Classification of $(\mathrm{k};3)$-arcs in $\mathrm{PG}(2,13)$ for $\mathrm{k}=5,6,7$

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Abstract: *An arc of size k and degreethree is a set of k points such that no four of them are collinear but some three of them are collinear. The aim of this paper is to determine the inequivalent $(k;3)$-arcs in $PG(2,13)$for $k=5,6,7$.*

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Introduction

The study of $(k;r)$-arc for $r \geq 3$ in $PG(2,q)$, the projective plane over Galois field $GF(q)$, as begun by Barlotti [1] in 1956.The case $r = 2$ was considered previously by Segre [2], [3].Recently; the case $r = 3$ has been studied by Marcugini [4] for $q = 7$.For $q = 11$,a maximal $(k;3)$-arcs has been found by Marcugini [5] and a full classification of $(k;3)$-arcs are given by Cook [6]. For$q = 13$, in [7] the maximal arc has been found. In [8] some new arcs of higher degree are givenin $PG(2,13)$.

In this paper, the inequivalent, incomplete $(k;3)$-arcsin $PG(2,13)$ for $k = 5,6,7$are classified and their stabilizer group type are determined.

Definition 1[9]:A$(k;r)$-arc Kin the projective plane $PG(2,q)$ is a set of k points of$PG(2,q)$ such that no $r+1$ points are collinear and some r points are collinear. A $(k;r)$-arc is complete if there is no $(k+1;r)$-arc containing it.If $r = 2$, then $K$is called ak-arc. The size of the largest $(k;r)$-arc in $PG(2,q)$is denoted$m_r(2,q)$.

Let Π denote any projective plane. A line ℓ of Π is an i-secant of a$(k;r)$-arc Kif $|\ell \cap K| = i$. Let$\mathcal{T}_i$denote the total number of i-secants to Kin Π, let$m_i = m_i(P)$ the number of i-secants through a point $P$of $K$and $\sigma_i = \sigma_i(Q)$ thenumber of i-secants through a point Qin $\Pi \backslash K$. LetK_{m_1,m_2,m_3}be set ofpoints of K with parameters $m_1, m_2$and m_3.

Lemma 1[9]: For a $(k;r)$-arc K, the following equations hold:

(i) $\sum_{i=1}^{r} m_i = q + 1$;

(ii) $\sum_{i=2}^{r} (i-1) m_i = k - 1$;

(iii) $\sum_{i=0}^{r} \sigma_i = q + 1$;

(iv) $\sum_{i=1}^{r} i\sigma_i = k$;

(v) $\sum_{i=0}^{r} \mathcal{T}_i = q^2 + q + 1$;

(vi) $\sum_{i=1}^{r} i\mathcal{T}_i = k(q+1)$;

Remark 1[9]: Let K be a $(k;r)$-arc in Π with stabilizer group G_K. The group G_K partitions the points of $\Pi\backslash K$ into different orbits. So, distinct $(k+1;r)$-arcs formed by adding one point from each orbit to K.

$(5;3)$-arcs

Let $\mathcal{S} = \{x_1, x_2, x_3, x_4, x_5\}$ be a set of five points that is contained in a $(k;3)$-arc, where $k \geq 5$. Then, some subset of four points in $\mathcal{S}$ is a 4-arc. Since a $(k;3)$-arc contains a 4-arc if $k \geq 5$ and the standard frame

$$\Upsilon_4 = \{e_1 = [1;0;0], e_2 = [0;1;0], e_3 = [0;0;1], e_4 = [1;1;1]\}$$

is projectively equivalent to every 4-arc, so Υ_4 is used as the base on which all $(r \geq 5;3)$-arcs are constructed.

Let $\mathcal{P}$ be a $(5;3)$-arc in $PG(2,13)$. Suppose that through a point P of $\mathcal{P}$ there pass m_i i-secant $(i = 1,2,3)$. From equation (i) and (ii) in Lemma 1 the following hold:

$$\left.\begin{aligned} m_1 + m_2 + m_3 &= 14 \\ m_2 + 2m_3 &= 4 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Let α, β, γ denote the number of points of $\mathcal{P}$ with respective m_i. The possible solutions for equation (1) are the following:

Table 1: Kind points of $\mathcal{P}$

Kind	m_1	m_2	m_3
α	10	4	0
β	11	2	1
γ	12	0	2

To determined $\mathcal{T}_i$, from equation and Table 1, the following are satisfied.

(i) $\mathcal{T}_3 = \frac{2\gamma+\beta}{3}$;

(ii) $\mathcal{T}_2 = 2\alpha + \beta$;

(iii) $\mathcal{T}_1 = 50 + (\beta + 2\gamma)$;

(iv) $\mathcal{T}_0 = 183 - (\mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3)$.

Since $\alpha + \beta + \gamma = 5$ and 3divided $2\gamma + \beta$, so the maximum value for$2\gamma + \beta$ is 10; that is, $2\gamma + \beta$ could be 0,3,6,9. According to this, the possible value of $\mathcal{T}_i$are the following.

Table 2: The possible values of $\mathcal{T}_i$

$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$
122	53	7	1
121	56	4	2
120	59	1	3

Remark2: The case $\mathcal{T}_3 = 3$ is not satisfied, because in this case the minimum value of the size of $\mathcal{P}$ is six.

For a point Q not on $\mathcal{P}$, the possible value of σ_i can be deduce from equations in Lemma 2[(iii),(iv)] is given in Table 3.

Table 3: The possible values of σ_i

Type	σ_0	σ_1	σ_2	σ_3
(i)	9	5	0	0
(ii)	10	3	1	0
(iii)	11	1	2	0
(iv)	11	2	0	1
(v)	12	0	1	1

Let $\mathcal{P}(s_1, s_2, s_3, s_4, s_5) = [s_1, s_2, s_3, s_4, s_5]$where $s_j$denote the number of points in $\Pi \backslash \mathcal{P}$ of type (j), j= i, ii, iii, iv, v.

Inequivalent $(5;3)$-arcs

Let $\mathcal{L}_1 = \mathcal{L}_1(x_0, x_1, x_2)$ be the line with equation $x_1 = x_2$which pass through e_1, e_4 and $\mathcal{L}_2 = \mathcal{L}_2(x_0, x_1, x_2)$be the line with equation $x_0 = 0$ which pass through e_2, e_3. To construct a $(5;3)$-arcs$\mathcal{P}_i$, the line$\mathcal{L}_1$has beenchosen to be the 3-secant of$\mathcal{P}_i$. So, there are twelve $(5;3)$-arcs.

(i) A $(5;3)$-arc with$\mathcal{T}_3 = 1$.Apart from the points e_1, e_4andP $= \mathcal{L}_1 \cap \mathcal{L}_2$, there are eleven other points on$\mathcal{L}_1$. Amongst these eleven$(5;3)$-arcs with $\mathcal{T}_3 = 1$ there are only three inequivalent as given bellow.

$$\mathcal{P}_1 = \{e_1, e_2, e_3, e_4, a_1 = [\nu^4; 1; 1]\};$$
$$\mathcal{P}_2 = \{e_1, e_2, e_3, e_4, a_2 = [\nu; 1; 1]\};$$
$$\mathcal{P}_3 = \{e_1, e_2, e_3, e_4, a_3 = [\nu^{10}; 1; 1]\},$$

Where ν is the primitive elements in the Galois field $GF(13) = F_{13}$.

Let partition$\mathcal{P}_i, i = 1,2,3$ into two subsets $\mathfrak{V}_1 = \{e_2, e_3\}$ and $\mathfrak{V}_2 = \{e_1, e_4, a_i\}$,
$i = 1,2,3$. Through each point of $\mathfrak{V}_1$ no 3-secant pass from it and through each point of $\mathfrak{V}_2$ pass one 3-secant, so $\mathcal{P}_{i_{10,4,0}} = \mathfrak{V}_1$and $\mathcal{P}_{i_{11,2,1}} = \mathfrak{V}_2$, $i = 1,2,3$. The projective group $G_{\mathcal{P}_i}$ of $\mathcal{P}_i, i = 1,2,3$ is the group of projectivities fixing $\mathfrak{V}_1$and $\mathfrak{V}_2$.

(ii) The $(5;3)$-arc with$\mathcal{T}_3 = 2$. From (i), the there is only one possibilityof $(5;3)$-arc with $\mathcal{T}_3 = 2$ which is form from Υ_4 with $a_4 = \mathcal{L}_1 \cap \mathcal{L}_2$; that is,

$$\mathcal{P}_4 = \{e_1, e_2, e_3, e_4, a_4 = [0; 1; 1]\}.$$

Remark3 [9]: In the projective plane, any ordered set of four independent points no three are collinear they are projectively equivalent. So, the uniqueness of the $(5;3)$-arc with$\mathcal{T}_3 = 2$ can be deduce. The points of the $(5;3)$-arc $\mathcal{P}_4$, partition into two kinds$\mathcal{P}_{4_{12,0,2}} = \{a_4\}and\mathcal{P}_{4_{11,2,1}} = \{e_1, e_2, e_3, e_4\}$.

Theorem 1: In$PG(2,13)$, there are four inequivalent, incomplete five arcs of degree three partition into two different types $\mathcal{P}_i(s_1, s_2, s_3, s_4, s_5)$, as summarized in Table 4.

Table 4: Full details of $(5;3)$-arcs in PG$(2,13)$

		Number of points of $\mathcal{P}_i$ of each type				
	$[\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3]$	α	β	γ	$\mathcal{P}_i(s_1, s_2, s_3, s_4, s_5)$	Stabilizer
$\mathcal{P}_1$	[122,53,7,1]	2	3	0	[90, 71, 6, 10, 1]	$\boldsymbol{Z_2}$
$\mathcal{P}_2$	[122,53,7,1]	2	3	0	[90, 71, 6, 10, 1]	$\boldsymbol{Z_2 \times Z_2}$
$\mathcal{P}_3$	[122,53,7,1]	2	3	0	[90, 71, 6,10, 1]	$\boldsymbol{Z_6}$
$\mathcal{P}_4$	[121,56,4,2]	0	4	1	[110,44, 2,22, 0]	$\boldsymbol{D_4}$

Inequivalent$(6;3)$-arcs

Let $\mathcal{H}$ be a $(6;3)$-arc in $PG(2,13)$ and P be a point in $\mathcal{H}$. From equations (i) and (ii) in Lemma 2 the following are satisfied.

$$\left.\begin{aligned} m_1 + m_2 + m_3 &= 14 \\ m_2 + 2m_3 &= 5 \end{aligned}\right\} \cdots\cdots \quad (2)$$

Let α, β, γ denote the number of points of $\mathcal{H}$ with respective m_i. The possible solutions for equation (2) are the following:

Table 5: Kind points of $\mathcal{H}$

Kind	m_1	m_2	m_3
α	9	5	0
β	10	3	1
γ	11	1	2

The possible value of $\mathcal{T}_i$ and σ_i, $i = 0,1,2,3$is in Table 6 and 7.

Table 6: The possible values of $\mathcal{T}_i$

$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$
113	57	12	1
112	60	9	2
111	63	6	3
110	66	3	4

Table 7: The possible values of σ_i

Type	σ_0	σ_1	σ_2	σ_3
(i)	8	6	0	0
(ii)	9	4	1	0
(iii)	10	2	2	0
(iv)	10	3	0	1
(v)	11	1	1	1
(vi)	11	0	3	0
(vii)	12	0	0	2

Theorem 2: In$PG(2,13)$, there are 62 inequivalent, incomplete six arcs of degree three. These 62 arcs$\mathcal{H}_i$ are partition into 14 different types$\mathcal{H}_i(s_1, s_2, s_3, s_4, s_5, s_6, s_7)$ as given in Table 8.

Table 8: Details of inequivalent $(6;3)$-arcs

No	$\mathcal{H}_i(s_1,s_2,s_3,s_4,s_5,s_6,s_7)$	$\hat{n}$	$[\mathcal{T}_0,\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3]$	$n{:}\,G$	α	β	γ
1	[48, 99, 15, 8, 4, 3, 0]	1	[113, 57, 12, 1]	$1{:}\,\boldsymbol{S_3}$	3	3	0
2	[49, 96, 18, 8, 3, 3, 0]	1	[113, 57, 12, 1]	$1{:}\,\boldsymbol{Z_3}$	3	3	0
3	[50, 93, 21, 8, 2, 3, 0]	4	[113, 57, 12, 1]	$4{:}\,\boldsymbol{Z_2}$	3	3	0
4	[51, 90, 24, 8, 1, 3, 0]	11	[113, 57, 12, 1]	$11{:}\,\boldsymbol{I}$	3	3	0
5	[52, 87, 27, 8, 0, 3, 0]	14	[113, 57, 12, 1]	$14{:}\,\boldsymbol{I}$	3	3	0
6	[63, 81, 9, 20, 3, 0, 1]	1	[112, 60, 9, 2]	$1{:}\,\boldsymbol{S_3}\times\boldsymbol{Z_3}$	0	6	0
7	[64, 78, 12, 20, 2, 0, 1]	1	[112, 60, 9, 2]	$1{:}\,\boldsymbol{D_4}$	0	6	0
8	[64, 79, 11, 18, 1, 4, 0]	5	[112, 60, 9, 2]	$5{:}\,\boldsymbol{Z_2}$	1	4	1
9	[65, 75, 15, 20, 1, 0, 1]	1	[112, 60, 9, 2]	$1{:}\,\boldsymbol{Z_2}$	0	6	0
10	[65, 76, 14, 18, 0, 4, 0]	13	[112, 60, 9, 2]	$10{:}\,\boldsymbol{I}2{:}\,\boldsymbol{Z_2}1{:}\,\boldsymbol{Z_4}$	1	4	1
11	[66, 72, 18, 20, 0, 0, 1]	3	[112, 60, 9, 2]	$1{:}\,\boldsymbol{Z_2}1{:}\,\boldsymbol{Z_3}1{:}\,\boldsymbol{Z_6}$	0	6	0
12	[80, 60, 3, 30, 1, 3, 0]	1	[111, 63, 6, 3]	$1{:}\,\boldsymbol{S_3}$	0	3	3
13	[81, 57, 6, 30, 0, 3, 0]	5	[111, 63, 6, 3]	$5{:}\,\boldsymbol{Z_3}$	0	3	3
14	[100, 30, 3, 44, 0, 0, 0]	1	[110, 66, 3, 4]	$1{:}\,\boldsymbol{S_4}$	0	0	6

In Table 8, a cell $n{:}\,G$ means that n is the number of $(6;3)$-arcs of stabilizer group isomorphic to G and a symbol $\hat{n}$means the number $(6;3)$-arc of type $\mathcal{H}_i(s_1,s_2,s_3,s_4,s_5,s_6,s_7)$.

Corollary1: From the five different kinds of $\alpha,\ \beta,\ \gamma$ as given in Table 8 the following holds.

1- If $= 3,\ \beta = 3,\ \gamma = 0$, then $\mathcal{H}_i = \mathcal{H}_{i_{9,5,0}} \cup \mathcal{H}_{i_{10,3,1}}$.

2- If $\alpha = 0,\ \beta = 6,\ \gamma = 0$, then $\mathcal{H}_i = \mathcal{H}_{i_{10,3,1}}$.

3- If $\alpha = 1,\ \beta = 4,\ \gamma = 1$, then $\mathcal{H}_i = \mathcal{H}_{i_{9,5,0}} \cup \mathcal{H}_{i_{10,3,1}} \cup \mathcal{H}_{i_{11,1,2}}$.

4- If $\alpha = 0,\ \beta = 3,\ \gamma = 3$, then $\mathcal{H}_i = \mathcal{H}_{i_{10,3,1}} \cup \mathcal{H}_{i_{11,1,2}}$.

5- If $\alpha = 0,\ \beta = 0,\ \gamma = 6$, then $\mathcal{H}_i = \mathcal{H}_{i_{11,1,2}}$.

Inequivalent$(7;3)$-arcs

Let $\mathcal{J}$ be a $(7;3)$-arc in $PG(2,13)$ and P be a point in $\mathcal{J}$. From equations (i) and (ii) in Lemma 2 the following are satisfied.

$$\left.\begin{aligned} m_1 + m_2 + m_3 &= 14 \\ m_2 + 2m_3 &= 6 \end{aligned}\right\} \cdots\cdots\cdots\cdots \quad (3)$$

Let $\alpha, \beta, \gamma, \delta$ denote the number of points of $\mathcal{J}$ with respective m_i. The possible solutions for equation (3) are the following:

Table 9: Kind points of $\mathcal{J}$

Kind	m_1	m_2	m_3
α	8	6	0
β	9	4	1
γ	10	2	2
δ	11	0	3

The possible value of $\mathcal{T}_i$ and $\sigma_i,\ i = 0,1,2,3$ is in Table 10 and 11.

Table 10: The possible values of $\mathcal{T}_i$

$\mathcal{T}_0$	$\mathcal{T}_1$	$\mathcal{T}_2$	$\mathcal{T}_3$
105	59	18	1
104	62	15	2
103	65	12	3
102	68	9	4
101	71	6	5
100	74	3	6

Table 11: The possible values of σ_i

Type	σ_0	σ_1	σ_2	σ_3
(i)	7	7	0	0
(ii)	8	5	1	0
(iii)	9	3	2	0
(iv)	9	4	0	1
(v)	10	1	3	0
(vi)	10	2	1	1
(vii)	11	0	2	1
(viii)	11	1	0	2

Theorem 3: In$PG(2,13)$, there are 1349 inequivalent, incomplete seven arcs of degree three. These 1349 arcs $\mathcal{J}_i$ are partition into 99 different types$\mathcal{J}_i(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8)$as given in Table 12.

Table 12: Details of inequivalent $(7;3)$-arcs

	$\mathcal{J}_i(s_1,s_2,s_3,s_4,s_5,s_6,s_7,s_8)$	$\hat{n}$	$[\mathcal{T}_0,\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3]$	$n{:}\,G$	α β γ δ
1	[18, 94, 43, 7, 10, 2, 2, 0]	1	[105, 59, 18, 1]	$1{:}\,\boldsymbol{Z_2}$	4 3 0 0
2	[18, 95, 41, 6, 11, 4, 1, 0]	1	[105, 59, 18, 1]	$\mathbf{1}{:}\,\boldsymbol{I}$	4 3 0 0
3	[19, 92, 44, 6, 10, 4, 1, 0]	3	[105, 59, 18, 1]	$\mathbf{2}{:}\,\boldsymbol{I}$ $\mathbf{1}{:}\,\boldsymbol{Z_2}$	4 3 0 0
4	[19, 93, 42, 5, 11, 6, 0, 0]	1	[105, 59, 18, 1]	$\mathbf{1}{:}\,\boldsymbol{I}$	4 3 0 0
5	[20, 88, 49, 7, 8, 2, 2, 0]	14	[105, 59, 18, 1]	$\mathbf{12}{:}\,\boldsymbol{Z_2}$ $\mathbf{2}{:}\,\boldsymbol{Z_4}$	4 3 0 0
6	[20, 89, 47, 6, 9, 4, 1, 0]	10	[105, 59, 18, 1]	$\mathbf{9}{:}\,\boldsymbol{I}$ $1{:}\,\boldsymbol{Z_2}$	4 3 0 0
7	[20, 90, 45, 5, 10, 6, 0, 0]	3	[105, 59, 18, 1]	$\mathbf{3}{:}\,\boldsymbol{I}$	4 3 0 0
8	[21, 86, 50, 6, 8, 4, 1, 0]	14	[105, 59, 18, 1]	$\mathbf{14}{:}\,\boldsymbol{I}$	4 3 0 0
9	[21, 87, 48, 5, 9, 6, 0, 0]	21	[105, 59, 18, 1]	$\mathbf{21}{:}\,\boldsymbol{I}$	4 3 0 0
10	[22, 82, 55, 7, 6, 2, 2, 0]	5	[105, 59, 18, 1]	$\mathbf{5}{:}\,\boldsymbol{Z_2}$	4 3 0 0
11	[22, 83, 53, 6, 7, 4, 1, 0]	18	[105, 59, 18, 1]	$\mathbf{18}{:}\,\boldsymbol{I}$	4 3 0 0
12	[22, 84, 51, 5, 8, 6, 0, 0]	34	[105, 59, 18, 1]	$\mathbf{34}{:}\,\boldsymbol{I}$	4 3 0 0
13	[23, 80, 56, 6, 6, 4, 1, 0]	28	[105, 59, 18, 1]	$\mathbf{27}{:}\,\boldsymbol{I}$ $\mathbf{1}{:}\,\boldsymbol{Z_2}$	4 3 0 0
14	[23, 81, 54, 5, 7, 6, 0, 0]	49	[105, 59, 18, 1]	$\mathbf{49}{:}\,\boldsymbol{I}$	4 3 0 0
15	[24, 76, 61, 7, 4, 2, 2, 0]	10	[105, 59, 18, 1]	$\mathbf{8}{:}\,\boldsymbol{Z_2}\mathbf{2}{:}\,\boldsymbol{Z_4}$	4 3 0 0
16	[24, 77, 59, 6, 5, 4, 1, 0]	17	[105, 59, 18, 1]	$\mathbf{16}{:}\,\boldsymbol{I}$ $\mathbf{1}{:}\,\boldsymbol{Z_2}$	4 3 0 0
17	[24, 78, 57, 5, 6, 6, 0, 0]	62	[105, 59, 18, 1]	$\mathbf{62}{:}\,\boldsymbol{I}$	4 3 0 0
18	[24, 96, 28, 16, 6, 2, 4, 0]	1	[104, 62, 15, 2]	$1{:}\,\boldsymbol{Z_2}\times\boldsymbol{Z_2}$	2 4 1 0
19	[25, 74, 62, 6, 4, 4, 1, 0]	11	[105, 59, 18, 1]	$\mathbf{8}{:}\,\boldsymbol{I}$ $\mathbf{3}{:}\,\boldsymbol{Z_2}$	4 3 0 0
20	[25, 75, 60, 5, 5, 6, 0, 0]	44	[105, 59, 18, 1]	$\mathbf{44}{:}\,\boldsymbol{I}$	4 3 0 0
21	[26, 70, 67, 7, 2, 2, 2, 0]	2	[105, 59, 18, 1]	$\mathbf{2}{:}\,\boldsymbol{Z_2}$	4 3 0 0
22	[26, 71, 65, 6, 3, 4, 1, 0]	7	[105, 59, 18, 1]	$\mathbf{5}{:}\,\boldsymbol{I}$ $\mathbf{2}{:}\,\boldsymbol{Z_2}$	4 3 0 0
23	[26, 72, 63, 5, 4, 6, 0, 0]	14	[105, 59, 18, 1]	$\mathbf{14}{:}\,\boldsymbol{I}$	4 3 0 0
24	[26, 92, 30, 14, 6, 6, 2, 0]	3	[104, 62, 15, 2]	$\mathbf{1}{:}\,\boldsymbol{I}$ $\mathbf{2}{:}\,\boldsymbol{Z_2}$	2 4 1 0
25	[27, 68, 68, 6, 2, 4, 1, 0]	1	[105, 59, 18, 1]	$\mathbf{1}{:}\,\boldsymbol{Z_2}$	4 3 0 0
26	[27, 69, 66, 5, 3, 6, 0, 0]	6	[105, 59, 18, 1]	$\mathbf{5}{:}\,\boldsymbol{I}$ $\mathbf{2}{:}\,\boldsymbol{Z_3}$	4 3 0 0
27	[27, 89, 33, 14, 5, 6, 2, 0]	4	[104, 62, 15, 2]	$\mathbf{4}{:}\,\boldsymbol{I}$	2 4 1 0
28	[27, 90, 30, 14, 8, 6, 0, 1]	2	[104, 62, 15, 2]	$\mathbf{2}{:}\,\boldsymbol{I}$	1 6 0 0
29	[27, 90, 31, 13, 6, 8, 1, 0]	3	[104, 62, 15, 2]	$\mathbf{3}{:}\,\boldsymbol{I}$	2 4 1 0
30	[28, 84, 40, 16, 2, 2, 4, 0]	2	[104, 62, 15, 2]	$1{:}\,\boldsymbol{Z_2}\times\boldsymbol{Z_2}\mathbf{1}{:}\,\boldsymbol{D_4}$	2 4 1 0
31	[28, 86, 36, 14, 4, 6, 2, 0]	17	[104, 62, 15, 2]	$\mathbf{6}{:}\,\boldsymbol{I}$ $\mathbf{11}{:}\,\boldsymbol{Z_2}$	2 4 1 0
32	[28, 87, 33, 14, 7, 6, 0, 1]	5	[104, 62, 15, 2]	$\mathbf{4}{:}\,\boldsymbol{I}$ $\mathbf{1}{:}\,\boldsymbol{Z_2}$	1 6 0 0
33	[28, 87, 34, 13, 5, 8, 1, 0]	8	[104, 62, 15, 2]	$\mathbf{8}{:}\,\boldsymbol{I}$	2 4 1 0

34	[28, 88, 32, 12, 6, 10, 0, 0]	10	[104, 62, 15, 2]	$\mathbf{4}:\boldsymbol{I}$ $\mathbf{4}:\boldsymbol{Z_2}\mathbf{1}:\boldsymbol{Z_4}$ $1:\boldsymbol{Z_2}\times\boldsymbol{Z_2}$	2 4 1 0
35	[29, 83, 39, 14, 3, 6, 2, 0]	1	[104, 62, 15, 2]	$\mathbf{1}:\boldsymbol{I}$	2 4 1 0
36	[29, 84, 36, 14, 6, 6, 0, 1]	11	[104, 62, 15, 2]	$\mathbf{9}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}\mathbf{1}:\boldsymbol{Z_3}$	1 6 0 0
37	[29, 84, 37, 13, 4, 8, 1, 0]	27	[104, 62, 15, 2]	$\mathbf{37}:\boldsymbol{I}$	2 4 1 0
38	[29, 85, 35, 12, 5, 10, 0, 0]	35	[104, 62, 15, 2]	$\mathbf{33}:\boldsymbol{I}$ $\mathbf{2}:\boldsymbol{Z_2}$	2 4 1 0
39	[30, 80, 42, 14, 2, 6, 2, 0]	13	[104, 62, 15, 2]	$\mathbf{6}:\boldsymbol{I}$ $\mathbf{7}:\boldsymbol{Z_2}$	2 4 1 0
40	[30, 81, 39, 14, 5, 6, 0, 1]	28	[104, 62, 15, 2]	$\mathbf{25}:\boldsymbol{I}$ $\mathbf{3}:\boldsymbol{Z_2}$	1 6 0 0
41	[30, 81, 40, 13, 3, 8, 1, 0]	50	[104, 62, 15, 2]	$\mathbf{50}:\boldsymbol{I}$	2 4 1 0
42	[30, 82, 38, 12, 4, 10, 0, 0]	69	[104, 62, 15, 2]	$\mathbf{25}:\boldsymbol{I}$ $\mathbf{8}:\boldsymbol{Z_2}$ $1:\boldsymbol{Z_2}\times\boldsymbol{Z_2}$	2 4 1 0
43	[31, 77, 45, 14, 1, 6, 2, 0]	1	[104, 62, 15, 2]	$\mathbf{1}:\boldsymbol{I}$	2 4 1 0
44	[31, 78, 42, 14, 4, 6, 0, 1]	33	[104, 62, 15, 2]	$\mathbf{31}:\boldsymbol{I}$ $\mathbf{2}:\boldsymbol{Z_2}$	1 6 0 0
45	[31, 78, 43, 13, 2, 8, 1, 0]	33	[104, 62, 15, 2]	$\mathbf{33}:\boldsymbol{I}$	2 4 1 0
46	[31, 79, 41, 12, 3, 10, 0, 0]	95	[104, 62, 15, 2]	$\mathbf{87}:\boldsymbol{I}$ $\mathbf{8}:\boldsymbol{Z_2}$	2 4 1 0
47	[32, 74, 48, 14, 0, 6, 2, 0]	3	[104, 62, 15, 2]	$\mathbf{3}:\boldsymbol{Z_2}$	2 4 1 0
48	[32, 75, 45, 14, 3, 6, 0, 1]	26	[104, 62, 15, 2]	$\mathbf{21}:\boldsymbol{I}$ $\mathbf{4}:\boldsymbol{Z_2}\mathbf{1}:\boldsymbol{Z_6}$	1 6 0 0
49	[32, 75, 46, 13, 1, 8, 1, 0]	14	[104, 62, 15, 2]	$\mathbf{14}:\boldsymbol{I}$	2 4 1 0
50	[32, 76, 44, 12, 2, 10, 0, 0]	64	[104, 62, 15, 2]	$\mathbf{62}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}\mathbf{1}:\boldsymbol{Z_4}$	2 4 1 0
51	[33, 72, 48, 14, 2, 6, 0, 1]	11	[104, 62, 15, 2]	$\mathbf{10}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}$	1 6 0 0
52	[33, 72, 49, 13, 0, 8, 1, 0]	4	[104, 62, 15, 2]	$\mathbf{4}:\boldsymbol{I}$	2 4 1 0
53	[33, 73, 47, 12, 1, 10, 0, 0]	17	[104, 62, 15, 2]	$\mathbf{15}:\boldsymbol{I}$ $\mathbf{2}:\boldsymbol{Z_2}$	2 4 1 0
54	[34, 69, 51, 14, 1, 6, 0, 1]	2	[104, 62, 15, 2]	$\mathbf{1}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}$	1 6 0 0
55	[34, 70, 50, 12, 0, 10, 0, 0]	1	[104, 62, 15, 2]	$\mathbf{1}:\boldsymbol{I}$	2 4 1 0
56	[35, 87, 18, 24, 3, 6, 3, 0]	1	[103, 65, 12, 3]	$\mathbf{1}:\boldsymbol{Z_3}$	1 3 3 0
57	[36, 84, 21, 24, 2, 6, 3, 0]	1	[103, 65, 12, 3]	$\mathbf{1}:\boldsymbol{S_3}$	0 6 0 1
58	[36, 85, 19, 23, 3, 8, 2, 0]	1	[103, 65, 12, 3]	$\mathbf{1}:\boldsymbol{I}$	1 3 3 0
59	[36, 86, 17, 22, 4, 10, 1, 0]	3	[103, 65, 12, 3]	$2:\boldsymbol{Z_2}$ $1:\boldsymbol{Z_2}\times\boldsymbol{Z_2}$	0 6 0 1
60	[37, 82, 21, 24, 4, 6, 1, 1]	5	[103, 65, 12, 3]	$\mathbf{1}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}$	0 5 2 0
61	[37, 83, 20, 22, 3, 10, 1, 0]	2	[103, 65, 12, 3]	$\mathbf{2}:\boldsymbol{I}$	1 3 3 0
62	[38, 78, 26, 25, 2, 4, 2, 1]	2	[103, 65, 12, 3]	$\mathbf{2}:\boldsymbol{Z_2}$	0 5 2 0
63	[38, 78, 27, 24, 0, 6, 3, 0]	4	[103, 65, 12, 3]	$\mathbf{3}:\boldsymbol{S_3}\mathbf{1}:\boldsymbol{D_6}$	0 6 0 1
64	[38, 79, 24, 24, 3, 6, 1, 1]	3	[103, 65, 12, 3]	$\mathbf{2}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}$	0 5 2 0
65	[38, 79, 25, 23, 1, 8, 2, 0]	2	[103, 65, 12, 3]	$\mathbf{2}:\boldsymbol{I}$	1 3 3 0
66	[38, 80, 22, 23, 4, 8, 0, 1]	6	[103, 65, 12, 3]	$\mathbf{6}:\boldsymbol{I}$	0 5 2 0
67	[38, 80, 23, 22, 2, 10, 1, 0]	12	[103, 65, 12, 3]	$\mathbf{8}:\boldsymbol{I}$ $\mathbf{3}:\boldsymbol{Z_2}$ $1:\boldsymbol{Z_2}\times\boldsymbol{Z_2}$	1 3 3 0 0 6 0 1
68	[38, 81, 21, 21, 3, 12, 0, 0]	11	[103, 65, 12, 3]	$\mathbf{11}:\boldsymbol{I}$	1 3 3 0
69	[39, 76, 27, 24, 2, 6, 1, 1]	9	[103, 65, 12, 3]	$\mathbf{7}:\boldsymbol{I}$ $\mathbf{2}:\boldsymbol{Z_2}$	0 5 2 0
70	[39, 77, 25, 23, 3, 8, 0, 1]	20	[103, 65, 12, 3]	$\mathbf{20}:\boldsymbol{I}$	0 5 2 0
71	[39, 77, 26, 22, 1, 10, 1, 0]	19	[103, 65, 12, 3]	$\mathbf{20}:\boldsymbol{I}$	1 3 3 0
72	[39, 78, 24, 21, 2, 12, 0, 0]	39	[103, 65, 12, 3]	$\mathbf{39}:\boldsymbol{I}$	1 3 3 0

					0 6 0 1
73	[40, 72, 32, 25, 0, 4, 2, 1]	1	[103, 65, 12, 3]	$\mathbf{1}:\boldsymbol{Z_4}$	0 5 2 0
74	[40, 73, 30, 24, 1, 6, 1, 1]	5	[103, 65, 12, 3]	$\mathbf{5}:\boldsymbol{I}$	0 5 2 0
75	[40, 74, 28, 23, 2, 8, 0, 1]	37	[103, 65, 12, 3]	$\mathbf{37}:\boldsymbol{I}$	0 5 2 0
76	[40, 74, 29, 22, 0, 10, 1, 0]	22	[103, 65, 12, 3]	$\mathbf{10}:\boldsymbol{I}$ $\mathbf{11}:\boldsymbol{Z_2}$ $2:\boldsymbol{Z_2}\times\boldsymbol{Z_2}$	1 3 3 0 0 6 0 1
77	[40, 75, 27, 21, 1, 12, 0, 0]	40	[103, 65, 12, 3	$\mathbf{39}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_3}$	1 3 3 0
78	[41, 70, 33, 24, 0, 6, 1, 1]	3	[103, 65, 12, 3]	$\mathbf{2}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}$	0 5 2 0
79	[41, 71, 31, 23, 1, 8, 0, 1]	29	[103, 65, 12, 3]	$\mathbf{29}:\boldsymbol{I}$	0 5 2 0
80	[41, 72, 30, 21, 0, 12, 0, 0]	29	[103, 65, 12, 3]	$\mathbf{25}:\boldsymbol{I}$ $\mathbf{2}:\boldsymbol{Z_2}\mathbf{2}:\boldsymbol{Z_3}$	1 3 3 0 0 6 0 1
81	[42, 68, 34, 23, 0, 8, 0, 1]	8	[103, 65, 12, 3]	$\mathbf{8}:\boldsymbol{I}$	0 5 2 0
82	[48, 72, 12, 35, 0, 6, 3, 0]	1	[102, 68, 9, 4]	$\mathbf{1}:\boldsymbol{S_3}$	0 3 3 1
83	[48, 74, 8, 33, 2, 10, 1, 0]	1	[102, 68, 9, 4]	$\mathbf{1}:\boldsymbol{Z_2}$	0 3 3 1
84	[48, 75, 6, 32, 3, 12, 0, 0]	1	[102, 68, 9, 4]	$\mathbf{1}:\boldsymbol{S_3}$	1 0 6 0
85	[49, 69, 14, 36, 1, 4, 2, 1]	1	[102, 68, 9, 4]	$\mathbf{1}:\boldsymbol{Z_2}$	0 2 5 0
86	[49, 70, 12, 35, 2, 6, 1, 1]	1	[102, 68, 9, 4]	$\mathbf{1}:\boldsymbol{I}$	0 2 5 0
87	[49, 72, 9, 32, 2, 12, 0, 0]	1	[102, 68, 9, 4]	$\mathbf{1}:\boldsymbol{I}$	0 3 3 1
88	[50, 68, 13, 34, 2, 8, 0, 1]	5	[102, 68, 9, 4]	$\mathbf{4}:\boldsymbol{I}$ $\mathbf{1}:\boldsymbol{Z_2}$	0 2 5 0
89	[50, 68, 14, 33, 0, 10, 1, 0]	8	[102, 68, 9, 4]	$\mathbf{8}:\boldsymbol{Z_2}$	0 3 3 1
90	[50, 69, 12, 32, 1, 12, 0, 0]	4	[102, 68, 9, 4]	$\mathbf{4}:\boldsymbol{Z_2}$	1 0 6 0
91	[51, 64, 18, 35, 0, 6, 1, 1]	3	[102, 68, 9, 4]	$\mathbf{3}:\boldsymbol{I}$	0 2 5 0
92	[51, 65, 16, 34, 1, 8, 0, 1]	12	[102, 68, 9, 4]	$\mathbf{9}:\boldsymbol{I}$ $\mathbf{3}:\boldsymbol{Z_2}$	0 2 5 0
93	[51, 66, 15, 32, 0, 12, 0, 0]	13	[102, 68, 9, 4]	$\mathbf{13}:\boldsymbol{I}$	1 3 3 0 0 6 0 1
94	[52, 62, 19, 34, 0, 8, 0, 1]	8	[102, 68, 9, 4]	$\mathbf{8}:\boldsymbol{I}$	0 2 5 0
95	[63, 54, 3, 47, 2, 6, 0, 1]	1	[101, 71, 6, 5]	$\mathbf{1}:\boldsymbol{Z_6}$	0 0 6 1
96	[64, 50, 8, 48, 0, 4, 1, 1]	1	[101, 71, 6, 5]	$1:\boldsymbol{Z_2}\times\boldsymbol{Z_2}$	0 0 6 1
97	[64, 52, 5, 45, 0, 10, 0, 0]	3	[101, 71, 6, 5]	$\mathbf{2}:\boldsymbol{Z_2}$ $\mathbf{1}:\boldsymbol{Z_4}$	0 1 4 2
98	[65, 48, 9, 47, 0, 6, 0, 1]	1	[101, 71, 6, 5]	$\mathbf{1}:\boldsymbol{Z_2}$	0 0 6 1
99	[80, 30, 0, 60, 0, 6, 0, 0]	1	[100, 74, 3, 6]	$\mathbf{1}:\boldsymbol{S_4}$	0 0 3 4

Corollary 2: There are twelve different kinds of $\alpha, \beta, \gamma, \delta$ as given in Table 12. So, the following holds.

1- If $\alpha = 0, \beta = 0, \gamma = 3, \delta = 4$or $\alpha = 0, \beta = 0, \gamma = 6, \delta = 1$, then $\mathcal{I}_i = \mathcal{I}_{i_{10,2,2}} \cup \mathcal{I}_{i_{11,0,3}}$.

2- If $\alpha = 0,\ \beta = 1,\ \gamma = 4,\ \delta = 2$ or $\alpha = 0,\ \beta = 3,\ \gamma = 3,\ \delta = 1$, then $\mathcal{I}_i = \mathcal{I}_{i_{9,4,1}} \cup \mathcal{I}_{i_{10,2,2}} \cup \mathcal{I}_{i_{11,0,3}}$.

3- If $\alpha = 0,\ \beta = 2,\ \gamma = 5,\ \delta = 0$ or $\alpha = 0,\ \beta = 5,\ \gamma = 2,\ \delta = 0$, then $\mathcal{I}_i = \mathcal{I}_{i_{9,4,1}} \cup \mathcal{I}_{i_{10,2,2}}$.

4- If $\alpha = 0,\ \beta = 6,\ \gamma = 0,\ \delta = 1$, then $\mathcal{I}_i = \mathcal{I}_{i_{9,4,1}} \cup \mathcal{I}_{i_{11,0,3}}$.

5- If $\alpha = 1,\ \beta = 0,\ \gamma = 6,\ \delta = 0$, then $\mathcal{I}_i = \mathcal{I}_{i_{8,6,0}} \cup \mathcal{I}_{i_{10,2,2}}$.

6- If $\alpha = 1,\ \beta = 3,\ \gamma = 3,\ \delta = 0$ or $\alpha = 2,\ \beta = 4,\ \gamma = 1,\ \delta = 0$, then $\mathcal{I}_i = \mathcal{I}_{i_{8,6,0}} \cup \mathcal{I}_{i_{9,4,1}} \cup \mathcal{I}_{i_{10,2,2}}$.

7- If $\alpha = 1,\ \beta = 6,\ \gamma = 0,\ \delta = 0$ or $\alpha = 4,\ \beta = 3,\ \gamma = 0,\ \delta = 0$, then $\mathcal{I}_i = \mathcal{I}_{i_{8,6,0}} \cup \mathcal{I}_{i_{9,4,1}}$.

Example 1: In $PG(2,13)$, the unique $(7;3)$-arc with stabilizer group G_χ isomorphic to $\boldsymbol{S_4}$ is:

$$\chi = \{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}, \boldsymbol{e_4}, [\boldsymbol{v};\mathbf{1};\mathbf{1}], [\mathbf{1};\mathbf{0};\mathbf{1}], [\mathbf{1};\mathbf{1};\mathbf{0}]\}.$$

The group G_χ is generated by the following two linear transformations:

$$\phi_1 = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \phi_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 2 & 1 & 1 \end{pmatrix},$$

where ${\phi_1}^4 = {\phi_2}^2 = (\phi_1 \ \ \phi_2 \ )^3 = 1.$

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تصنيف الاقواس من النمط $(k;3)$ في $PG(2,13)$ عندما $k=5,6,7$

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المستخلص

القوس ذو سعة k ومن الدرجة الثالثة هو مجموعة مكونة من k من النقاط بحيث لايوجد اربعة من هذه النقاط على استقامة واحدة ولكن يوجد ثلاثة منها على استقامة واحدة. الهدف من هذا البحث هو تحديد الاقواس من النمط $(k;3)$ في المستوي الاسقاطي $PG(2,13)$ عندما $k=5,6,7$.

الكلمات الدليلية: المستوي الاسقاطي , القوس.