TRFFIC FLOW MODELS WITH THE CONSISTENCY OF ACLASS

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Abstract

This note discusses a recently discovered internal inconsistency of a class of higher-order traffic flow models. It shows that this class of models are mathematically consistent, and that the supposed internal inconsistency is not part of their intrinsic property. Rather, it is a result of insisting on the universality of a special conservation law and imposing incompatible solutions on a equations.

Keywords Conservation law; Continuum traffic models

Higher-order continuum traffic models, as exemplified by the payne-whitham (pw) theory (payne,1971;whitham, 1974), are subject to increasing scrutiny in recent literature. While meritsand dis-meritsof these models are bhile meritsand dis-meritsof these models are being debated by various authors (e.g.,Daganzo, 1995; pa-pageorgiou,1998), a new flaw has recently been discovered Heidemann (1999). Unlike the he merits of higher-order models, which are largely centered on the physical character of these models and the utility of such models in practical applications, the flaw exposed by Heidemann is of a mathematical nature. Based on a time series expansion of solutions around an equilibrium state, Heidemann showed that the pw-like higher-order models have an internal inconsistency-their governing equations are not mathematically consistent. This internal in-consistency, if true, undermines the validity of an entire class of higher-order models and therefore deserves a critical examination.

What we refer as pw-like models are a class of traffic models governing equations inciude the conservation of cars.$P_t + (q)_x = 0 \ldots\ldots\ldots\ldots\ldots..(1)$

And speed evolution

$$v_t + vv_x = \frac{v_x(P) - v}{\tau} - \frac{C^2(P)}{P} P_x \ldots\ldots\ldots\ldots\ldots (2)$$

The parameter τ is relaxation time, usually treated as a constant but can be a function of vehicle concentration. The function $v_*(P)$ is the so-called equilibrium speed-concentration, whose meaning will become clearer when we discuss its origin.$C(P)$ is the traffic sound speed . It is the relative speed at which a small disturbance travels against a uniform, moving traffic stream. Various forms for $C(P)$ were proposed in literature, including$-\sqrt{v/\tau}$ (whith am, 1974,v is a parameter), $-\sqrt{-(V_*^t/(P))}/2\pi$ (payne, 1971 $V_*^t(P)$ is the derivative of V_* with respect to P) and $Pv_*^t(P)$ (zhang,1998).

In both equations, vehicle concentration$P(x,t)$,travel speed $v(x,t)$, and flow rate $q(x,t)$are suitably defined average quantities in a proper neighborhood of position x and tim t (Fig.I):

$$q(x,t) = \frac{dx}{dA}, \qquad P(x,t) = \frac{dt}{dA}, \qquad v(x,t) = \frac{dx}{dt},$$

Where dA is the total area encircled by the closed curve φ, dt the total time spent in φ and dx are total distance traveled in φ by all vehicles passing through φ (Daganzo,1997).by definition, flow, concentration and speed are related by.

$q = Pv \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.(3)$

With (3)the conservation equation becomes.

$P_t + (Pv)_x = 0$ … … … … … … . (4)

(3)and(4),being derived from physical principles, are about the only undisputed facts in a continuum traffic theory. Because they hold true in any one -dimensional material flow these equations are not expected to reveal the true character of the special kind of fluid that we try to model-vehicular traffic. the character of vehicular traffic, as it turns out can be conveniently studied on the (P, v)plane ,also known as the (P, v)phase plane. Although varying by location and time,$P - v$ plots of observed data are quite similar to the one shownfig.2(a)-overall speed decreases as concentration or occupancy(a surrogate of concentration) increases, but data points are scattered around a general trend line (the dashed line).upon closer examination, one can find

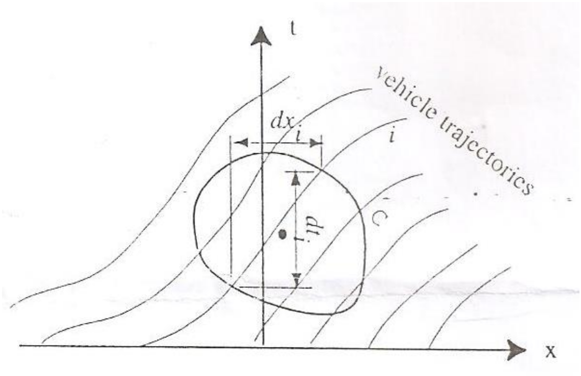

Fig. 1. Time-space diagram.

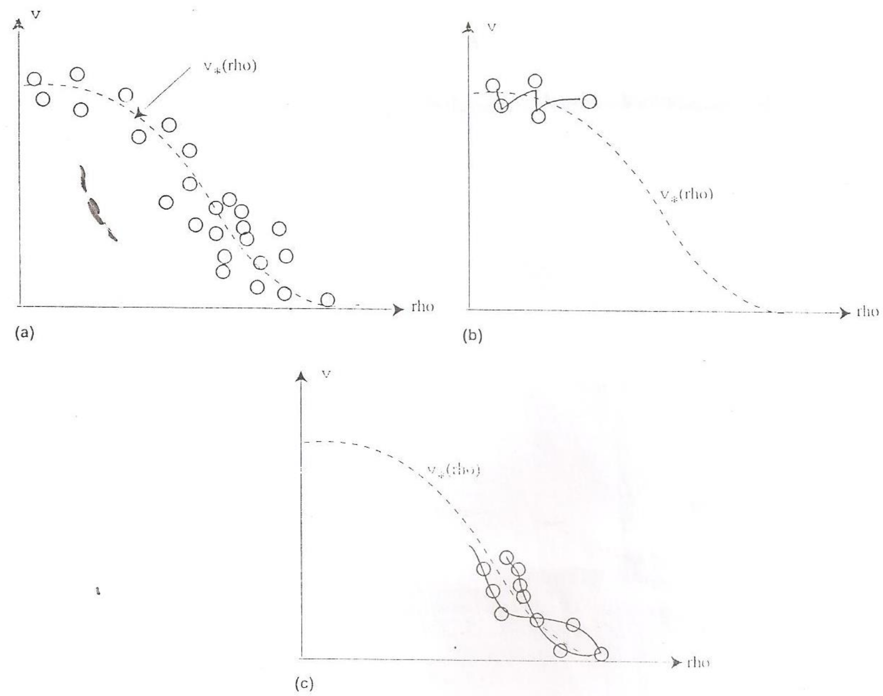

Fig. 2. ρ-v phase plot.

Random zig-zag phase transitions(Fig.2(b)) or transitions that have well defined structures(Fig.2(c)) in such plots. The latter indicates that the scatter cannot be fully explained by measurement noise alone. In other words, a phase curve obtained through statistical averaging suppresses finer traffic dynamics represented by the fine structures in the scatter. Nevertheless, an average trend represented by a smooth function

$V_*(P)\left(V_*(0) = V_f, V_*(P_j)\right) = 0\ and\ V_*^I(P) < 0, v_f$ Free flow speed,P_J-jam concentration) captures the main features(i.e., first-order effects)of traffic dynamics and constitutes a useful approximation. With such an approximation, that is, restricting v to $v_*(P)$, the conservation equation becomes the well known kinematic wave theory of Lighthill and whitham (1955) and Richards(1956)(the LWR model).

$$P_t + f_*(P)_x = 0, v = v_*(P), q = Pv = Pv_*(P) \equiv f_*(P) \ldots (5)$$

Which supplemented with proper initial/boundary conditions, is by itseif suffient to solve for $P(x,t), v(x,t) and\ q(x,t)$ for all x and t.This new form of the conservation equation, however, is not universally valid. Its validity depends on the correctness of $v = v_*(P)$.

The LWR theory, although quite capable in modeling certain main traffic features (e.g., for- mation and dissolution of shocks), is inadequate in describing some other more complicated traffic patterns such as stop-start waves.This because the approximation $v = v_*(P)$suppresses all other phase transitions not on this curve. The ability to model stop-start waves requires a finer representation of $P - v$ phase transitions, and(2)gives one such representation. In(2) the left-hand side(LHS) is traffic acceleration, the first term of the right-hand side(RHS) captures relaxation, and the second term of the RHS models drivers reaction to traffic conditions ahead. This equation basically states that for any concentration level P,travel speed can differ from $v_*(P)$ due to driver anticipation or initial conditions, but cannot differ too much because of the pull exerted by relaxation. The interplay between relaxation and anticipation produces the phase scatter that is necessary (may not be sufficient)for modeling stop-start waves.

Together with proper initial/boundary data, the partial differential equations comprising the pw-like models,(2) and(4),form well-posed mathematical problems. In fact, we know that this is a hyperbolic system, whose solutions exist(Lax,1972;Li and zhang, 2001) and include shock and rarefaction waves (whitham,1974; Del Castillo,1993;zhang,1999 ,2000). Thus, there does not appear to be any internal inconsistency in these models. The internal inconsistency arises when one imposes another equation, Eq. (5), to the PW-like models and insist on certain type of solutions that are not compatible with the system of equations. Noticing that(5) holds true only when $v = v_*(P)$, and for any properly prescribed initial data(5) alone can solve for$P(x,t)$,(2) and (4) must have special kinds of solutions to make (5),(2) and (4) a compatible system. In fact one can find these solutions. With $v = v_*(P)$ (4) and (5) are equivalent, as they should be. Further-more, (2) reduces to.

$$\left(P\big(v_*^I(P)\big)^2 - \frac{C^2(P)}{P}\right)P_x = 0 \dots\dots(6)$$

Clearly, if $C^2(P) \neq Pv_*^I(P)$, which is the case discussed in Heidemann (1999), we have $P_x = 0$ from(5) we have$P_x = 0$ Therefore the solution of(2) and (4) are constants:

$P = P_0 . v = v_*(P_0)$ As

$P = P_0 + Pexp\{ikx + \omega t\} \dots\dots\dots\dots(7)$
$v = v_*(P_0) + v\ exp\{ikx + \omega t\} \dots\dots\dots.(8)$

Onto these equations.

When $C^2(P) = Pv_*^I(P)$ (6) holds regard less of the value Thus the entire system of equations reduces to a single equation, Eq.(5). In this case we can solve this equation for any kind of P including the one given by (8). But v must be given by $v_*(P)$ instead of an arbitrary time-space expansion.

The pw-like models indeed have some flaws(e.g.,Dagan%0, 1995;zhang, 2000). But these flaws are of a physical nature. Mathematically they are consistent theories. The alleged inconsistency is a result of insisting on the universality of $P_t + f_*(P)_x = 0$ and imposing arbitrary solutions the system of three equations:

$$P_t + (Pv)_x = 0. \quad P_t + f_*(P)_x = 0, \qquad v_t + vv_x = \frac{v_*(P) - v}{\tau} - \frac{C^2(P)}{P} P_x$$

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