Minimax Estimation Of The Parameter Of The Maxwell Distribution Under Quadratic Loss Function

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Abstract: *This paper is concerned with the problem of finding the minimax estimators of the parameter θ of the Maxwell distribution (MW) for quadratic loss functions by applying the theorem of Lehmann [1950]. Through simulation study the performance of this method compared with the classical methods containing the Maximum Likelihood and moment Estimators with respect to Mean squared-errors (MSEs) .We reach to that the Minimax estimator with small positive values of c gives the best results, followed by the Maximum likelihood estimator.*

Keywords: Minimax; Bayes estimator; Maximum likelihood estimator; moment estimator; Mean squared error; Quadratic loss function

1.Introduction

The minimax estimation was introduced by Wald (1950) from the concept of game theory. It opens a new dimension in statistical estimation and enriches the method of point estimations. Von Neumann (1944) introduced the word minimax. Podder et al. (2004) studied the minimax estimator of the parameter of the Pareto distribution under Quadratic and MLINEX loss functions. Also, Dey (2008) studied the minimax estimator of the parameter for the Rayleigh distribution under quadratic loss function. Shadrokh and Pazira (2010) studied the minimax estimator of the parameter for the minimax distribution under several loss functions. Masoud Yarmohammadi and Hassan Pazira (2010) estimate the parameter θ (when λ is known) by using the technique of minimax approach of the Parameter of the Burr Type Xii Distribution [1]. Iman Makhdoom (2011) studied the Minimax estimation of the parameter of the Generalized Exponential distribution.

The Maxwell distribution was first introduced in the literature by J.C. Maxwell (1860) and again described by Boltzman (1870) with a few assumptions. This distribution plays an important role in Physics and other allied sciences. It defines the speed of molecules in thermal equilibrium under some conditions as defined in statistical mechanics. For example, this distribution explains many fundamental gas properties in kinetic theory of gases.

The probability density function (pdf) of Maxwell distribution is given by:

$$f(x|\theta) = \frac{4}{\sqrt{\pi}} . \frac{1}{\theta^{3/2}} x^2 \, e^{-x^2/\theta} \quad ; \quad 0 < X, \theta \qquad (1)$$

Where θ is the parameter of the distribution. The cumulative distribution function (cdf) is given by:

$$F(x) = \frac{1}{\Gamma\left(\frac{3}{2}\right)} \Gamma\left(\frac{x^2}{\theta}, \frac{3}{2}\right) \quad ; \quad 0 < X, \theta \qquad (2)$$

Where $\Gamma(x,\alpha) = \int_0^x e^{-u}u^{\alpha-1}du$ is the incomplete gamma function. [2]

In this paper, we shall estimate the parameter of the Maxwell distribution using the technique of Minimax approach under quadratic loss function, which is essentially a Bayesian approach and compare this estimator with the classical estimators (the Moment and the Maximum likelihood estimator (MLE)).

This paper is organized as follows: In Section 2, we find the classical estimators of the parameter of the Maxwell distribution. The minimax estimator of the parameter of the Maxwell distribution under quadratic loss function is obtained in section 3. In Section 4, a simulation study is carried out to compare these estimators. The simulation results and discussions are provided in Section 5.

2. Classical Estimations

In this section, we obtain the classical estimators of θ using:

2.1 Moment estimation

Let $X_1, X_2, \ldots X_n$ be a random sample from density (1) the r^{th} moment is given by: [3]

$$E(x^r) = \frac{2}{\sqrt{\pi}}\,\theta^{\frac{r}{2}}\Gamma\left(\frac{r+2}{2}\right) \quad ; \quad r > -3 \tag{3}$$

In particular, the mean of Maxwell random variable is given by:

$$\mu = 2\sqrt{\frac{\theta}{\pi}}$$

So, using the moment estimator for θ:

$$\hat{\theta}_{mo} = \frac{\pi}{4}\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2 = \frac{\pi}{4}\bar{x}^2 \tag{4}$$

2.2 Maximum likelihood estimation (MLE)

The likelihood function for the Maxwell pdf is given by:

$$L(x,\theta) = \left(\frac{4}{\sqrt{\pi}}\right)^n \cdot \frac{1}{\theta^{3n/2}} \prod_{i=1}^{n} x^2 \ e^{-\sum x^2/\theta} \qquad (5)$$

By taking the log and differentiating partially with respect to θ, we get:

$$\frac{\partial \ln L(x,\theta)}{\partial \theta} = \frac{-3n}{2\theta} + \frac{\sum_{i=1}^{n} x_i^2}{\theta^2} \qquad (6)$$

Then the MLE of θ is the solution of equation (6) after equating the first derivative to zero. Hence:

$$\hat{\theta}_{MLE} = \frac{2\sum_{i=1}^{n} x_i^2}{3n} \qquad (7)$$

3.Minimax Estimations:

In this section, we derive the minimax estimators of the parameter θ for the Maxwell distribution by applying a theorem, which is due to Hodge and Lehmann (1950) and can be stated as follows:

Lehmann's Theorem:

Let τ = {Fθ; θεΘ} be a family of distribution functions and D a class of estimators of θ. Suppose that d* ε D is a Bayes' estimator against a prior distribution ξ*(θ) on the parameter space Θ and the risk function R(d*, θ) = constant on Θ; then d* is a minimax estimator of θ. [4]

The main results are contained in the following Theorem.

Theorem 3.1

Let x = (x_1, x_2, . . . , x_n) be n independently and identically distributed random variables drawn from the density (1). Then:

$$\hat{\theta}_{MQL} = \frac{\Gamma\left(\frac{3n+2c}{2}\right)\sum_{i=1}^{n} x_i}{\Gamma\left(\frac{3n+2c+2}{2}\right)} \qquad (8)$$

Is the minimax estimator of the parameter for the Maxell distribution for the quadratic loss function (QLF) of the type:[4],[5]

$$L(\theta,\hat{\theta}) = \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 \quad (9)$$

First we have to prove Theorem 3.1. We use Lehmann's Theorem, which was stated before. Here, we consider the Quadratic loss function (QLF) of the form (9) which is a non-negative symmetric and continuous loss function of θ and $\hat{\theta}$. This theorem can be proven through applying these two steps:

1. Finding the Bayes estimator $\hat{\theta}$ of θ .
2. Showing that the risk function of $\hat{\theta}$ is a constant.

The Theorem will be followed. Let us assume that θ has Jeffrey's non-informative prior density defined as:

$$g(\theta) \propto \frac{1}{\theta^c} \quad ; \theta > 0, c \in R^+ \quad (10)$$

$$g(\theta) = k\frac{1}{\theta^c}, \text{ where k a constant} \quad (11)$$

Combining the likelihood function (5) and the prior $g(\theta)$ in (11), The posterior distribution for the parameter θ given the data (x_1, x_2… x_n) is:

$$h(\theta|\mathbf{x}) = \frac{\prod_{i=1}^{n} f(x_i|\theta)g(\theta)}{\int_{\forall\theta}\prod_{i=1}^{n} f(x_i|\theta)g(\theta)d\theta} \quad (12)$$

$$= \frac{(\frac{4}{\sqrt{\pi}})^n \cdot \frac{1}{\theta^{3n/2}}\prod_{i=1}^{n} x^2\, e^{-\sum x^2/\theta}\frac{k}{\theta^c}}{\int_0^\infty (\frac{4}{\sqrt{\pi}})^n \cdot \frac{1}{\theta^{3n/2}}\prod_{i=1}^{n} x^2\, e^{-\sum x^2/\theta}\frac{k}{\theta^c}d\theta}$$

$$= \frac{\frac{e^{-\sum x^2/\theta}}{\theta^{(3n+2c)/2}}}{\int_0^\infty \frac{e^{-\sum x^2/\theta}}{\theta^{(3n+2c)/2}}d\theta}$$

Let

$$y = \frac{\sum_{i=1}^{n} x_i^2}{\theta}$$

Then the posterior distribution became as follows:

$$= \frac{y^{\frac{3n+2c}{2}} e^{-y}}{-\sum_{i=1}^{n} x_i^2 \Gamma(\frac{3n+2c-2}{2})}$$

$$h(\theta|\mathbf{x}) = \frac{(\frac{\sum_{i=1}^{n} x_i^2}{\theta})^{\frac{3n+2c}{2}} e^{-\frac{\sum_{i=1}^{n} x_i^2}{\theta}}}{-\sum_{i=1}^{n} x_i^2 \Gamma(\frac{3n+2c-2}{2})}$$

According to the QLF of the form (9), we can obtain the risk function such that:

$$\hat{\theta} = \frac{E\left(\frac{1}{\theta}\middle|X\right)}{E\left(\frac{1}{\theta^2}\middle|X\right)} = \frac{\int_0^{\infty}\frac{1}{\theta} h(\theta|\mathbf{x})\, d\theta}{\int_0^{\infty}\frac{1}{\theta^2} h(\theta|\mathbf{x})\, d\theta}$$

$$\hat{\theta} = \frac{\int_0^{\infty}\frac{1}{\theta}\left[\frac{(\frac{\sum_{i=1}^{n} x_i^2}{\theta})^{\frac{3n+2c}{2}} e^{-\frac{\sum_{i=1}^{n} x_i^2}{\theta}}}{-\sum_{i=1}^{n} x_i^2 \Gamma(\frac{3n+2c-2}{2})}\right] d\theta}{\int_0^{\infty}\frac{1}{\theta^2}\left[\frac{(\frac{\sum_{i=1}^{n} x_i^2}{\theta})^{\frac{3n+2c}{2}} e^{-\frac{\sum_{i=1}^{n} x_i^2}{\theta}}}{-\sum_{i=1}^{n} x_i^2 \Gamma(\frac{3n+2c-2}{2})}\right] d\theta}$$

On simplification, we get the minimax estimator ($\hat{\theta}_{MQL}$)

$$\hat{\theta}_{MQL} = \frac{\Gamma\left(\frac{3n+2c}{2}\right)\sum_{i=1}^{n} x_i^2}{\Gamma\left(\frac{3n+2c+2}{2}\right)} = \frac{\sum_{i=1}^{n} x_i^2}{\left(\frac{3n+2c}{2}\right)} = \frac{2\sum_{i=1}^{n} x_i^2}{3n+2c} \qquad (13)$$

Now, we should prove that, the risk function of the estimator $\hat{\theta}_{MQL}$ is a constant:

$$R_{\hat{\theta}_{MQL}}(\theta) = E[L(\theta \mid \hat{\theta}_{MQL})]$$

$$= \frac{1}{\theta^2}\left[\theta^2 - 2\theta E(\hat{\theta}) + E(\hat{\theta})^2\right]$$

$$= \frac{1}{\theta^2}\left[\theta^2 - 2\theta E\left(\frac{\Gamma\left(\frac{3n+2c}{2}\right)\sum_{i=1}^{n} x_i^2}{\Gamma\left(\frac{3n+2c+2}{2}\right)}\right) + E\left(\frac{\Gamma\left(\frac{3n+2c}{2}\right)\sum_{i=1}^{n} x_i^2}{\Gamma\left(\frac{3n+2c+2}{2}\right)}\right)^2\right]$$

$$= 1 - \frac{1}{\theta}\left[\frac{4}{3n+2c} E\left(\sum_{i=1}^{n} x_i^2\right) - \frac{1}{\theta}\frac{4}{(3n+2c)^2} E\left(\sum_{i=1}^{n} x_i^2\right)^2\right]$$

$$= 1 - \frac{1}{\theta}\left[\frac{4}{3n+2c} E(T) - \frac{1}{\theta}\frac{4}{(3n+2c)^2} E(T)^2\right] \qquad (14)$$

Where $T=\sum_{i=1}^{n} {x_i}^2$

Now,
If Y has a Gamma $(\frac{3}{2},\frac{1}{\theta})$ distribution then, $X = \sqrt{Y}$ will be distributed as MW(θ). So the statistic ${x_i}^2$ will be distributed as Gamma $(\frac{3}{2},\frac{1}{\theta})$ distribution and T is distributed with parameters $\frac{3n}{2}$ and $\frac{1}{\theta}$ .i.e.' T~ Gamma $(\frac{3n}{2},\frac{1}{\theta})$.
Here, $E(T) = \frac{3n}{2}\theta$ and $E(T)^2 = \frac{3n\theta^2(2+3n)}{4}$
Substituting for $E(T)$ and $E(T)^2$ in (14) gives:

$$R_{\hat{\theta}_{MQL}}(\theta) = 1 - \frac{1}{\theta}\left[\frac{4}{3n+2c}\left(\frac{3n}{2}\theta\right) - \frac{1}{\theta}\frac{4}{(3n+2c)^2}\left(\frac{3n\theta^2(2+3n)}{4}\right)\right]$$

$$= 1 - \frac{9n^2+12nc-6n}{(3n+2c)^2} \quad \text{, which is constant.}$$

So according to the Lehmann's theorem it follows that:
$\hat{\theta}_{MQL} = \frac{2\sum_{i=1}^{n} {x_i}^2}{3n+2c}$ is the minimax estimator of the parameter θ of the Maxwell distribution under the quadratic loss function of the form (9).

4. Simulation Study

Mean Squared Errors (MSEs) are considered to compare the different estimators of the parameter θ of and are obtained by the method of Maximum likelihood, moment and Minimax for Quadratic Loss function methods. In this simulation study, the number of replication used was I = 3000 samples of sizes n = 5, 10, 20, 30, 50 from the Maxwell distribution with θ = 1 and 3, using the different values of c, c= ±1, ±1.5, ±3.The MSE of an estimator is defined by:

$$MSE(\theta) = \frac{\sum_{i=1}^{I}(\hat{\theta}_i-\theta)^2}{I} \quad (15)$$

The results were summarized and tabulated in tables 1 and 2 for the MSE and $\hat{\theta}$ of the different estimators for all sample sizes and also presented them in Figures 1 and 2.

However, in most of the cases, the minimax estimator under the quadratic loss function (MQL) when c has a positive value is better than the classical estimator.

Table 1. Biases and MSEs of different estimators for the parameter θ of the Maxwell distribution when $\theta = 1$

n	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{mo}$	$\hat{\theta}_{MQL}$					
				C = -3	**C = -1.5**	**C = -1**	**C = 1**	**C = 1.5**	**C = 3**
5	E $(\hat{\theta})$	0.99750	1.03240	1.66250	1.24690	1.15099	0.88017	0.83127	0.71250
	Bias	-0.00250	0.03240	0.66250	0.24690	0.15099	-0.11983	-0.16873	-0.28750
	Var $(\hat{\theta})$	0.13459	0.15135	0.37397	0.21034	0.17921	0.10480	0.09348	0.06864
	MSE	0.13460	0.15240	0.81288	0.27130	0.20201	0.11916	0.12195	0.15130
10	E $(\hat{\theta})$	1.00290	1.01960	1.25360	1.11430	1.07450	0.94020	0.91174	0.83580
	Bias	0.00290	0.01960	0.25360	0.11430	0.07450	-0.05980	-0.08826	-0.16420
	Var $(\hat{\theta})$	0.06919	0.07582	0.10819	0.08548	0.07947	0.06083	0.05721	0.04809
	MSE	0.06920	0.07620	0.17250	0.09854	0.08502	0.06441	0.06500	0.07505
20	E $(\hat{\theta})$	0.99660	1.00480	1.10730	1.04903	1.03090	0.96440	0.94913	0.90599
	Bias	-0.00340	0.00480	0.10730	0.04903	0.03090	-0.03560	-0.05087	-0.09401
	Var $(\hat{\theta})$	0.03279	0.03544	0.04044	0.03630	0.03506	0.03067	0.02971	0.02707
	MSE	0.03280	0.03546	0.05195	0.03870	0.03601	0.03194	0.03230	0.03591
30	E $(\hat{\theta})$	0.99960	1.00540	1.07101	1.03410	1.02230	0.97788	0.96740	0.93713
	Bias	-0.00040	0.00540	0.07101	0.03410	0.02230	-0.02212	-0.03260	-0.06287
	Var $(\hat{\theta})$	0.02271	0.02440	0.02607	0.02434	0.02375	0.02173	0.02127	0.01996
	MSE	0.02271	0.02443	0.03111	0.02550	0.02425	0.02222	0.02233	0.02391
50	E $(\hat{\theta})$	1.00105	1.00400	1.04280	1.02150	1.01460	0.98790	0.98142	0.96250
	Bias	0.00105	0.00400	0.04280	0.02150	0.01460	-0.01210	-0.01858	-0.03750
	Var $(\hat{\theta})$	0.01349	0.01442	0.01463	0.01404	0.01389	0.01313	0.01296	0.01246
	MSE	0.01349	0.01444	0.01646	0.01450	0.01410	0.01328	0.01331	0.01387

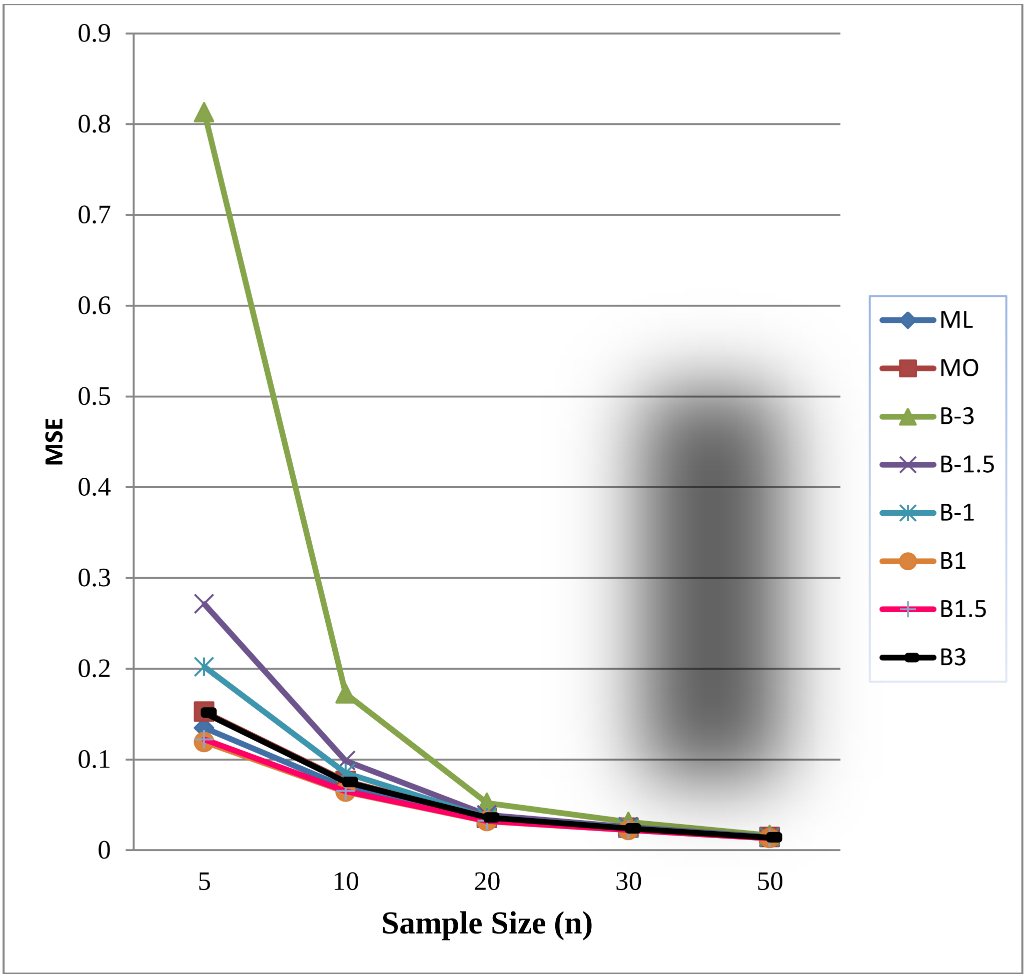

Fig. 1: MSE's of the MLE, moment, and minimax estimators with different values of c when $\theta = 1$.

Table 2. Biases and MSEs of different estimators for the parameter θ of the Maxwell distribution when θ = 3.

n	Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{mo}$	$\hat{\theta}_{MQL}$					
				C = -3					**C = -3**
5	E $(\hat{\theta})$	2.9926	3.0972	4.9876	3.7407	3.45297	2.64051	2.4938	2.13756
	Bias	-0.00740	0.09720	1.98760	0.74070	0.45297	-0.35949	-0.50620	-0.86244
	Var $(\hat{\theta})$	1.21150	1.36252	3.36535	1.89296	1.61295	0.94321	0.84126	0.61810
	MSE	1.21155	1.37197	7.3159	2.4416	1.81813	1.07244	1.0975	1.3619
10	E $(\hat{\theta})$	3.0087	3.0588	3.7609	3.343	3.2236	2.8207	2.7352	2.5073
	Bias	0.00870	0.05880	0.76090	0.34300	0.22360	-0.17930	-0.26480	-0.49270
	Var $(\hat{\theta})$	0.62302	0.68252	0.97343	0.76915	0.71520	0.54755	0.51489	0.43265
	MSE	0.6231	0.68598	1.5524	0.8868	0.7652	0.5797	0.58501	0.6754
20	E $(\hat{\theta})$	2.9897	3.0145	3.3219	3.1471	3.0928	2.8933	2.8474	2.7179
	Bias	-0.01030	0.01450	0.32190	0.14710	0.09280	-0.10670	-0.15260	-0.28210
	Var $(\hat{\theta})$	0.29479	0.31889	0.36396	0.32666	0.31549	0.27612	0.26739	0.24362
	MSE	0.2949	0.3191	0.46758	0.3483	0.3241	0.2875	0.29068	0.3232
30	E $(\hat{\theta})$	2.9988	3.0163	3.21303	3.1022	3.06699	2.93363	2.9021	2.8114
	Bias	-0.00120	0.01630	0.21303	0.10220	0.06699	-0.06637	-0.09790	-0.18860
	Var $(\hat{\theta})$	0.20430	0.21953	0.23459	0.21866	0.21371	0.19556	0.19138	0.17960
	MSE	0.2043	0.2198	0.27997	0.2291	0.2182	0.19996	0.20096	0.21517
50	E $(\hat{\theta})$	3.0031	3.0121	3.1283	3.0644	3.0437	2.9636	2.9443	2.8876
	Bias	0.00310	0.01210	0.12830	0.06440	0.04370	-0.03640	-0.05570	-0.11240
	Var $(\hat{\theta})$	0.12139	0.12981	0.13170	0.12635	0.12469	0.11818	0.11668	0.11222
	MSE	0.1214	0.12996	0.14816	0.1305	0.1266	0.1195	0.11978	0.12485

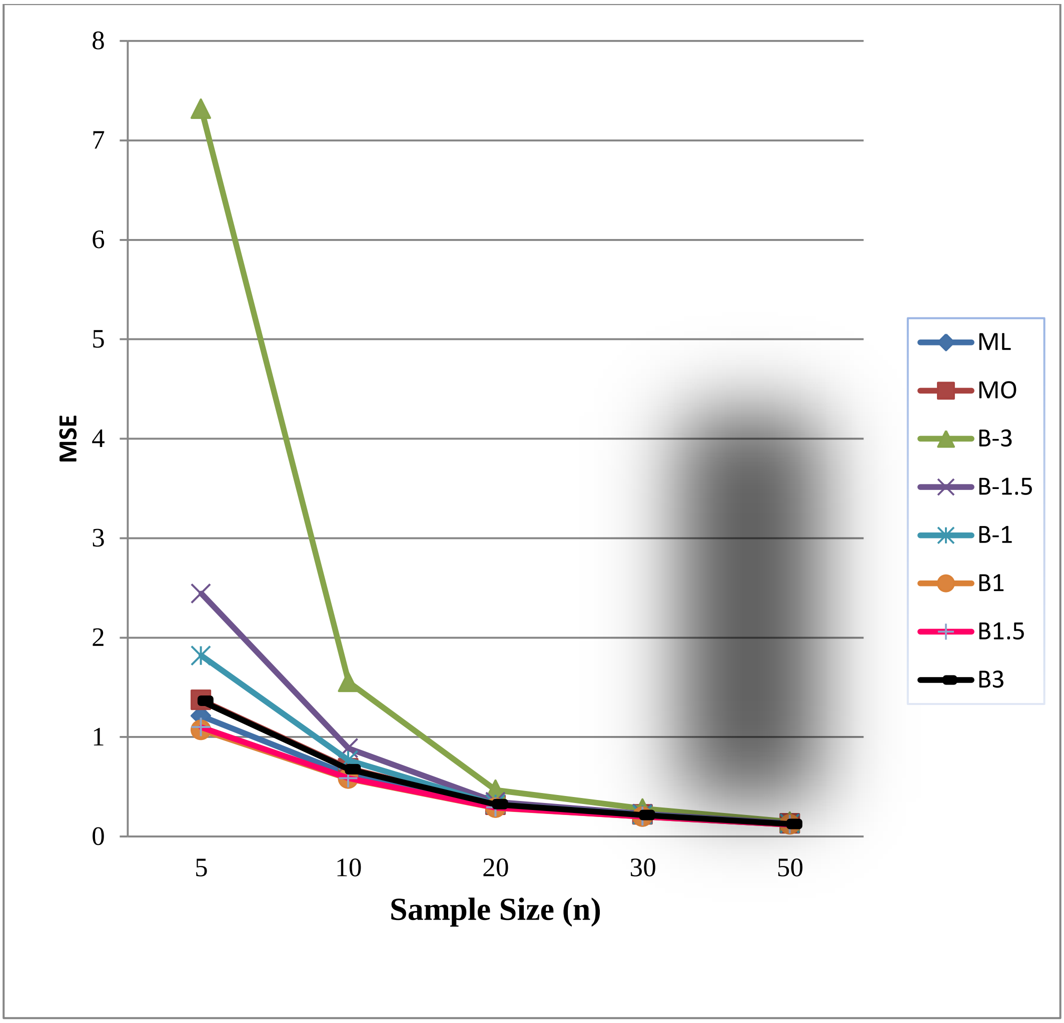

Fig. 1: MSE's of the MLE, moment, and minimax estimators with different values of c when $\boldsymbol{\theta} = 1$.

5. Discussion and Conclusion.

Through the simulation study it appears from tables 1 and 2 that $E(\hat{\theta}_{MQL}) < \boldsymbol{\theta}$ when c has a positive value and $E(\hat{\theta}_{MQL}) > \boldsymbol{\theta}$ for negative values of c, and It can be seen from Tables 1, 2 with Figures 1, 2, that the minimax estimator under squared error loss function is the best estimator when c=1, followed by the same estimator with c=1.5, and then Maximum Likelihood Estimator.

Also we can see that the minimax estimator under squared error loss function and the classical maximum likelihood estimator have approximately are identical when the value of c is positive and sample sizes n>20. Also, it can be seen from the results of the simulation study that the MLE and mo estimators are appear to be better than minimax estimator under quadratic loss function when the value of **c** is negative for all sample sizes.

We can also notice that, in minimax estimator when c has a positive value, MSE decrease as **θ** increases, but for each of MLE, and the minimax estimators for negative values of c, the MSE's inecrease as **θ** increases.

Thus, we suggest using the minimax estimator under squared error loss function with positive value of c for estimating **θ** parameter of the Maxwell distribution especially with a small sample sizes.

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تقدير Minimax لمعلمة توزيع ماكسويل تحت دالة الخسارة التربيعية

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المستخلص

يهتم هذا البحث بمشكلة إيجاد مقدرات Minimax للمعلمة θ لتوزيع ماكسويل (MW) لدوال الخسارة التربيعية من خلال تطبيق نظرية Lehmann [1950]. من خلال دراسة المحاكاة تم مقارنة أداء هذه الطريقة مع الطرائق التقليدية المتضمنة تقديرات الإمكان الأعظم والعزوم بالإعتماد على متوسط مربعات الخطأ (MSEs) وقد تم التوصل إلى أن مقدر Minimax مع قيم صغيرة موجبة الى c يعطي أفضل تقدير، يليه مقدر الإمكان الأعظم.