(\mu\xi)\ \ sin(\mu\xi) -$$
$$a\, k\, \alpha^{n+1}\, \beta\, \mu\ \ cos^{(n\, +\, 1)\beta\, -\, 1}(\mu\xi)\ \ sin(\mu\xi) + b\, k^2\alpha\, \beta(\beta -$$

$$1)\ \mu^2 cos^{\beta-2}(\mu\xi) - [b\,k^2\alpha\,\beta^2\,\mu^2 - c\,\alpha]\,cos^{\beta}(\mu\xi) - c\ \alpha^{n+1}cos^{(n+1)\beta}(\mu\xi) = 0 \qquad (53)$$

From Eq.(53), equating exponents $(n+1)\beta$ and $\beta-1$ yield

$$(n+1)\beta = \beta - 1 \qquad (54)$$

so that

$$\beta = \frac{-1}{n} \qquad (55)$$

when the exponent pair $(n+1)\beta - 1 = \beta - 2$, is equated gave the same value of $\beta = \frac{-1}{n}$, Thus setting their coefficients to zero yields:

$$c\ \alpha^{n+1} + \lambda k\ \alpha\,\beta\,\mu = 0$$

$$b\,k^2\alpha\,\beta(\beta-1)\,\mu^2 - a\,k\,\alpha^{n+1}\,\beta\,\mu = 0 \qquad (56)$$

By solving the algebraic system (49), we get,

$$\lambda = -\frac{b\,c\,(n+1)}{a},\ \alpha = \left(\frac{b\ (n+1)}{a\,n} k\,\mu\right)^{\frac{1}{n}} \qquad (57)$$

Then by substituting Eq. (57) into Eq. (7) , the exact soliton solution of equation (51) can be written in the form

$$u(x,t) = \left[\frac{b(n+1)}{a\,n}\ k\,\mu\ \ sec\ \left(\mu k\left(x + \frac{bc(n+1)}{a} t\right)\right)\right]^{\frac{1}{n}} \qquad (58)$$

3.7 Cubic modified Boussinesq equation

Consider the cubic modified Boussinesq equation [27],

$$u_{tt} + u_{xxt} + \frac{2}{9} u_{xxxx} - (u^3)_{xx} = 0 \qquad (59)$$

Mohamed et al [27] tried to solve Eq.(59) by applied the Homotopy Perturbation method and Padé approximants. Eq.(59) has an exact solution[27],

$$u(x,t) = 1 + \tanh\frac{3}{2}(x - 2t) \tag{60}$$

The traveling wave hypothesis as given by

$$\xi = kx - \lambda t \tag{61}$$

The nonlinear partial differential equation (60) is carried to an ordinary differential equation

$$\lambda^2 \ U'' - k^2\,\lambda\, U''' + \frac{2}{9}\ k^4\, U'''' - 3\,k^2\,(U^2U')' = 0 \tag{62}$$

Integrating Eq.(62) twice with zero constant, Eq.(62) reduces to

$$\lambda^2 \ U - k^2\,\lambda\, U' + \frac{2}{9}\ k^4\, U'' - \ k^2\, U^3 = 0 \tag{63}$$

Applying Sine-cosine method to solve Eq.(63), and seeking the solution in (16) then

$$\lambda^2 \ \alpha\, cos^{\beta}(\mu\xi) + k^2\,\lambda \ \ \alpha\,\beta\,\mu\ \ cos^{\beta-1}(\mu\xi)\ \ sin(\mu\xi)\ + \frac{2}{9}\,k^4\,\alpha\,\beta(\beta-1)\,\mu^2\ \ cos^{\beta-2}(\mu\xi) - \alpha\,\beta^2\,\mu^2\ \ cos^{\beta}(\mu\xi) - k^2\,\alpha^3\,cos^{3\beta}(\mu\xi) = 0 \tag{64}$$

Equating the exponents and the coefficients of each pair of the cosine functions we find the following algebraic system:

$$\begin{gathered}\beta - 2 = 3\beta \text{ , then } \beta = -1 \\ \lambda^2\ \alpha - \alpha\,\beta^2\,\mu^2 \quad = 0 \\ \frac{2}{9}\,k^4\,\alpha\,\beta(\beta-1)\,\mu^2 - \ k^2\,\alpha^3\ = 0\end{gathered} \tag{65}$$

By solving the algebraic system (32), we get,

$$\lambda = \mp\,\mu \text{ , } \alpha = \frac{2}{3}\ k\ \ \mu \tag{66}$$

Then by substituting Eq.(66) into Eq. (16) then, the exact soliton solution of equation (59) can be written in the form:

$$u_{1,2}(x,t) = \frac{2}{3}\ k\ \ \mu\quad sec(\,\mu\,(kx \mp\ \mu t)) \tag{67}$$

for $k = \frac{3}{2}$, $\mu\ = 1$, then:

$$u_{1,2}(x,t) = \ sec\ \left(\frac{3}{2}x \mp t\right) \tag{68}$$

Figure (1) represents the soliatry of the solution $u_2(x,t) = sec\left(\frac{3}{2}x - t\right)$ at $-10 < x < 10$, and $0 < t < 1$.

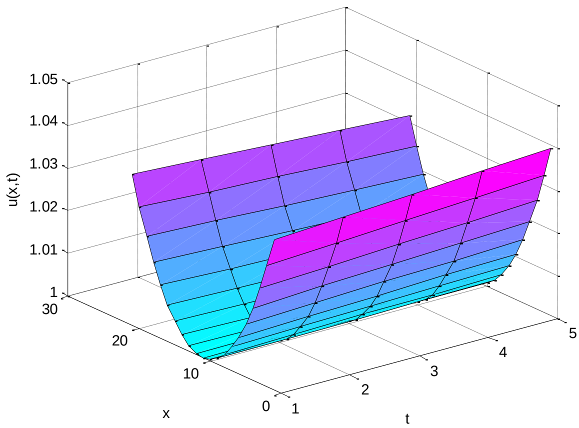

Fig. (1)

3.8 Cubic modified Boussinesq equation

Consider the cubic modified Boussinesq equation[27],

$$u_{tt} - u_{xxxx} - (u^3)_{xx} = 0 \tag{69}$$

Mohamed [27] tried to solve Eq.(69) by applied the Homotopy Perturbation method and Padé approximants. The exact solution of Eq.(69) is;

$$u(x,t) = \sqrt{2}\,\mathrm{sech}(x-t) \tag{70}$$

The nonlinear partial differential equation (69) is carried to an ordinary differential equation using the transformation

$$\xi = kx - \lambda t \tag{71}$$

then

$$\lambda^2\ U'' - k^4\ U'''' - 3\ k^2\ (U^2U')' = 0 \tag{72}$$

Integrating Eq.(72) twice and assuming the constant of integration equal to zero, then

$$\lambda^2\ U - k^4\ U'' - \ k^2\ U^3 = 0 \tag{73}$$

By applying Sine-cosine method to solve Eq.(73), and seeking the solution in (16) then

$-\lambda^2 \alpha \beta \mu \cos^{\beta-1}(\mu\xi) \sin(\mu\xi) + k^4 [\alpha \beta(\beta-1)(\beta-2)\mu^3 \cos^{\beta-3}(\mu\xi) \sin(\mu\xi) - \alpha \beta^3 \mu^3 \cos^{\beta-1}(\mu\xi) \sin(\mu\xi)] + 3 k^2 \alpha^3 \beta \mu \cos^{3\beta-1}(\mu\xi) \sin(\mu\xi) = 0$ (74)

Equating the exponents and the coefficients of each pair of the sine functions we find the following algebraic system:

$\beta - 3 = 3\beta - 1$, then $\beta = -1$

$$-\lambda^2 \alpha \beta \mu - \alpha \beta^3 \mu^3 k^4 = 0$$

$$k^4 \alpha \beta(\beta-1)(\beta-2)\mu^3 + 3 k^2 \alpha^3 \beta \mu = 0 \quad (75)$$

By solving the algebraic system (75), we get,

$\lambda = \mp i k^2 \mu , \quad \alpha = \mp i\sqrt{2} k \mu$ (76)

Then by substituting Eq.(76) into Eq. (15) then, the exact soliton solution of equation (69) can be written in the form:

$u(x,t) = \mp i\sqrt{2} k \mu \ sec(\mu k(x \pm i k \mu t))$ (77)

or

$u(x,t) = \mp \sqrt{2} k \mu \ \mathrm{sech}(\mu k(ix \mp k \mu t))$ (78)

for $k = \mu = 1$, Eq.(78) becomes

$u(x,t) = \mp \sqrt{2} \ \mathrm{sech}(ix \mp t)$ (79)

4. Conclusion

The sine-cosine function method has been implemented to find the solution for nonlinear partial differential equations. New exact solutions were found by the proposed method. Thus, we can say that the proposed method can be extended to solve the problems of nonlinear partial differential equations which arising in the theory of solitons and other areas.

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طريقة دالة الجيب – الجيب تمام للحلول التامة للمعادلات التفاضلية الجزئية وغير الخطية

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المستخلص:

في هذا البحث تم الحصول على الحلول التامة للمعادلات التفاضلية الجزئية وغير الخطية باستخدام طريقة دالة الجيب – الجيب تمام. تم تطبيق طريقة الحل للحصول على الحلول التامة لعدد من المعادلات التفاضلية الحزئية وغير الخطية مثل معادلة $K(n + 1, n + 1)$ ومعادلة شرودنجر-هيروتا و معادلة كاردنر ومعادلة KdV المعدلة و معادلة بيرجرالقلقة ومعادلة بيرجر-فيشر العامة ومعادلة بوسنيسك الثلاثية المعدلة ، وهذه المعادلات مهمة ومشهورة في مجال معادلات الموجة المنعزلة.