## الخلاصة

في هذا البحث ،قمنا بعرض ومناقشة خوارزمية لطريقة تحليلية لحل منظومة خطية لمعادلات فولتيرا التكاملية من النوع الأول ( $LVIE1^{st}$ ). الخوارزمية للطريقة التحليلية لهذه المنظومة تقوم على تحويل لابلاس والمعكوس العام.كما وتم تطبيق الخوارزمية المقترحة على بعض الأمثلة للتدليل على كفاءتها وبساطتها.

## Abstract

In this research, we have presented and discussed an algorithm for analytical method to solve linear system of Volterra integral equations of the first kind ( $LVIE1^{st}$ ). Algorithm of the analytical method for this system based on the Laplace transform and generalized inverse. The proposed algorithm has been applied to some examples to demonstrate the efficiency and simplicity.


## 1. Introduction

Biazar, Babolian and Islam [1] used Adomian decomposition method to solve linear and non-linear systems of Volterra integral equations of the first kind, where the number of the unknown functions ( n ) equal to the number of equations ( m ).

In this paper, we introduce an algorithm to solve a system of $LVIE1^{st}$ not necessarily ( n ) equal to ( m ) by using generalized inverse.

The general form for the system of $LVIE1^{st}$ is :

$$\int_0^x \sum_{j=1}^{n} k_{ij}(x,t)\, u_j(t)\, dt = f_i(x) \qquad i=1,\dots,m \qquad \text{....(1)}$$

where $f_i$ , $i = 1, \dots, m$ are known functions, $k_{ij}(x,t)$, $i = 1, \dots, m$ , $j = 1, \dots, n$ are the kernels of the ith integral equation and $u_j$ , $j = 1, \dots, n$ are unknown functions.

## 2. Preliminary Remarks

In this section, we summaries some important properties of Laplace transform [2,3], we shall need in order to reduce a system of eq.(1) to a matrix form also some definitions about generalized inverse and pseudo inverse [4,5].

## 2.1 Important properties

i) The convolution property: If $F_1(s)=L\{f_1\}$ and $F_2(s)=L\{f_2\}$ then,

$$L\left\{\int_0^x f_1(x-y)\, f_2(y)\, dy\right\} = F_1(s)\; F_2(s) \qquad \ldots(2)$$

ii) If $F(s)=L\{f\}$ then

$$L\left\{\int_0^x f(y)\, dy\right\} = \frac{F(s)}{s} \qquad \ldots(3)$$

## 2.2 Definition

For $A \in R^{m\times n}$ and $A^+ \in R^{n\times m}$, the following equations are used to define a generalized inverse, a reflexive generalized inverse, and a pseudo inverse of A ( Bullion and Odell 1971 )[6] :

$A\, A^+\, A = A$ … (4)

$A^+\, A\, A^+ = A^+$ … (5)

$(A\, A^+)^T = A\, A^+$ that is $A\, A^+$ is symmetric … (6)

$(A^+ A)^T = A^+\, A$ that is $A^+\, A$ is symmetric … (7)

Eqs.(4-7) are called the Penrose conditions (Penrose 1955) [7].

A generalized inverse of a matrix $A \in R^{m\times n}$ is a matrix $A^+ = A^{-1} \in R^{n\times m}$ satisfying eq.(4).

A reflexive generalized inverse of a matrix $A \in R^{m\times n}$ is a matrix $A^+ = A_r^{-1} \in R^{n\times m}$ satisfying eqs.(4 and 5) .

A pseudo inverse of a matrix $A \in R^{m\times n}$ is a matrix $A^+ = A \in R^{n\times m}$ satisfying eqs.(4 – 7).

A pseudo inverse is sometimes called the Moore – Penrose inverse after the pioneering works by Moore ( 1920, 1935 ) [8] and Penrose ( 1955 ) .

# 3. The Analytic Algorithm

## 3.1 Laplace Transform

A system of LVIE1st can be reduced to a matrix form by Laplace transform as follows:

Recall eq.(1):

$$\int_0^x \sum_{j=1}^{n} k_{ij}(x,t)\, u_j(t)\, dt = f_i(x) \qquad i=1,\dots,m$$

Take the Laplace transform of both sides, yields:

$$\sum_{j=1}^{n} L\left\{\int_0^x k_{ij}(x,t)\, u_j(t)\, dt\right\} = L\left\{f_i(x)\right\} \qquad i=1,\dots,m \qquad \dots(8)$$

Note that, the term $L\left\{\int_0^x k_{ij}(x,t)\, u_j(t)\, dt\right\}$ in the left hand side of the eq.(8) could not be evaluated unless $k_{ij}(x,t)$, $i=1,\dots,m$, $j=1,\dots,n$ are the difference kernel, that is $k_{ij}(x,t) = k_{ij}(x-t)$ or constant kernel.

If $k_{ij}(x,t)$ are difference kernels in eq.(8), then can be used the convolution property of Laplace transform of eq.(2) to get:

$$\sum_{j=1}^{n} K_{ij}(s)\, U_j(s) = F_i(s) \qquad i=1,\dots,m \qquad \dots(9)$$

where $K_{ij}(s) = L\left\{k_{ij}(x,t)\right\}$, $U_j(s) = L\left\{u_j(x)\right\}$ and $F_i(s) = L\left\{F_i(x)\right\}$ , $i=1,\dots,m$, $j=1,\dots,n$.

If $k_{ij}(x,t) = c$ where c is any constant, using eq.(3) to obtain:

$$\sum_{j=1}^{n} \frac{c\, U_j(s)}{s} = F_i(s) \qquad i=1,\dots,m \qquad \dots(10)$$

Consequently eqs.(9 and 10) are both systems of linear equations in $U_j(s)$, $j=1,\dots,m$. Solving it by generalized inverse to find $U_j(s)$, $j=1,\dots,m$.

Finally, using inverse Laplace transform on $u_j(x)$, $j=1,\dots,m$ to obtain the solution of the original system of LVIE1$^{st}$.

## 3.2 Generalized Inverse

Let A be m×n matrix of the rank(r) where r(A) < min { m , n } and A = FG such that F and G are two matrices also have rank(r). Then the generalized inverse $A^+$ of A can be obtained from the following relation [9]:

$$A^{+} = G^{T} ( F^{T} A G^{T} )^{-1} F^{T} \qquad \ldots(11)$$

Now construct the matrices F and G. Firstly the general Gaussian elimination procedure [10] is applied of the matrix A, we obtain a new matrix have the rows below the matrix all elements are zeros and the other rows up the matrix represent the matrix G. The number of rows of G is the rank of matrix A.

To find the matrix F. Firstly, we write the identity matrix ( I ) of order $m\times m$ if $n < m$ or $n\times n$ if $m < n$, then we apply the same operations which be applied on the matrix A ( to get the matrix G ), but we begin from the last to the first operation with change sign the addition or the subtraction. Finally, we find the matrix F from the first columns such that the number of columns in F equal to the number of rows in G.

This Analytic algorithm will be illustrated by some examples in the next section.

## 4. Examples:

The performance of proposed algorithm described in this paper will be tested it on three systems of $LVIE1^{st}$.

**Example (1):**

Consider a system of $LVIE1^{st}$ with the exact solutions: $f(x) = x^2$ and $g(x) = x$

$$\int_0^x [f(y) - g(y)]\, dy = \frac{x^3}{3} - \frac{x^2}{2}$$

$$\int_0^x [f(y) + (x - y)\, g(y)]\, dy = \frac{x^3}{3}$$

$$\int_0^x (x - y)\, f(y)\, dy = \frac{x^4}{12}$$

The algorithm starts. Taking Laplace transform with using eqs.(9 and 10) and simplify we get:

$$F(s) - G(s) = \frac{2}{s^3} - \frac{1}{s^2}$$

$$s\, F(s) + G(s) = \frac{3}{s^2}$$

$$F(s) = \frac{2}{s^3}$$

The above linear system can be described by the following matrix form:

$$AU = B \qquad \ldots(12)$$

where $A=\begin{pmatrix}1 & -1\\ s & 1\\ 1 & 0\end{pmatrix}$ , $U=\begin{pmatrix}F(s)\\ G(s)\end{pmatrix}$ and $B=\begin{pmatrix}(2-s)/s^3\\ 3/s^2\\ 2/s^3\end{pmatrix}$

Then, eq.(12) can be solved for the vector U, of coefficients by generalized inverse (subsection 3.2) can be summarized by the following steps:

Step 1:

Construct the matrices F and G such that A = F G as follows:

$$A=\begin{pmatrix}1 & -1\\ s & 1\\ 1 & 0\end{pmatrix}\xrightarrow[R_3-R_1]{R_2-sR_1}\begin{pmatrix}1 & -1\\ 0 & 1+s\\ 0 & 1\end{pmatrix}\xrightarrow{R_3-\frac{1}{1+s}R_2}\begin{pmatrix}1 & -1\\ 0 & 1+s\\ 0 & 0\end{pmatrix}=\begin{pmatrix}G\\ \ldots\\ 0\end{pmatrix},$$

Then

$$G=\begin{pmatrix}1 & -1\\ 0 & 1+s\end{pmatrix}$$

Now, to find F, let the identity matrix of order $3\times 3$ :

$$\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\xrightarrow{R_3+\frac{1}{1+s}R_2}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1/1+s & 1\end{pmatrix}\xrightarrow[R_3+R_1]{R_2+sR_1}\begin{pmatrix}1 & 0 & 0\\ s & 1 & 0\\ 1 & 1/1+s & 1\end{pmatrix}=\begin{pmatrix} & & 0\\ F & \vdots & 0\\ & & 1\end{pmatrix}$$

That is

$$F=\begin{pmatrix}1 & 0\\ s & 1\\ 1 & 1/1+s\end{pmatrix}$$

Step (2):

Using eq.(11) to find the generalized inverse $A^+$ of matrix A and we obtain:

$$A^+=\frac{1}{s^2+2s+3}\begin{pmatrix}1+s & 1+s & 2\\ -(s^2+s+1) & 2+s & 1-s\end{pmatrix}$$

Step (3):

$$U = A^{+} B = \begin{pmatrix} 2/s^3 \\ 1/s^2 \end{pmatrix} = \begin{pmatrix} F(s) \\ G(s) \end{pmatrix} \qquad \ldots(13)$$

Step (4):

Using inverse Laplace transformation for both side of eq.(13) we have:

$$\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} x^2 \\ x \end{pmatrix}$$

**Example (2):**

Consider the following system of integral equations of the first kind:

$$\int_0^x [f(y) + h(y)]\, dy = \frac{x^3}{3} + 2x$$

$$\int_0^x [f(y) + 2g(y) - h(y)]\, dy = x^3$$

where the exact solution is : $f(x) = x^2 + 1$ , $g(x) = x^2$ and $h(x) = 1$

By eq.(10) the above system can be written as matrix form as follows :

$$AU = B$$

where

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 2 & -1 \end{pmatrix} \quad , \quad B = \begin{pmatrix} 2/s^3 + 2/s^2 \\ 6/s^3 \end{pmatrix} \quad \text{and} \quad U = \begin{pmatrix} F(s) \\ G(s) \\ H(s) \end{pmatrix}$$

In this example let $A^T = C$ and then apply the same steps in the example(1) we have :

$$C^{+} = \frac{1}{6} \begin{pmatrix} 3 & 0 & 3 \\ 1 & 2 & -1 \end{pmatrix}$$

Now, we obtain:

$$C^{+} = (A^T)^{+} = (A^{+})^T$$

we get

$$A^{+} = \frac{1}{6}\begin{pmatrix} 3 & 1 \\ 0 & 2 \\ 3 & -1 \end{pmatrix}, \quad \text{then} \quad U = A^{+}B = \begin{pmatrix} 2/s^3 + 1/s^2 \\ 2/s^3 \\ 1/s^2 \end{pmatrix}$$

Finally, by inverse Laplace transform we have the exact solution of original system :

$$\begin{pmatrix} f(x) \\ g(x) \\ h(x) \end{pmatrix} = \begin{pmatrix} x^2 + 1 \\ x^2 \\ 1 \end{pmatrix}$$

**Example (3):**

Consider the following system:

$$\int_0^x f(y)\,dy = 2x^2$$

$$\int_0^x \left[f(y) - (x-y)(g(y) - h(y))\right] dy = 2x^2 - \frac{x^3}{3}$$

$$\int_0^x \left[g(y) - h(y)\right] dy = x^2$$

where the exact solution is: $f(x) = 4x$ , $g(x) = x$ and $h(x) = -x$

The generalized inverse $A^{+}$ is :

$$A^{+} = \frac{1}{4+2s^2}\begin{pmatrix} 4 & 2s & 2s \\ s & -1 & 1+s^2 \\ -s & 1 & -1-s^2 \end{pmatrix} \quad \text{and} \quad U = A^{+}B = \begin{pmatrix} 4/s^2 \\ 1/s^2 \\ -1/s^2 \end{pmatrix}$$

Then, the exact solution is $U = \begin{pmatrix} 4x \\ x \\ -x \end{pmatrix}$

In this example we solve linear system of equations with singular matrix.

## 5. Conclusions

It has been used a new approach based Moore – Penrose generalized inverse to solve linear system of Volterra integral equations of the first kind (LVIE1$^{st}$) where the matrix is not square and square matrix with singular matrix.

We described the method of solution by analytic algorithm. This algorithm makes the calculation clearly simple without the need to use computer programming and efficiency of this method for solving these problems has been approved by some examples.

## 6. References


[1] Biazar, J.,Babolian, E. & Islam, R., "Solution of a system of Volterra integral equations of the first kind by Adomian method", Applied Mathematics and Computation, 13, 249 – 253, 2003.

[2] Spiegel, M.R., "Laplace transforms", Mc. Graw – Hill, Inc., NewYork, 1965.

[3] Jerri, A.J., "Introduction to integral equations with applications", Marcel Dekker, Inc., NewYork and Basel, 1985.

[4] Rao, C.R., "Estimation of variance and covariance components in linear models", JASA67, 112 – 115, 1972.

[5] Searle, S.R., "Variance Components some history and a summary account of estimation methods", J., Animal Breeding & Genetics 106, 1 – 29, 1989.

[6] Boullion, T.L., and Odell, P.L., "Generalized Inverse Matrices" John Wiley &Sons, Inc., New York , 1971.

[7] Penrose, R.A.,"A Generalized Inverse for Matrices" Proceedings of the Cambridge philosophical society 51, 406 – 13, 1955.

[8] Moore, E.H.,"On the Reciprocal of the General Algebraic Matrix. (Abstract)" Bulletin of the American Mathematical Society26,394– 94, 1920.

[9] Ahmed, S.O., "Suggestion method to compute Minque estimators for variance components using Moore–Penrose generalized inverse", M.Sc.Thesis, Al-Mustansiria University, 2006.

[10] Burden, R.L. & Faires, J.D., "Numerical analysis", seventh edition Thomson Learning, Inc., Wadsworth Group, 2001.